Hyers-Ulam stability of some linear operators on a Hilbert space

Mahmood Haji Shaabani and Malihe Farzi Haromi

February 17, 2022

Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran.

E-mail: shaabani@sutech.ac.ir

Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran.

E-mail: m.farzi@sutech.ac.ir

Abstract

Suppose that T is a bounded operator from a Hilbert space H into H. In this paper, for an injective cohyponormal or complex symmetric operator T, we find a necessary and sufficient condition for T to have the Hyers-Ulam stability. Moreover, when T is injective, we find necessary and sufficient conditions for T^*T to have the Hyers-Ulam stability.

1 Introduction

The first stability problem concerning group homomorphisms was raised by Ulam [15] in a conference at Wisconsin University, Madison 1940. Suppose that G_1 is a group and G_2 is a metric group with a metric $d(.,.)$. For each $\varepsilon > 0$, does there exist a $\delta > 0$ so that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for each $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for each $x \in G_1$? If the answer is positive, the equation of homomorphism $H(xy) = H(x)H(y)$ is called stable. In 1941, Hyers [9] obtained the first important result, which

AMS Subject Classifications. 34K20

Key words and phrases: Hyers-Ulam stability, cohyponormal, complex symmetric, Fredholm.

we now call the Hyers-Ulam stability by giving an answer to the question of Ulam by considering approximately mappings for the case where G_1 and G_2 are Banach spaces. After Hyers result several mathematicians were devoted to study Hyers-Ulam stability for various equations. The result of Hyers has been generalized by Aoki [1] for additive mapping and by Rassias [12] which allows the Cauchy difference to be unbounded. In recent years a large amount of researchers have investigated the stability of many algebraic, differential, integral, operatorial, functional equations (see [5, 7, 8, 11, 13] and the references there in). The Hyers-Ulam stability of linear operators was considered for the first time by Miura et al. (see [7, 8, 10]). In [11], the authors remarked that a bounded linear operator between Banach spaces has the Hyers-Ulam stability if and only if it has closed range. Let H be a Hilbert space. The set of all bounded operators from H into itself is denoted by $B(H)$. For $T \in B(H)$, we use $N(T)$ to denote the set of all elements $x \in H$ that $T(x) = 0$ and $\mathrm{Ran}(T)$ to denote the set of all elements $T(x)$ that $x \in H$. In this paper, we devote to study Hyers-Ulam stability for some operators in $B(H)$.

2 Hyers-Ulam stability of linear operator T

Let X and Y be Banach spaces and T be a mapping from X into Y. We say that the mapping T has the Hyers-Ulam stability, if there exists a constant k so that for any $g \in T(X)$, $\varepsilon > 0$ and $f \in X$ satisfying $\|Tf - g\| \leq \varepsilon$, we can find an $f_0 \in X$ such that $Tf_0 = g$ and $\|f - f_0\| \leq k\varepsilon$. We call such $k > 0$ a Hyers-Ulam stability constant for T, and denote by K_T the infimum of all Hyers-Ulam stability constant for T. About these concepts, we recommend the research papers [9] and [16]. Miura at el. introduced these concepts in [10], and gave a characterization in order that the operator has the Hyers-Ulam stability, and they obtained a sufficient and necessary condition. One of their illustrative examples were discussed in the paper [14].

By the linearity of T, T has the Hyers-Ulam stability if and only if there exists a constant k with the following property: For given $f \in X$, there is an $f_0 \in X$ such that $Tf = Tf_0$ and $\|f_0\| \leq k\|Tf\|$. For $T \in B(X)$, we denote the null space of T by $N(T)$ and the range of T by $R(T)$ and consider the operator $\widetilde{T}$ from the quotient space $\frac{X}{N(T)}$ into X by $\widetilde{T}(f + N(T)) = Tf$, for all $f \in X$. Clearly $\widetilde{T}$ is an injective continuous linear operator from $\frac{X}{N(T)}$ onto $R(T)$ and from the Open Mapping Theorem the operator $\widetilde{T}^{-1}$ is continuous. In Proposition 2.1, one can see that the the operator $\widetilde{T}^{-1}$ from $T(X)$ into $\frac{X}{N(T)}$ is closely related to the Hyers-Ulam stability of T.

First, in the following proposition, we state a necessary and sufficient condition for T to have the Hyers-Ulam stability.

Proposition 2.1. ([11, Theorem 2]). *For a bounded linear operator T on a Banach space, the following statements are equivalent.*
(a) T has the Hyers-Ulam stability.
(b) T has closed range.
(c) $\widetilde{T}^{-1}$ is bounded.
Moreover, in this case $K_T = \|\widetilde{T}^{-1}\|$.

In the next proposition, we set some conditions on T_1 and T_2 such that T_1T_2 has the Hyers-Ulam stability.

Proposition 2.2. *Let T_1 and T_2 be in $B(H)$. Assume that T_1 and T_2 have the Hyers-Ulam stabilities. If $N(T_1) = \{0\}$, then T_1T_2 has the Hyers-Ulam stability.*

Proof. Let f be in H. Since T_2 has the Hyers-Ulam stability, there exists $f_0 \in H$ such that $T_2f_0 = T_2f$ and

$$\|f_0\| \leq k\|T_2f\|, \tag{1}$$

where k is the Hyers-Ulam stability constant for T_2. Suppose that k' is the Hyers-Ulam stability constant for T_1. Therefore, we can find $f_1 \in H$ such that $T_1f_1 = T_1T_2f$ and

$$\|f_1\| \leq k'\|T_1T_2f\|. \tag{2}$$

Since T_1 is injective, $f_1 = T_2f$. Hence by Equations (1) and (2), $\|f_0\| \leq k'k\|T_1T_2f\|$ and $T_1T_2f_0 = T_1T_2f$, so the proof is complete. □

In the following proposition, we investigate that when the operator T_2 has the Hyers-Ulam stability whenever T_1T_2 has the Hyers-Ulam stability.

Proposition 2.3. *Suppose that T_1 and T_2 belong to $B(H)$. Assume that T_1T_2 has the Hyers-Ulam stability. If T_1 is injective on $T_2(H)$, then T_2 has the Hyers-Ulam stability.*

Proof. Since T_1T_2 has the Hyers-Ulam stability, for given $f \in H$, there is $f_0 \in H$ such that $T_1T_2f = T_1T_2f_0$ and $\|f_0\| \leq k\|T_1T_2f\|$, where k is the Hyers-Ulam stability constant for T_1T_2. Since T_1 is a bounded operator,

$\|f_0\| \leq k\|T_1\|\|T_2 f\|$. Note that T_1 is injective on $T_2(H)$, so $T_2 f = T_2 f_0$. It follows immediately that T_2 has the Hyers-Ulam stability. $\square$

Recall that a bounded operator T on a Hilbert space H is said to be cohyponormal if $TT^* \geq T^*T$. Now for an injective cohyponormal operator T, we characterize this operator which has the Hyers-Ulam stability.

Theoram 2.4. *Let $T \in B(H)$ be a cohyponormal operator and $N(T) = \{0\}$. Then T has the Hyers-Ulam stability if and only if T is invertible.*

Proof. If T is invertible, then T has closed range. Hence by Proposition 2.1, T has the Hyers-Ulam stability.
Conversely, suppose that T has the Hyers-Ulam stability. Again by Proposition 2.1, T has closed range. Since $N(T) = \{0\}$ and T is cohyponormal, $N(T^*) = \{0\}$. Then by [2, Theorem 2.19, p. 35] and [2, Corollary 2.10, p. 10], T has dense range. Therefore, $\mathrm{Ran}(T) = H$ which shows that T is invertible. $\square$

A bounded linear operator T on a complex Hilbert space H is complex symmetric if there is a conjugation C (an isometric, antilinear and involution) such that $CT^*C = T$. The complex symmetric operators class was defined by Garcia and Putinar (see [3] and [4]) and includes the Volterra integration operators, Hankel operators and normal operators. In the next theorem, we see that for a complex symmetric operator T, an analogue of Theorem 2.4 holds.

Theorem 2.5. *Let $T \in B(H)$ be a complex symmetric operator and $N(T) = \{0\}$. Then T has the Hyers-Ulam stability if and only if T is invertible.*

Proof. Suppose that T has the Hyers-Ulam stability and T is complex symmetric with conjugation C. If $x \in N(T^*)$, then we obtain that $TCx = CT^*x = 0$. Since $N(T) = \{0\}$ and C is an isometry, $x = 0$. It means that $N(T^*) = \{0\}$. Then by [2, Theorem 2.19, p. 35] and [2, Corollary 2.10, p. 10], T has dense range and so $\mathrm{Ran}(T) = H$. Therefore, T is invertible.
Conversely, the result follows by the same idea which was stated in the proof of Theorem 2.4. $\square$

In the continuation of this paper, first we state two results which will be used to find necessary and sufficient conditions for T^*T to have the Hyers-Ulam stability.

Lemma 2.6. *Let H be a Hilbert space and $T \in B(H)$. If $N(T) = \{0\}$, then $N(T^*T) = \{0\}$.*

Proof. Suppose that there is $h \in H$ such that $T^*Th = 0$. We see that $0 = \|Th\|^2 = \langle Th, Th\rangle = \langle T^*Th, h\rangle$. Since $N(T) = \{0\}$, $h = 0$ and the result follows. □

For $T \in B(H)$, we define $|T| = (T^*T)^{1/2}$. A partial isometry is an operator U such that $\|Uh\| = \|h\|$ for each $h \in (N(U))^{\perp}$. Recall that for $T \in B(H)$, the polar decomposition $T = U|T|$ expresses T uniquely as the product of the positive operator $|T|$ and a partial isometry U with $N(U) = N(|T|)$ and which maps $\overline{\text{Ran}(|T|)}$ onto $\overline{\text{Ran}(T)}$. Note that a necessary and sufficient condition that U be an isometry (i.e., $\langle Uf, Ug\rangle = \langle f, g\rangle$ for all $f, g \in H$) is that $N(T) = \{0\}$ (see [6]).

Proposition 2.7. *Let $T \in B(H)$ and $N(T) = \{0\}$. The operator $|T|$ has the Hyers-Ulam stability if and only if T has the Hyers-Ulam stability.*

Proof. Assume that $|T|$ has the Hyers-Ulam stability. Suppose that $T = U|T|$ is the polar decomposition of T. Let $Tx_n \to y$ as $n \to \infty$. Then $U|T|x_n \to y$ as $n \to \infty$. It shows that $|T|x_n \to U^*y$ as $n \to \infty$ because U is an isometry (see [2, Proposition 2.17 (b), p. 35]). Since $|T|$ has closed range, there is $y_0 \in H$ such that $U^*y = |T|y_0$. Since $|T|x_n \to |T|y_0$ as $n \to \infty$, $U|T|x_n \to U|T|y_0$ as $n \to \infty$ and so T has closed range. The result follows from Proposition 2.1.
Conversely, suppose that T has the Hyers-Ulam stability. Since T is injective, by the polar decomposition $T = U|T|$, we can see that U is injective on $\text{Ran}(|T|)$. Then Proposition 2.3 shows that $|T|$ has the Hyers-Ulam stability.
□

Assume that H and H' are Hilbert spaces and $T : H \to H'$ is a bounded operator. The operator T is left semi-Fredholm if there exists a bounded linear operator $T' : H' \to H$ and a compact operator K on H so that $T'T = I + K$. Also, T is said to be right semi-Fredholm if there is a bounded operator $T' : H' \to H$ and a compact operator K' on H' so that $TT' = I + K'$. We say that an operator T is Fredholm if it is both left and right semi-Fredholm. It is not hard to see that T is left semi-Fredholm if and only if T^* is right semi-Fredholm [2].

Theorem 2.8. *Let H be a Hilbert space and $T \in B(H)$. If $N(T) = \{0\}$, then the following are equivalent.*

(a) The operator T^*T is Fredholm.

(b) The operator T^*T is invertible.

(c) The operator T^*T has the Hyers-Ulam stability.

(d) For each positive integer n, T^n has the Hyers-Ulam stability.

Proof. (b)⇔(c): By Lemma 2.6 and Theorem 2.4, it is clear.

(a)⇒(b): Since T^*T is Fredholm, by [2, Theorem 2.3, p. 350], Ran(T^*T) is closed. Hence by Proposition 2.1, T^*T has the Hyers-Ulam stability. Lemma 2.6 and Theorem 2.4 shows that T^*T is invertible.

(b)⇒(a): Since T^*T is self-adjoint, by [2, Theorem 2.3, p. 350], the result follows.

(d)⇒(c): Let for each positive integer n, the operator T^n has the Hyers-Ulam stability. Then T has the Hyers-Ulam stability. Proposition 2.7 shows that $|T|$ has the Hyers-Ulam stability. Since N(T) = $\{0\}$, it is easy to see that $|T|$ is njective. Then by Proposition 2.2, $|T|^2 = T^*T$ has the Hyers Ulam stability.

(c)⇒(d): Let $|T|^2 = T^*T$ have the Hyers-Ulam stability. Since by Lemma 2.6, N(T^*T) = $\{0\}$, we obtain N($|T|$) = $\{0\}$. Proposition 2.3 states that $|T|$ has the Hyers Ulam stability. Proposition 2.7 implies that T has the Hyers-Ulam stability. Then by Proposition 2.2 and induction, for each positive integer n, the operator T^n has the Hyers Ulam stability. □

References

[1] T. Aoki, On the stability of linear transformation in Banach spaces, *J. Math. Soc.* Japan, **2** (1950), 64-66.

[2] J. B. Conway, A Course in Functional Analysis, Second Edition, Springer-Verlag, New York, 1990.

[3] S. R. Garcia and M. Putinar, Complex symmetric operators and applications, *Trans. Amer. Math. Soc.* **358** (2006), 1285-1315.

[4] S. R. Garcia and M. Putinar, Complex symmetric operators and applications II, *Trans. Amer. Math. Soc.* **359** (2007), 3913-3931.

[5] P. Găvruţa, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994) 431-436.

[6] P. R. Halmos, A Hilbert Space Problem Book, Second Edition, Springer-Verlag, New York, 1982.

[7] O. Hatori, K. Kobayashi, T. Miura, H. Takagi and S. E. Takahasi, On the best constant of Hyers-Ulam stability, *J. Nonlinear Convex Anal.* **5** (2004), 387-393.

[8] G. Hirasawa and T. Miura, Hyers-Ulam stability of a closed operator in a Hilbert space, *Bull Korean Math. Soc.* **43** (2006), 107-117.

[9] D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, **27** (1941), 222-224.

[10] T. Miura, M. Miyajima and S. E. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, *Math. Nachr.* **258** (2003), 90-96.

[11] T. Miura, H. Takagi and S. E. Takahasi, Essential norms and stability constants of weighted composition operators on C(X), *Bull. Korean Math. Soc.* **40** (2003), 583-591.

[12] Th. M. Rassias, On the stability of the linear mappings in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297-300.

[13] Th. M. Rassias, C. Wang and T. Z. Xu, On the stability of multi-additive mappings in non-Archimedean normed spaces, *J. Comput. Anal. Appl.* **18** (2015).

[14] S. E. Takahasi, T. Miura and S. Miyajima, On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$, *Bull. Korean Math. Soc.* **39** (2002), 309-315.

[15] S. M. Ulam, A Collection of Mathematical Problems, Interscience, New York, 1960.

[16] S. M. Ulam, Problems in Modern Mathematics, Chapter VI, Science Editions, Wiley, New York, 1964.