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Original Research Paper

Common Fixed Points of a Pair of H^{β}-Hausdorff Multivalued Operators in b-Metric Space and Application to Integral Equations

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Abstract. A common fixed point theorem for a pair of H^{β}-Hausdorff multi-valued operators for $\beta \in [0, 1]$ is proved in a b-metric space. Our result is a proper extension and new variants of many well known contraction conditions existing in literature. As an application of our main result, we have proved an existence result for a common solution of a pair of nonlinear Volterra type integral equations.

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1 Introduction

The study of a metric function on the set of closed and bounded subsets of a metric space was initiated by Pompeiu in [30] and then continued by Hausdorff [18]. Such a metric function is reffered to as the Hausdorff-Pompeiu metric. On the other hand Bakhtin [7] introduced the concept of a b-metric space as a generalisation of metric space and proved Banach's contraction principle in a b-metric space. Some recent interesting results on contraction principles in a b-metric space and its applications can be found in [1, 4, 5, 8, 13, 16, 17, 22, 29, 32, 34]. Banach's contraction principle was extended to a multi-valued function in a metric space by Nadler [28] and in a b-metric space by Czerwik [11] using the Hausdorff-Pompieu metric H. Further generalized results of multivalued contractions can be found in [2, 12, 20, 21, 23, 25]. Czervik's contraction was also generalised in many directions, to name a few q-quasi contraction [6], Hardy Rogers contraction [27], weak quasi contraction [19], Ćirić contraction [26] etc . More results for multi-valued contraction mappings in a b-metric space can be found in [9, 10, 14, 24, 29]. In [15] the authors introduced the concept of H^β-Hausdorff-Pompeiu b-metric for some $0 \leq \beta \leq 1$ and proved fixed point theorems for multi-valued mappings belonging to various classes of multi-valued H^β-contractions in a b-metric space. The aim of this work is to prove common fixed point theorems for a pair of multivalued mappings in a b-metric space using H^β-Hausdorff Pompieu b-metric and thereby extend and introduce new variants of various fixed point results for multi-valued mappings existing in literature. An application of our main result is demonstrated by proving the existence of a common solution of a pair of nonlinear Volterra type integral equations.

2 Preliminaries

In this section we provide some preliminary definitions, lemmas and propositions required in our main results.

Definition 2.1. [7] Let X be a nonempty set and $d_s\colon X \times X \to [0, \infty)$ satisfy:

1. $d_s(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

2. $d_s(x,y) = d(y,x)$ for all $x, y \in X$;
3. there exists a real number $s \geq 1$ such that $d(x,y) \leq s[d_s(x,\ell) + d_s(\ell, y)]$ for all $x, y, \ell \in X$.

Then d_s is a b-metric on X and (X, d_s) is a b-metric space with coefficient s.

Let $CB^{d_s}(X)$ be the collection of all nonempty closed and bounded subsets of a b-metric space (X, d_s). For $A, B \in CB^{d_s}(X)$, define $d_s(x, A) = \inf\{d_s(x,a) : a \in A\}$, $\delta_{d_s}(A,B) = \sup_{a\in A} d_s(a,B)$ and $H_{d_s}(A,B) = \max\{\delta_{d_s}(A,B), \delta_{d_s}(B,A)\}$. Czerwik [11] has shown that H_{d_s} is a b-metric in the set $CB^{d_s}(X)$ and is called the the Hausdorff-Pompeiu b-metric induced by d_s. In [15], the authors introduced the function

$$\begin{aligned} H^\beta(A,B) &= \max\{\beta\delta_{d_s}(A,B) + (1-\beta)\delta_{d_s}(B,A), \beta\delta_{d_s}(B,A) \\ &\quad +(1-\beta)\delta_{d_s}(A,B)\} \end{aligned}$$

for some $\beta \in [0,1]$ and showed that H^β form a metric for the set $CB^{d_s}(X)$. They called this function the H^β-Hausdorff Pompieu b-metric induced by the b-metric d_s. Note that for $\beta = 0$ or 1, H^β metric is equal to the Hausdorff-Pompieu metric H.

Proposition 2.2. *[15] For any $x, y \in X$, $H^\beta(\{x\},\{y\}) = d_s(x,y)$.*

Definition 2.3. [26] The b-metric d_s is $*$-continuous if and only if for any $A \in CB^{d_s}(X)$ and sequence $\{x_n\}$ in (X, d_s) with $lim_{n\to\infty} x_n = x$, we have $lim_{n\to\infty} d_s(x_n, A) = d_s(x, A)$.

Proposition 2.4. *[9] For any $A \subseteq X$,*

$$a \in \bar{A} \Longleftrightarrow d_s(a, A) = 0.$$

Lemma 2.5. *[26] Let $\{x_n\}$ be a sequence in (X, d_s). Then for any $n \in N$ and $k \in \{1, 2, 3 ... 2^n - 1, 2^n\}$, we have*

$$d(x_0, x_k) \leq s^n \sum_{i=0}^{k-1} d(x_i, x_{i+1}).$$

Lemma 2.6. *[26] Let $\{x_n\}$ be a sequence in (X, d_s). If there exists $\lambda \in [0,1)$ such that $d_s(x_n, x_{n+1}) \le \lambda d_s(x_{n-1}, x_n)$ for all $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

Following the technique of [26], we now prove the following lemma.

Lemma 2.7. *If for some $\lambda, \epsilon \in [0,1)$, with $\lambda < \epsilon$, $d_s(x_n, x_{n+1}) \le \lambda d_s(x_{n-1}, x_n) + \epsilon^n$ for all $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Note that $d_s(x_n, x_{n+1}) \le \lambda d_s(x_{n-1}, x_n) + \epsilon^n$ implies

$$d_s(x_n, x_{n+1}) \le \lambda^n \, d_s(x_0, x_1) + \frac{\epsilon^{n+1}}{\epsilon - \lambda} \tag{1}$$

for all $n \in N$. Also for all $m, k \in N$ and $p = [log_2 k]$, we have

$$\begin{aligned} & d_s(x_{m+1}, x_{m+k}) \le s\, d_s(x_{m+1}, x_{m+2}) + s^2\, d_s(x_{m+2}, x_{m+2^2}) \\ & + s^3\, d_s(x_{m+2^2}, x_{m+2^3}) + \cdots \\ & \le \sum_{n=1}^{p} s^n\, d_s(x_{m+2^{n-1}}, x_{m+2^n}) + s^{p+1} d_s(x_{m+2^p}, x_{m+k}). \end{aligned} \tag{2}$$

Then using Lemma 2.5 and (1), we get

$$\begin{aligned} & d_s(x_{m+1}, x_{m+k}) \le \sum_{n=1}^{p} s^n \Big\{ s^n \sum_{i=m}^{m+2^{n-1}-1} d_s(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}) \Big\} \\ & + s^{2(p+1)} \sum_{i=m}^{m+k-2^p-1} d_s(x_{2^p+i}, x_{2^p+i+1}) \\ & \le \sum_{n=1}^{p+1} s^{2n} \sum_{i=m}^{m+2^{n-1}-1} d_s(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}) \\ & \le \sum_{n=1}^{p+1} s^{2n} d_s(x_0, x_1) \sum_{i=0}^{2^{n-1}-1} \Big\{ \lambda^{m+2^{n-1}+i} + \frac{\epsilon^{m+2^{n-1}+i+1}}{\epsilon - \lambda} \Big\} \\ & \le \frac{d_s(x_0, x_1)\lambda^m}{1-\lambda} \sum_{n=1}^{p+1} s^{2n} \lambda^{2^{n-1}} + \frac{d_s(x_0, x_1)\epsilon^m}{(\epsilon - \lambda)(1-\epsilon)} \sum_{n=1}^{p+1} s^{2n} \epsilon^{2^{n-1}} \\ & \le \frac{d_s(x_0, x_1)\lambda^m}{1-\lambda} \sum_{n=1}^{p+1} \lambda^{2n\, log_\lambda s + 2^{n-1}} + \frac{d_s(x_0, x_1)\epsilon^m}{(\epsilon - \lambda)(1-\epsilon)} \sum_{n=1}^{p+1} \epsilon^{2n\, log_\epsilon s + 2^{n-1}} \end{aligned}$$

Note that $lim_{n\to\infty} 2n\, log_\lambda\, s + 2^{n-1} = \infty$ and $lim_{n\to\infty} 2n\, log_\epsilon\, s + 2^{n-1} = \infty$. So for fixed $M > 0$, there exists $n_1, n_2 \in N$ such that $2n\, log_\lambda\, s + 2^{n-1} \geq M$ for all $n \geq n_1$ and $2n\, log_\epsilon\, s + 2^{n-1} \geq M$ for all $n \geq n_2$, that is $\lambda^{2n\, log_\lambda\, s + 2^{n-1}} < \lambda^M$ for all $n \geq n_1$ and $\epsilon^{2n\, log_\epsilon\, s + 2^{n-1}} < \lambda^M$ for all $n \geq n_2$. Thus the series $\sum_{n=1}^{p+1} \lambda^{2n\, log_\lambda\, s + 2^{n-1}}$ and $\sum_{n=1}^{p+1} \epsilon^{2n\, log_\epsilon\, s + 2^{n-1}}$ are convergent. Let $\sum_{n=1}^{p+1} \lambda^{2n\, log_\lambda\, s + 2^{n-1}} = S_1$ and $\sum_{n=1}^{p+1} \epsilon^{2n\, log_\lambda\, s + 2^{n-1}} = S_2$. Then we get

$$d_s(x_{m+1}, x_{m+k}) \;\; \leq \;\; \frac{d_s(x_0, x_1)\lambda^m}{1-\lambda} S_1 + \frac{d_s(x_0, x_1)\epsilon^m}{(\epsilon - \lambda)(1-\epsilon)} S_2$$

for all $m, k \in N$. Thus sequence $\{x_n\}$ is a Cauchy sequence.

3 Main Results

We introduce pairwise H^β-Hausdorff functions as follows:

Definition 3.1. Let $S, T : X \to CB^{d_s}(X)$. For any $x \in X$, $y \in Tx($ or $Sx)$ and any $\epsilon > 0$ if there exist $z \in Sy($ or $Ty)$ such that

$$d(y, z) \leq H^\beta(Tx, Sy) + \epsilon \;\; \text{or respectively} \;\; d(y, z) \leq H^\beta(Sx, Ty) + \epsilon \quad (3)$$

then we say that T and S are pairwise H^β-Hausdorff functions .

For $S = T$, we get the following:

Definition 3.2. For any $x \in X$, $y \in Tx$ and any $\epsilon > 0$ if there exists $z \in Ty$ such that

$$d(y, z) \leq H^\beta(Tx, Ty) + \epsilon \tag{4}$$

then we say that T is a H^β-Hausdorff function.

Remark 3.3. (i) For $\beta = 1$, $T : X \to CB(X)$ is always a H^β-Hausdorff function.

(ii) If for any $0 \leq \beta_1 \leq 1$, the function $T : X \to CB(X)$ is a H_1^β-Hausdorff function then for any $0 \leq \beta_1 \leq \beta_2 \leq 1$, the function $T : X \to CB(X)$ is a H_2^β-Hausdorff function.

Example 3.4. Let $X = [0, \frac{33}{48}] \bigcup \{1\}$,

$$d_s(x,y) = |x-y|^2 \quad \text{for all } x, y \in X.$$

and $S, T : X \to CB(X)$ be as follows :

$$S(x) = \begin{cases} \{\frac{x}{4}\}, & \text{for } x \in (0, \frac{33}{48}] \\ \{\frac{33}{48}, 1\}, & \text{for } x \in \{0, 1\}, \end{cases}$$

$$T(x) = \begin{cases} \{\frac{x}{2}\}, & \text{for } x \in (0, \frac{33}{48}] \\ \{\frac{1}{3}, \frac{33}{48}, 1\}, & \text{for } x \in \{0, 1\}. \end{cases}$$

We will show that the functions S and T satisfies (3). We will consider the values of x in X as follows :

(i) $x \in (0, \frac{33}{48}]$. In this case Sx and Ty are singleton sets and so (3) is obviously true.

(ii) $x = 0$. $Sx = \{\frac{33}{48}, 1\}$. If $y = \frac{33}{48}$, $Ty = \{\frac{33}{96}\}$, then we have $z = \frac{33}{96}$ and $d_s(y,z) = \frac{1089}{9216}$, $\delta_s(Sx, Ty) = \frac{3969}{9216}$, $\delta_s(Ty, Sx) = \frac{1089}{9216}$ and $H^{\frac{3}{4}}(Sx, Ty) = \frac{3249}{9216}$. Thus (3) is true for all $\epsilon > 0$. If $y = 1$, $Ty = \{\frac{1}{3}, \frac{33}{48}, 1\}$ then inequality (3) holds with $z = 1$.

(iii) $x = 1$. $Sx = \{\frac{33}{48}, 1\}$ and the result follows in the same way as in (ii) above.

(iv) $x = 0$. $Tx = \{\frac{1}{3}, \frac{33}{48}, 1\}$. If $y = \frac{1}{3}$, $Sy = \{\frac{1}{12}\}$, then we have $z = \frac{1}{12}$ and $d_s(y,z) = \frac{9}{144}$, $\delta_s(Tx, Sy) = \frac{121}{144}$, $\delta_s(Sy, Tx) = \frac{9}{144}$ and $H^{\frac{3}{4}}(Sx, Ty) = \frac{93}{144}$. Thus (3) is true for all $\epsilon > 0$. If $y = \frac{33}{48}$, $Sy = \{\frac{33}{192}\}$ then we take $z = \frac{33}{192}$ and Then $d_s(y,z) = \frac{1089}{4096}$, $\delta_s(Tx, Sy) = \frac{2809}{4096}$, $\delta_s(Sy, Tx) = \frac{961}{36864}$ and $H^{\frac{3}{4}}(Tx, Sy) = \frac{19201}{36864}$. Thus (3) is true for all $\epsilon > 0$. If $y = 1$, $Sy = \{\frac{33}{48}, 1\}$ and inequality (3) holds with $z = 1$.
Thus S and T are pairwise H^{β}-Hausdorff functions for $\beta = \frac{3}{4}$. However S and T are not pairwise H^{β}-Hausdorff functions for $\beta = \frac{1}{2}$, as we see that inequality (3) is not satisfied for $x = 0$, $Tx = \{\frac{1}{3}, \frac{33}{48}, 1\}$ and $y = \frac{33}{48}$. In fact S and T are not pairwise H^{β}-Hausdorff functions for $\frac{34}{95} < \beta < \frac{61}{95}$.

3.1 Fixed point results

We now present our main result.

Theorem 3.5. *Let (X, d_s) be a complete b-metric space with constant $s \geq 1$, d_s be $*$-continuous, $T, S : X \to P_{cl,b}(X)$ be multivalued pairwise H^{β}-Hausdorff functions for some $\frac{1}{2} \leq \beta \leq 1$. If there exist nonnegative real numbers α, γ, δ satisfying $\alpha + 2\gamma + 2s\delta < 1$, $s(\gamma + \delta) < \beta$ and*

$$\begin{aligned} H^{\beta}(Tx, Sy) \quad &\leq \quad \alpha\, d_s(x, y) + \gamma\, [d_s(x, Tx) + d_s(y, Sy)] \\ &\quad\quad +\delta[d_s(x, Sy) + d_s(y, Tx)] \end{aligned} \tag{5}$$

for all $x, y \in X$, then S and T has a common fixed point.

Proof. Let $x_0 \in X$, $x_1 \in Tx_0$ and $0 < \epsilon < 1$. By (3) there exists $x_2 \in Sx_1$, such that $d(x_1, x_2) \leq H^{\beta}(Tx_0, Sx_1) + \epsilon$. By (3) again , there exists $x_3 \in Tx_2$, such that $d(x_2, x_3) \leq H^{\beta}(Sx_1, Tx_2) + \epsilon^2$

Continuing this way we construct the sequence $< x_n >$ such that,

$$x_{2n+1} \in Tx_{2n}, x_{2n+2} \in Sx_{2n+1}; \tag{6}$$

$$d_s(x_{2n+1}, x_{2n+2}) \leq H^{\beta}(Tx_{2n}, Sx_{2n+1}) + \epsilon^{2n+1}; \tag{7}$$

$$d_s(x_{2n+2}, x_{2n+3}) \leq H^{\beta}(Sx_{2n+1}, Tx_{2n+2}) + \epsilon^{2n+2}. \tag{8}$$

Then we have

$$\begin{aligned} & d_s(x_{2n+1}, x_{2n+2}) \leq H^{\beta}(Tx_{2n}, Sx_{2n+1}) + \epsilon^{2n+1} \\ & \leq \alpha\, d_s(x_{2n}, x_{2n+1}) + \gamma[d_s(x_{2n}, Tx_{2n}) + d_s(x_{2n+1}, Sx_{2n+1})] \\ & +\delta\, [d_s(x_{2n}, Sx_{2n+1}) + d_s(x_{2n+1}, Tx_{2n})] \\ & \leq \alpha\, d_s(x_{2n}, x_{2n+1}) + \gamma[d_s(x_{2n}, x_{2n+1}) + d_s(x_{2n+1}, x_{2n+2})] \\ & +\delta\, d_s(x_{2n}, x_{2n+2}) + d_s(x_{2n+1}, x_{2n+1})] + \epsilon^{2n+1} \\ & \leq \alpha\, d(x_{2n}, x_{2n+1}) + \gamma[d_s(x_{2n}, x_{2n+1}) + d_s(x_{2n+1}, x_{2n+2})] \\ & +\delta\, s[d_s(x_{2n}, x_{2n+1}) + d_s(x_{2n+1}, x_{2n+2})] + \epsilon^{2n+1}. \end{aligned}$$

Thus we have

$$d_s(x_{2n+1}, x_{2n+2}) \quad \leq \quad \frac{\alpha + \gamma + s\delta}{1 - \gamma - s\delta} d_s(x_{2n}, x_{2n+1}) + \epsilon^{2n+1}.$$

Again

$$\begin{aligned}
&d_s(x_{2n+2}, x_{2n+3}) \leq H^\beta(Sx_{2n+1}, Tx_{2n+2}) + \epsilon^{2n+2}\\
&\leq \alpha\, d_s(x_{2n+1}, x_{2n+2}) + \gamma[d_s(x_{2n+2}, Tx_{2n+2}) + d_s(x_{2n+1}, Sx_{2n+1})]\\
&+\delta[d_s(x_{2n+2}, Sx_{2n+1}) + d_s(x_{2n+1}, Tx_{2n+2})] + \epsilon^{2n+2}\\
&\leq \alpha\, d_s(x_{2n+1}, x_{2n+2}) + \gamma[d_s(x_{2n+2}, x_{2n+3}) +, d_s(x_{2n+1}, x_{2n+2})]\\
&+\delta[d_s(x_{2n+2}, x_{2n+2}) + d_s(x_{2n+1}, x_{2n+3})] + \epsilon^{2n+2}\\
&\leq \alpha\, d_s(x_{2n+1}, x_{2n+2}) + \gamma[d_s(x_{2n+2}, x_{2n+3}) + d_s(x_{2n+1}, x_{2n+2}]\\
&+\delta\, s[d_s(x_{2n+1}, x_{2n+2}) + d_s(x_{2n+2}, x_{2n+3})] + \epsilon^{2n+2}.
\end{aligned}$$

Thus we have

$$d(x_{2n+2}, x_{2n+3}) \quad \leq \quad \frac{\alpha + \gamma + s\delta}{1 - \gamma - s\delta} d_s(x_{2n+1}, x_{2n+2}).$$

Thus we have

$$d_s(x_n, x_{n+1}) \quad \leq \quad \lambda d_s(x_{n-1}, x_n) + \epsilon^n$$

where, $\lambda = \frac{\alpha+\gamma+s\delta}{1-\gamma-s\delta} < 1$.

By Lemma 2.7 the sequence $< x_n >$ is a Cauchy sequence. Since (X, d_s) is complete, there exist $\hbar \in X$ such that the sequence $< x_n >$ converges to $\hbar$. We will show that $\hbar \in T\hbar \bigcap S\hbar$.

By the definition of H^β, we have

$$\begin{aligned}
&\beta\delta_s(Sx_{2n+1}, T\hbar) + (1-\beta)\delta_s(T\hbar, Sx_{2n+1}) \leq H^\beta(Sx_{2n+1}, T\hbar)\\
&\leq \alpha\, d_s(x_{2n+1}, \hbar) + \gamma[d_s(\hbar, T\hbar + d_s(x_{2n+1}, Sx_{2n+1})]\\
&+\delta[d_s(\hbar, Sx_{2n+1}) + d_s(x_{2n+1}, T\hbar)] + \epsilon^{2n+2}\\
&\leq \alpha\, d_s(x_{2n+1}, \hbar) + \gamma[d_s(\hbar, T\hbar + d_s(x_{2n+1}, x_{2n+2})]\\
&+\delta[d_s(\hbar, x_{2n+2}) + d_s(x_{2n+1}, T\hbar)] + \epsilon^{2n+2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{n\to\infty} \beta\delta_s(Sx_{2n+1}, T\hbar) + (1-\beta)\delta_s(T\hbar, Sx_{2n+1})\\
&\leq \lim[\alpha\, d_s(x_{2n+1}, \hbar) + \gamma[d_s(\hbar, T\hbar + d_s(x_{2n+1}, x_{2n+2})]\\
&+\delta[d_s(\hbar, x_{2n+2}) + d_s(x_{2n+1}, T\hbar)] + \epsilon^{2n+2}]\\
&\leq (\gamma + \delta)\, d_s(\hbar, T\hbar).
\end{aligned}$$

Since

$$\begin{aligned}&\lim_{n\to\infty}\beta\delta_s(Sx_{2n+1},T\hbar)+\lim_{n\to\infty}(1-\beta)\delta_s(T\hbar,Sx_{2n+1})\\&\leq\lim_{n\to\infty}\beta\delta_s(Sx_{2n+1},T\hbar)+(1-\beta)\delta_s(T\hbar,Sx_{2n+1}),\end{aligned}$$

we have

$$\lim_{n\to\infty}\beta\delta_s(Sx_{2n+1},T\hbar)+\lim_{n\to\infty}(1-\beta)\delta_s(T\hbar,Sx_{2n+1})\leq(\gamma+\delta)\,d_s(\hbar,T\hbar).$$

This implies

$$\lim_{n\to\infty}\beta\delta_s(Sx_{2n+1},T\hbar)\leq(\gamma+\delta)\,d_s(\hbar,T\hbar). \tag{9}$$

Again we have

$$\begin{aligned}&\beta\delta_s(Tx_{2n},S\hbar)+(1-\beta)\delta_s(S\hbar,Tx_{2n})\leq H^{\beta}(Tx_{2n},S\hbar)\\&\leq\alpha\,d_s(x_{2n},\hbar)+\gamma[d_s(x_{2n},Tx_{2n})+d_s(\hbar,S\hbar)]\\&+\delta[d_s(x_{2n},S\hbar)+d_s(\hbar,Tx_{2n})]+\epsilon^{2n+1}\\&\leq\alpha\,d_s(x_{2n},\hbar)+\gamma[d_s(x_{2n},x_{2n+1})+d_s(\hbar,S\hbar)]\\&+\delta[d_s(x_{2n},S\hbar)+d_s(\hbar,x_{2n+1})]+\epsilon^{2n+1}\end{aligned}$$

It follows that

$$\begin{aligned}&\lim_{n\to\infty}\beta\delta_s(Tx_{2n},S\hbar)+(1-\beta)\delta_s(S\hbar,Tx_{2n})\\\leq\;&\lim[\alpha\,d_s(x_{2n},\hbar)+\gamma[d_s(x_{2n},x_{2n+1})+d_s(\hbar,S\hbar)]\\&+\delta[d_s(x_{2n},S\hbar)+d_s(\hbar,x_{2n+1})\}]+\epsilon^{2n+1}]\\&\leq(\gamma+\delta)\,d_s(\hbar,S\hbar).\end{aligned}$$

Since

$$\begin{aligned}&\lim_{n\to\infty}\beta\delta_s(Tx_{2n},S\hbar)+\lim_{n\to\infty}(1-\beta)\delta_s(S\hbar,Tx_{2n})\\&\leq\lim_{n\to\infty}\beta\delta_s(Tx_{2n},S\hbar)+(1-\beta)\delta_s(S\hbar,Tx_{2n}),\end{aligned}$$

we have

$$\lim_{n\to\infty}\beta\delta_s(Tx_{2n},S\hbar)+\lim_{n\to\infty}(1-\beta)\delta_s(S\hbar,Tx_{2n})\leq(\gamma+\delta)\,d_s(\hbar,S\hbar).$$

This implies

$$\lim_{n\to\infty} \beta\delta_s(Tx_{2n}, S\hbar) \leq (\gamma+\delta)\, d_s(\hbar, S\hbar). \tag{10}$$

Now

$$d_s(\hbar, T\hbar) \leq s[d_s(\hbar, x_{2n+2}) + \delta_s(Sx_{2n+1}, T\hbar)]$$
$$d_s(\hbar, S\hbar) \leq s[d_s(\hbar, x_{2n+1}) + \delta_s(Tx_{2n}, S\hbar))]$$

Using (9) and (10) we get

$$\begin{aligned} & d_s(\hbar, T\hbar) \leq s \lim_{n\to\infty} d_s(\hbar, x_{2n+2}) + s \lim_{n\to\infty} \delta_s(Sx_{2n+1}, T\hbar) \\ \leq \quad & \frac{s(\gamma+\delta)}{\beta}\, d_s(\hbar, T\hbar). \\ & d_s(\hbar, S\hbar) \leq s \lim_{n\to\infty} d_s(\hbar, x_{2n+1}) + s \lim_{n\to\infty} \delta_s(Tx2n, S\hbar) \\ \leq \quad & \frac{s(\gamma+\delta)}{\beta}\, d_s(\hbar, T\hbar). \end{aligned}$$

Since $s(\gamma+\delta) < \beta$, we get $d_s(\hbar, T\hbar) = 0$ and $d_s(\hbar, S\hbar) = 0$. Since T and S are closed we have $\hbar \in T$ and $\hbar \in S$.

Example 3.6. Let $X = [0, \frac{5}{12}] \bigcup \{2\}$, $d_s(x,y) = |x-y|^2$ for all $x, y \in X$ and $S, T : X \to CB(X)$ be as follows :

$$S(x) = \begin{cases} \{\frac{x}{4}\}, & \text{for } x \in [0, \frac{5}{12}] \\ \\ \{0, \frac{1}{3}, 2\}, & \text{for } x = 2, \end{cases}$$
$$T(x) = \begin{cases} \{\frac{x}{4}\}, & \text{for } x \in [0, \frac{5}{12}] \\ \\ \{0, \frac{5}{12}, 2\}, & \text{for } x = 2, \end{cases}$$

We will show that the functions S and T satisfy contraction condition (5) for $\beta = \frac{1}{2}$.

Case 1. $x, y \in [0, \frac{5}{12}]$. By Lemma ( ), we have

$$\begin{aligned} H^{\frac{1}{2}}(Sx, Ty) &= H^{\frac{1}{2}}(\{\frac{x}{4}\}, \{\frac{y}{4}\}) \\ &= d_s(\frac{x}{4}, \frac{y}{4}) \\ &= |\frac{x}{4} - \frac{y}{4}|^2 \\ &\leq \alpha_1 |x-y|^2, \qquad \text{for any } \alpha \geq \tfrac{1}{16} \\ &= \alpha\, d_s(x, y). \end{aligned}$$

Case 2. $x \in [0, \frac{5}{12}], y = 2$. We have
$d_s(x, y) = |2-x|^2$. The minimum value of $d_s(x, y)$ for $x \in [0, \frac{5}{12}]$ is $\frac{361}{144}$.
$\delta_s(Sx, Ty) = \delta_s(\{\frac{x}{4}\}, \{0, \frac{5}{12}, 2\}) = \frac{x^2}{16}$.
$\delta_s(Ty, Sx) = \delta_s(\{0, \frac{5}{12}, 2\}, \{\frac{x}{4}\}) = (2 - \frac{x}{4})^2$.
$H^{\frac{1}{2}}(Sx, Ty) = \frac{1}{2}\big(\frac{x^2}{16} + (2 - \frac{x}{4})^2\big)$
The maximum value of $H^{\frac{1}{2}}(Sx, Ty)$ for $x \in [0, \frac{5}{12}]$ is 2 (at $x = 0$). Thus $H^{\frac{1}{2}}(Sx, Ty) \leq \alpha\, d_s(x, y)$ for any $\alpha \geq \frac{288}{361}$.

Case 3. $x = 2, y \in [0, \frac{5}{12}]$. We have
$d_s(x, y) = |2-y|^2$. The minimum value of $d_s(x, y)$ for $y \in [0, \frac{5}{12}]$ is $\frac{361}{144}$.
$\delta_s(Sx, Ty) = \delta_s(\{0, \frac{5}{12}, 2\}, \{\frac{y}{4}\}) = (2 - \frac{y}{4})^2$.
$\delta_s(Ty, Sx) = \delta_s(\{\frac{x}{4}\}, \{0, \frac{5}{12}, 1\}) = \frac{y^2}{16}$.
$H^{\frac{1}{2}}(Sx, Ty) = \frac{1}{2}\big(\frac{y^2}{16} + (2 - \frac{y}{4})^2\big)$
The maximum value of $H^{\frac{1}{2}}(Sx, Ty)$ for $y \in [0, \frac{5}{12}]$ is 2 (at $y = 0$). Thus $H^{\frac{1}{2}}(Sx, Ty) \leq \alpha\, d_s(x, y)$ for any $\alpha \geq \frac{288}{361}$.

Thus S and T satisfy contraction condition (5) for $\beta = \frac{1}{2}$, $\frac{288}{361} \leq \alpha < 1$ and $\gamma = \delta = 0$. Simple calculations shows that S and T are pairwise H^{β}-Hausdorff functions. All conditions of Theorem 3.5 are satisfied and 0 is a common fixed point of S and T. However we see that at $x = 0, y = 2$, S and T do not satisfy contraction condition (5) for $\beta = 1$ and so do not satisfy Nadler's contraction and Czerwic's contraction.

Remark 3.7. In Example 3.6 above, simple calculations show that S and T do not satisfy contraction condition (5) for $\frac{62}{100} < \beta \leq 1$. However in view of Remark 3.3(i), there may exist functions S and T which satisfy contraction condition (5) for $\beta = 1$ but may not satisfy for $\beta < 1$. Thus for $\beta = 1$ Theorem 3.5 is an extension of Nadler's contraction [28], Czervik's contraction [11] and many of its generalisations. For $\beta < 1$ Theorem 3.5 provides new variants of Nadler's contraction [28], Czervik's contraction [11] and many of its generalisations.

Taking $\gamma = \delta = 0$ in Theorem 3.5, we get the following extension and new variants of Nadler's contraction and Czerwik's contraction :

Corollary 3.8. *Let (X, d_s) be a complete b-metric space with constant $s \geq 1$, $T, S : X \to P_{cl,b}(X)$ be multivalued pairwise H^β-Hausdorff functions for some $\frac{1}{2} \leq \beta \leq 1$ and satisfying the following condition:*

$$H^\beta(Tx, Sy) \leq \alpha d_s(x, y).$$

for all $x, y \in X$ and $0 \leq \alpha < 1$, then S and T has a common fixed point.

Taking $\alpha = \delta = 0$ in Theorem 3.5, we get the following extension and new variants of Kannan's contraction :

Corollary 3.9. *Let (X, d_s) be a complete b-metric space with constant $s \geq 1$, $T, S : X \to P_{cl,b}(X)$ be multivalued pairwise H^β-Hausdorff functions for some $\frac{1}{2} \leq \beta \leq 1$ and satisfying the following condition;*

$$H^\beta(Tx, Sy) \leq \gamma[d_s(x, Tx) + d_s(y, Sy)$$

for all $x, y \in X$ and some real number γ with $0 \leq \gamma < \frac{1}{2}$, then S and T has a common fixed point.

Taking $\alpha = \gamma = 0$ in Theorem 3.5, we get the following extension and new variants of Kannan's contraction :

Corollary 3.10. *Let (X, d_s) be a complete b-metric space with constant $s \geq 1$, $T, S : X \to P_{cl,b}(X)$ be multivalued pairwise H^β-Hausdorff functions for some $\frac{1}{2} \leq \beta \leq 1$ and satisfying the following condition;*

$$H^\beta(Tx, Sy) \leq \gamma[d_s(x, Sy) + d_s(y, Tx)$$

for all $x, y \in X$ and some real number δ with $0 \leq s\delta < \frac{1}{2}$, then S and T has a common fixed point.

3.2 Application to integral equation

In this section, motivated by the applications given in [3, 31, 33], we establish the sufficient conditions for the existence of a common solution of a pair of nonlinear Volterra type integral equations.

For some real numbers a, b with $0 \le a < b$ and $I = [a, b]$, let $X = C(I, \mathbb{R})$ be the Banach space of real continuous functions defined on I equipped with a norm given by $\|x\| = \max_{t\in I} |x(t)|$. For some and some $p \ge 1$, define a b-metric d_s on X by

$$d_s(x, y) = \max_{t\in I} |x(t) - y(t)|^p, \text{ for all } x, y \in X.$$

Then $(X, d_s, 2^{p-1})$ is a complete b-metric space. Consider two Fredholm integral equations

$$\begin{cases} x(t) = q(t) + \int_a^{\mu(t)} \mathcal{P}(t,s)\mathcal{K}_1(t,s,x(s))ds + \int_a^{\sigma(t)} \mathcal{Q}(t,s)\mathcal{K}_2(t,s,y(s))ds \\ y(t) = q(t) + \int_a^{\mu(t)} \mathcal{P}(t,s)\mathcal{K}_2(t,s,y(s))ds + \int_a^{\sigma(t)} \mathcal{Q}(t,s)\mathcal{K}_1(t,s,y(s))ds \end{cases} \tag{11}$$

for all $t, s \in I = [a, b] \subseteq \mathbb{R}$, $|\lambda| > 0$, $\mathcal{K}_{i=1,2} : I \times I \times X \to \mathbb{R}$ and $q : I \to \mathbb{R}$ and $\mathcal{P}, \mathcal{Q} : I \times I \to \mathbb{R}$ are continuous functions and $\mu, \sigma : I \to I$.

Suppose $T, S : X \to X$ be self-mappings defined by

$$\begin{cases} Tx(t) = q(t) + \int_a^{\mu(t)} \mathcal{P}(t,s)\mathcal{K}_1(t,s,x(s))ds + \int_a^{\sigma(t)} \mathcal{Q}(t,s)\mathcal{K}_2(t,s,y(s))ds \\ Sy(t) = q(t) + \int_a^{\mu(t)} \mathcal{P}(t,s)\mathcal{K}_2(t,s,y(s))ds + \int_a^{\sigma(t)} \mathcal{Q}(t,s)\mathcal{K}_1(t,s,y(s))ds, \end{cases} \tag{12}$$

for all $x, y \in X$, where $t \in I$. It is obvious that $\hbar(t)$ is a solution of (11) if and only if it is a common fixed point of T and S.

Theorem 3.11. *Suppose that the following hypotheses hold:*

(H_1) : *$T(X)$ and $S(X)$ are closed in X;*

(H_2) : *There exist nonnegative real numbers α, γ, δ satisfying $\alpha + 2\gamma + 2^p\delta < 1$, $2^{p-1}(\gamma + \delta) < \beta$ such that*

$$|\mathcal{K}_1(t, s, x(s)) - \mathcal{K}_2(t, s, y(s))|^p \le N(T, S, p, t)$$

(H_3) : $\int_a^{\mu(t)} \mathcal{P}(t,s)ds + \int_a^{\sigma(t)} \mathcal{Q}(t,s)ds \leq \frac{1}{2^{p-1}}$.

where,

$$N(T,S,p,t) = \alpha|x(t)-y(t)|^p + \gamma[|x(t)-Tx(t)|^p + |y(t)-Sy(t)|^p]$$
$$+\delta[|x(t)-Sy(t)|^p + |y(t)-Tx(t)|^p].$$

Then the system (11) of integral equations has a common solution in X.

Proof: Using (H_2) and (H_3) we have

$$d_s(Tx,Sy) = \max_{t\in I}|Tx(t)-Sy(t)|^p$$
$$\leq \max_{t\in I}|\int_a^{\mu(t)} \mathcal{P}(t,s)\mathcal{K}_1(t,s,x(s))ds + \int_a^{\sigma(t)} \mathcal{Q}(t,s)\mathcal{K}_2(t,s,y(s))ds$$
$$-\int_a^{\mu(t)} \mathcal{P}(t,s)\mathcal{K}_2(t,s,y(s))ds - \int_a^{\sigma(t)} \mathcal{Q}(t,s)\mathcal{K}_1(t,s,y(s))ds|^p$$
$$\leq \max_{t\in I} 2^{p-1}\{|\int_a^{\mu(t)} \mathcal{P}(t,s)\mathcal{K}_1(t,s,x(s))ds$$
$$-\int_a^{\mu(t)} \mathcal{P}(t,s)\mathcal{K}_2(t,s,y(s))ds|^p$$
$$+|\int_a^{\sigma(t)} \mathcal{Q}(t,s)\mathcal{K}_2(t,s,y(s))ds - \int_a^{\sigma(t)} \mathcal{Q}(t,s)\mathcal{K}_1(t,s,y(s))ds|^p\}$$
$$\leq \max_{t\in I} 2^{p-1}\{|\int_a^{\mu(t)} \mathcal{P}(t,s)(\mathcal{K}_1(t,s,x(s)) - \mathcal{K}_2(t,s,y(s)))ds|^p$$
$$+|\int_a^{\sigma(t)} \mathcal{Q}(t,s)(\mathcal{K}_2(t,s,y(s)) - \mathcal{K}_1(t,s,y(s)))ds|^p\}$$
$$\leq \max_{t\in I} 2^{p-1}\{\int_a^{\mu(t)} |\mathcal{P}(t,s)|^p|(\mathcal{K}_1(t,s,x(s)) - \mathcal{K}_2(t,s,y(s)))|^p ds$$
$$+\int_a^{\sigma(t)} |\mathcal{Q}(t,s)|^p|(\mathcal{K}_2(t,s,y(s)) - \mathcal{K}_1(t,s,y(s)))|^p ds\}$$

$$\leq \max_{t\in I} 2^{p-1}\{\int_a^{\mu(t)} |\mathcal{P}(t,s)|^p N(T,S,p,t)ds$$
$$+ \int_a^{\sigma(t)} |\mathcal{Q}(t,s)|^p N(T,S,p,t)ds$$
$$\leq \max_{t\in I} 2^{p-1} N(T,S,p,t)\{\int_a^{\mu(t)} |\mathcal{P}(t,s)|^p ds + \int_a^{\sigma(t)} |\mathcal{Q}(t,s)|^p ds$$
$$\leq \max_{t\in I} N(T,S,p,t)$$
$$\leq \alpha\, d_s(x,y) + \gamma[d_s(x,Tx) + d_s(y,Sy)] + \delta[d_s(x,Sy) + d_s(y,Tx)].$$

Thus conditions of Theorem 3.5 are satisfied. Theorem 3.5 therefore ensures a common fixed point of T and S, which in turn is a common solution of the pair of integral equations (11).

Remark 3.12. Taking $\mathcal{Q}(t,s) = 0, \mathcal{P}(t,s) = 1, q(t) = 0, \mu(t) = t$ and $a = 0$ in (11), we get the Volterra-type integral equations considered in Rasham et al [31] and Alshoraify et al [3].

Remark 3.13. Taking $\mathcal{Q}(t,s) = 0, \mu(t) = 1$ and $a = 0$ in (11), we get the Fredholm-type integral equations (III.3) considered in Shoaib et al [33].

Remark 3.14. Taking $\mathcal{Q}(t,s) = 0, \mathcal{P}(t,s) = 1$ and $\mu(t) = b$ in (11), we get the Fredholm-type integral equations (III.1) considered in Shoaib et al [33].

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