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Original Research Paper

Hyers-Ulam Stability of Some Linear Operators on a Hilbert Space

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Abstract. Suppose that T is a bounded operator from a Hilbert space H into H. In this paper, for an injective cohyponormal or complex symmetric operator T, we find a necessary and sufficient condition for T to have the Hyers-Ulam stability. Moreover, when T is injective, we find necessary and sufficient conditions for T^*T to have the Hyers-Ulam stability.

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1 Introduction

The first stability problem concerning group homomorphisms was raised by Ulam [16] in a conference at Wisconsin University, Madison 1940. Suppose that G_1 is a group and G_2 is a metric group with a metric $d(.,.)$. For each $\varepsilon > 0$, does there exist a $\delta > 0$ so that if a function $h : G_1 \to G_2$

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satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for each $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for each $x \in G_1$? If the answer is positive, the equation of homomorphism $H(xy) = H(x)H(y)$ is called stable. In 1941, Hyers [9] obtained the first important result, which we now call the Hyers-Ulam stability by giving an answer to the question of Ulam by considering approximately mappings for the case where G_1 and G_2 are Banach spaces. After Hyers result several mathematicians were devoted to study Hyers-Ulam stability for various equations. The result of Hyers has been generalized by Aoki [1] for additive mapping and by Rassias [13] which allows the Cauchy difference to be unbounded. In recent years a large amount of researchers have investigated the stability of many algebraic, differential, integral, operatorial, functional equations (see [5, 7, 8, 10, 12, 14] and the references there in). The Hyers-Ulam stability of linear operators was considered for the first time by Miura et al. (see [7, 8, 11]). In [12], the authors remarked that a bounded linear operator between Banach spaces has the Hyers-Ulam stability if and only if it has closed range. Let H be a Hilbert space. The set of all bounded operators from H into itself is denoted by $B(H)$. For $T \in B(H)$, we use $N(T)$ to denote the set of all elements $x \in H$ that $T(x) = 0$ and $\mathrm{Ran}(T)$ to denote the set of all elements $T(x)$ that $x \in H$. In this paper, we devote to study Hyers-Ulam stability for some operators in $B(H)$.

2 Hyers-Ulam Stability of Linear Operator T

Let X and Y be Banach spaces and T be a mapping from X into Y. We say that the mapping T has the Hyers-Ulam stability, if there exists a constant k so that for any $g \in T(X)$, $\varepsilon > 0$ and $f \in X$ satisfying $\|Tf - g\| \leq \varepsilon$, we can find an $f_0 \in X$ such that $Tf_0 = g$ and $\|f - f_0\| \leq k\varepsilon$. We call such $k > 0$ a Hyers-Ulam stability constant for T, and denote by K_T the infimum of all Hyers-Ulam stability constant for T. About these concepts, we recommend the research papers [9] and [17]. Miura at el. introduced these concepts in [11], and gave a characterization in order that the operator has the Hyers-Ulam stability, and they obtained a sufficient and necessary condition. One of their illustrative examples were discussed in the paper [15].

By the linearity of T, T has the Hyers-Ulam stability if and only if there exists a constant k with the following property: For given $f \in X$, there is an $f_0 \in X$ such that $Tf = Tf_0$ and $\|f_0\| \leq k\|Tf\|$. For $T \in B(X)$, we denote the null space of T by $N(T)$ and the range of T by $R(T)$. We consider the operator $\widetilde{T}$ from the quotient space $\frac{X}{N(T)}$ into X by $\widetilde{T}(f + N(T)) = Tf$, for all $f \in X$. Clearly $\widetilde{T}$ is an injective continuous linear operator from $\frac{X}{N(T)}$ onto $R(T)$ and from the Open Mapping Theorem the operator $\widetilde{T}^{-1}$ is continuous. In Proposition 2.1, one can see that the the operator $\widetilde{T}^{-1}$ from $T(X)$ into $\frac{X}{N(T)}$ is closely related to the Hyers-Ulam stability of T.

First, in the following proposition, we state a necessary and sufficient condition for T to have the Hyers-Ulam stability.

Proposition 2.1. (See [12, Theorem 2]) *For a bounded linear operator T on a Banach space, the following statements are equivalent.*
(a) T has the Hyers-Ulam stability.
(b) T has closed range.
(c) $\widetilde{T}^{-1}$ is bounded.
Moreover, in this case $K_T = \|\widetilde{T}^{-1}\|$.

In the next proposition, we set some conditions on T_1 and T_2 such that T_1T_2 has the Hyers-Ulam stability.

Proposition 2.2. *Let T_1 and T_2 be in $B(H)$. Assume that T_1 and T_2 have the Hyers-Ulam stabilities. If $N(T_1) = \{0\}$, then T_1T_2 has the Hyers-Ulam stability.*

Proof. Let f be in H. Since T_2 has the Hyers-Ulam stability, there exists $f_0 \in H$ such that $T_2f_0 = T_2f$ and

$$\|f_0\| \leq k\|T_2f\|, \tag{1}$$

where k is the Hyers-Ulam stability constant for T_2. Suppose that k' is the Hyers-Ulam stability constant for T_1. Therefore, we can find $f_1 \in H$

such that $T_1 f_1 = T_1 T_2 f$ and

$$\|f_1\| \leq k' \|T_1 T_2 f\|. \tag{2}$$

Since T_1 is injective, $f_1 = T_2 f$. Hence by Equations (1) and (2), $\|f_0\| \leq k'k\|T_1T_2f\|$ and $T_1T_2f_0 = T_1T_2f$, so the proof is complete. □

In the following proposition, we investigate that when the operator T_2 has the Hyers-Ulam stability whenever T_1T_2 has the Hyers-Ulam stability.

Proposition 2.3. *Suppose that T_1 and T_2 belong to $B(H)$. Assume that T_1T_2 has the Hyers-Ulam stability. If T_1 is injective on $T_2(H)$, then T_2 has the Hyers-Ulam stability.*

Proof. Since T_1T_2 has the Hyers-Ulam stability, for given $f \in H$, there is $f_0 \in H$ such that $T_1T_2f = T_1T_2f_0$ and $\|f_0\| \leq k\|T_1T_2f\|$, where k is the Hyers-Ulam stability constant for T_1T_2. Since T_1 is a bounded operator, $\|f_0\| \leq k\|T_1\|\|T_2f\|$. Note that T_1 is injective on $T_2(H)$, so $T_2f = T_2f_0$. It follows immediately that T_2 has the Hyers-Ulam stability. □

Recall that a bounded operator T on a Hilbert space H is said to be cohyponormal if $TT^* \geq T^*T$. Now for an injective cohyponormal operator T, we characterize this operator which has the Hyers-Ulam stability.

Theorem 2.4. *Let $T \in B(H)$ be a cohyponormal operator and $N(T) = \{0\}$. Then T has the Hyers-Ulam stability if and only if T is invertible.*

Proof. If T is invertible, then T has closed range. Hence by Proposition 2.1, T has the Hyers-Ulam stability.
Conversely, suppose that T has the Hyers-Ulam stability. Again by Proposition 2.1, T has closed range. Since $N(T) = \{0\}$ and T is cohyponormal, $N(T^*) = \{0\}$. Then by [2, Theorem 2.19, p. 35] and [2, Corollary 2.10, p. 10], T has dense range. Therefore, $\mathrm{Ran}(T) = H$ which shows that T is invertible. □

A bounded linear operator T on a complex Hilbert space H is complex symmetric if there is a conjugation C (an isometric, antilinear and involution) such that $CT^*C = T$. The complex symmetric operators class was defined by Garcia and Putinar (see [3] and [4]) and includes the Volterra integration operators, Hankel operators and normal operators. In the next theorem, we see that for a complex symmetric operator T, an analogue of Theorem 2.4 holds.

Theorem 2.5. *Let $T \in B(H)$ be a complex symmetric operator and $N(T) = \{0\}$. Then T has the Hyers-Ulam stability if and only if T is invertible.*

Proof. Suppose that T has the Hyers-Ulam stability and T is complex symmetric with conjugation C. If $x \in N(T^*)$, then we obtain that $TCx = CT^*x = 0$. Since $N(T) = \{0\}$ and C is an isometry, $x = 0$. It means that $N(T^*) = \{0\}$. Then by [2, Theorem 2.19, p. 35] and [2, Corollary 2.10, p. 10], T has dense range and so $\mathrm{Ran}(T) = H$. Therefore, T is invertible.
Conversely, the result follows by the same idea which was stated in the proof of Theorem 2.4. □

In the continuation of this paper, first we state two results which will be used to find necessary and sufficient conditions for T^*T to have the Hyers-Ulam stability.

Lemma 2.6. *Let H be a Hilbert space and $T \in B(H)$. If $N(T) = \{0\}$, then $N(T^*T) = \{0\}$.*

Proof. Suppose that there is $h \in H$ such that $T^*Th = 0$. We see that $0 = \|Th\|^2 = \langle Th, Th\rangle = \langle T^*Th, h\rangle$. Since $N(T) = \{0\}$, $h = 0$ and the result follows. □

For $T \in B(H)$, we define $|T| = (T^*T)^{1/2}$. A partial isometry is an operator U such that $\|Uh\| = \|h\|$ for each $h \in (N(U))^{\perp}$. Recall that for $T \in B(H)$, the polar decomposition $T = U|T|$ expresses T uniquely as the product of the positive operator $|T|$ and a partial isometry U with $N(U) = N(|T|)$ and which maps $\overline{\mathrm{Ran}(|T|)}$ onto $\overline{\mathrm{Ran}(T)}$. Note that a necessary and sufficient condition that U be an isometry (i.e., $\langle Uf, Ug\rangle = \langle f, g\rangle$ for all $f, g \in H$) is that $N(T) = \{0\}$ (see [6]).

Proposition 2.7. *Let $T \in B(H)$ and $N(T) = \{0\}$. The operator $|T|$ has the Hyers-Ulam stability if and only if T has the Hyers-Ulam stability.*

Proof. Assume that $|T|$ has the Hyers-Ulam stability. Suppose that $T = U|T|$ is the polar decomposition of T. Let $Tx_n \to y$ as $n \to \infty$. Then $U|T|x_n \to y$ as $n \to \infty$. It shows that $|T|x_n \to U^*y$ as $n \to \infty$ because U is an isometry (see [2, Proposition 2.17 (b), p. 35]). Since $|T|$ has closed range, there is $y_0 \in H$ such that $U^*y = |T|y_0$. Since $|T|x_n \to |T|y_0$ as $n \to \infty$, $U|T|x_n \to U|T|y_0$ as $n \to \infty$ and so T has closed range. The result follows from Proposition 2.1.
Conversely, suppose that T has the Hyers-Ulam stability. Since T is injective, by the polar decomposition $T = U|T|$, we can see that U is injective on Ran$(|T|)$. Then Proposition 2.3 shows that $|T|$ has the Hyers-Ulam stability. □

Assume that H and H' are Hilbert spaces and $T : H \to H'$ is a bounded operator. The operator T is left semi-Fredholm if there exists a bounded linear operator $T' : H' \to H$ and a compact operator K on H so that $T'T = I + K$. Also, T is said to be right semi-Fredholm if there is a bounded operator $T' : H' \to H$ and a compact operator K' on H' so that $TT' = I + K'$. We say that an operator T is Fredholm if it is both left and right semi-Fredholm. It is not hard to see that T is left semi-Fredholm if and only if T^* is right semi-Fredholm [2].

Theorem 2.8. *Let H be a Hilbert space and $T \in B(H)$. If $N(T) = \{0\}$, then the following are equivalent.*

- *(a) The operator T^*T is Fredholm.*
- *(b) The operator T^*T is invertible.*
- *(c) The operator T^*T has the Hyers-Ulam stability.*
- *(d) For each positive integer n, T^n has the Hyers-Ulam stability.*

Proof. (b)⇔(c): By Lemma 2.6 and Theorem 2.4, it is clear.

(a)⇒(b): Since T^*T is Fredholm, by [2, Theorem 2.3, p. 350], Ran(T^*T) is closed. Hence by Proposition 2.1, T^*T has the Hyers-Ulam stability. Lemma 2.6 and Theorem 2.4 shows that T^*T is invertible.

(b)⇒(a): Since T^*T is self-adjoint, by [2, Theorem 2.3, p. 350], the result follows.

(d)⇒(c): Let for each positive integer n, the operator T^n has the Hyers-Ulam stability. Then T has the Hyers-Ulam stability. Proposition 2.7 shows that $|T|$ has the Hyers-Ulam stability. Since $\mathrm{N}(T) = \{0\}$, it is easy to see that $|T|$ is njective. Then by Proposition 2.2, $|T|^2 = T^*T$ has the Hyers Ulam stability.

(c)⇒(d): Let $|T|^2 = T^*T$ have the Hyers-Ulam stability. Since by Lemma 2.6, $\mathrm{N}(T^*T) = \{0\}$, we obtain $\mathrm{N}(|T|) = \{0\}$. Proposition 2.3 states that $|T|$ has the Hyers Ulam stability. Proposition 2.7 implies that T has the Hyers-Ulam stability. Then by Proposition 2.2 and induction, for each positive integer n, the operator T^n has the Hyers Ulam stability. □

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