Journal of Mathematical Extension
Vol. 16, No. 12, (2022) (3)1-9
URL: https://doi.org/10.30495/JME.2022.2300
ISSN: 1735-8299
Original Research Paper

Topological Invariants and Curvature

G. H. Arab
Karaj Branch, Islamic Azad University

M. Toomanian*
Karaj Branch, Islamic Azad University

Abstract. It is widely known that the fundamental group of a Lie group, and in general a symmetric space, is abelian. In the current paper it is demonstrated that any finitely generated abelian group is the fundamental group of a compact Lie group. In addition, it is proved that for any arbitrary group there is a differentiable manifold of dimension greater than 3 whose fundamental group is that arbitrary group.

AMS Subject Classification: 22E15; 22EXX; 14F35; 14H30; 57T20.
Keywords and Phrases: Lie groups, Fundamental group, Finitely generated abelian groups, Homotopy groups, Sectional curvature.

1 Introduction

Let M be a Riemannian manifold. The relation between the curvature of M and its topology is of fundamental importance. This is particularly more evident in theorems and results by Hadamard, Bonnet-Myers and Preissman [2, 8], that show the relationship between curvature and the fundamental group. For instance, the universal covering of a complete

Received: January 2008; Accepted: February 2009
*Corresponding Author

Riemannian manifold with negative sectional curvature is diffeomorphic to Euclidean space. Moreover, if this manifold is compact, its fundamental group cannot be abelian. A special but still important case of this problem is the relation between a compact Lie group and its fundamental group. This relation has been extensively studied by many authors at different times, (see for example [2] and its references for observing some of these studies). Along this line, we study some relations between Lie groups and their fundamental groups. The reader is referred to [2, 5, 6], for undefined terms and concepts.

In the second section, it is shown that the fundamental group $\pi_1(G)$ is a finitely generated abelian group when G is a compact Lie group. Samelson [9] uses differential forms as well as differential geometry to show that the fundamental group of a compact semi-simple Lie group is finite. Taking another approach to the above problem, the authors prove, by Bonnet-Myers' theorem, that if the Lie group G has a bi-invariant metric and its Lie algebra $\mathcal{G}$ has trivial center, then its fundamental group is finite.

In the third section for each finitely generated abelian group G, the authors find a Lie group whose fundamental group is G. This is carried out on the basis of the decomposition of finitely generated abelian groups and the fact that the functor π_1 (between the category of topological spaces and the category of groups) commutes with products and maps.

Hatcher [5] shows that for any group G, there exists a CW-Complex, whose fundamental group is G. In the current paper, this problem for differentiable manifolds with dimensions greater than 3 is solved by drawing on his construction method.

By generalizing the fundamental group or the first homotopy group of the manifold M to the n-th homotopy group, including homotopy classes of maps of n-dimensional sphere into M, Samelson [10] has shown that the second homotopy group $\pi_2(G)$ of a Lie group G is trivial. Also, Hadamard's theorem, which deals with the relationship between the topology of a manifold and its geometry, states that for the manifold M with negative sectional curvature $K < 0$, all higher order homotopy groups are trivial and at the level of homotopy, the information about the topology of M is contained in its fundamental group. In the fourth section, by drawing on the Preissman and Synge theorems the authors present the results on the curvature of a metric of a compact Lie group as well as m-torus. The reader can find the counterpart of the above concepts and theorems in Finsler geometry in [3, 8].

2 Fundamental Group of a Lie Group

Though if is well-known that the fundamental group of a Lie group is abelian, here we give a proof for the sake of completeness. We will use this fact in the sequel.

Lemma 2.1. *The fundamental group of a Lie group is abelian.*

Proof. Let G be a Lie group with operation $\centerdot$ and identity element e. Let $\Omega(G,e)$ denote the set of all loops in G based at e. If $f,g \in \Omega(G,e)$, let us define a loop $f \centerdot g$ by the rule $(f \centerdot g)(s) = f(s) \centerdot g(s)$. This operation makes $\Omega(G,e)$ into a group and induces a group operation $\centerdot$ on $\pi_1(G,e)$. Since

$$f * g = (f \centerdot c_e) * (c_e \centerdot g) = (f * c_e) \centerdot (c_e * g) \simeq f \centerdot g,$$

where c_e is the constant loop at e, the two group operations $*$ and $\centerdot$ on $\pi_1(G,e)$ are the same. Besides

$$g * f = (c_e \centerdot g) * (f \centerdot c_e) = (c_e * f) \centerdot (g * c_e) \simeq f \centerdot g.$$

Therefore $f * g \simeq f \centerdot g \simeq g * f$. This shows $\pi_1(G,e)$ is abelian. $\square$

Corollary 2.2. *Let G be a Lie group, then $H_1(G) = \pi_1(G,e)$, where $H_1(G)$ is the first homology group of G.*

Proof. The first homology group, $H_1(G)$, by the very definition is $\pi_1(G)/[\pi_1(G), \pi_1(G)]$, i.e., the abelianized fundamental group. Since $\pi_1(G)$ is abelian, we have $H_1(G) = \pi_1(G)$. $\square$

R. Bott and L. W. Tu [1], have proved that the first homology group of a compact Lie group is finitely generated. A direct consequence of their result and Corollary 2.2 is the following proposition.

Proposition 2.3. *Let G be a compact Lie group, then $\pi_1(G)$ is finitely generated.*

Proof. By [1, Theorem 5.1 and Proposition 5.3.1], $H_1(G)$ is finitely generated. By Corollary 2.2, $\pi_1(G) = H_1(G)$ is also finitely generated. $\square$

Studying fundamental groups of real connected compact semi-simple Lie groups, H. Samelson [9], discovered that these fundamental groups are in fact finite. The next result is at the same time, a counterpart and a generalization of the Samelson's theorem.

Theorem 2.4. *Let G be a connected Lie group whose Lie algebra $\mathcal{G}$ has trivial center. If G has a bi-invariant metric, then G and its universal cover $\tilde{G}$ are compact. In particular, the fundamental group $\pi_1(G)$ is finite.*

Proof. For $X, Y \in \mathcal{G}$, then $< X, X >$, $< X, Y >$, $< Y, Y >$ are constant on G, so

$$\begin{aligned} 2 < Y, \nabla_X X > &=< X, [Y, X] > + < X, [Y, X] > - < Y, [X, X] > \\ &= 2 < X, [Y, X] > . \end{aligned}$$

By bi-invariance $< X, [Y, X] >= - < [X, X], Y >= 0$.

For fixed X, $< Y, \nabla_X X >= 0, \forall Y$. So $\nabla_X X = 0, \forall X \in \mathcal{G}$. It follows that for $X, Y \in \mathcal{G}$,

$0 = \nabla_{X+Y}(X + Y) = \nabla_Y X + \nabla_X Y$, $\nabla_X Y = -\nabla_Y X = \frac{1}{2}[X, Y]$, and so

$$\begin{aligned} R(X, Y)X &= \nabla_Y \nabla_X X - \nabla_X \nabla_Y X + \nabla_{[X,Y]} X \\ &= \frac{1}{4}[X, [X, Y]] + \frac{1}{2}[[X, Y], X] \\ &= \frac{1}{4}[[X, Y], X]. \end{aligned}$$

Therefore

$$R(X, Y, X, Y) = \frac{1}{4} < [[X, Y], X], Y >= \frac{1}{4}|[X, Y]|^2.$$

If $e_1, \ldots, e_n$ is an orthonormal basis of $\mathcal{G} = T_e G$, then

$Ric_e(X) = \frac{1}{4(n-1)} \sum_i |[X, e_i]|^2 > 0$, for any $X \in \mathcal{G}$ with $X \neq 0$, since $\mathcal{G}$ has trivial center.

Since S^{n-1} is compact, there exists $c > 0$ such that $Ric_e(X) \geq c > 0$, for all $X \in \mathcal{G}$ with $|X| = 1$. Now apply Bonnet-Myers' theorem to conclude that G and its universal cover $\tilde{G}$ are compact. Hence the number of sheets of the covering and hence the number of elements in the fundamental group $\pi_1(G)$ are finite. $\square$

Remark 2.5. Any compact connected Lie group has a bi-invariant metric and semisimple Lie group implies that $\mathcal{G}$ has trivial center.

3 Finitely Generated Abelian Groups and Lie Groups

In this section, we study the reverse implications, which have already been proved in the second section. We begin with the first result, which deals with "the realization problem" of abelian groups, that is: "for every abelian group A, find a Lie group G, such that $\pi_1(G) = A$". We begin with our first result, which gives a partial answer to this question.

Theorem 3.1. *Every finitely generated abelian group is the fundamental group of a Lie group.*

Proof. A finitely generated abelian group is a finite product of cyclic groups, so it suffices to realize cyclic groups and then take the product of the Lie groups that realize them. For an infinite cyclic group one can just take the circle group $U(1)$. For a finite cyclic group C_n of order n, this embeds in $SU(n)$ as the subgroup consisting of scalar multiples of the identity matrix where the scalar is an n-th root of unity. These diagonal matrices are contained in the center of $SU(n)$, so C_n is a normal subgroup of $SU(n)$ and one can form the quotient group $SU(n)/C_n$. Since $SU(n)$ is simply connected, it is the universal cover of $SU(n)/C_n$, therefore C_n is the fundamental group of $SU(n)/C_n$. $\square$

Remark 3.2. In this paper "Lie group" we mean "finite dimensional Lie group". We have already seen that for compact Lie groups, the fundamental group is a finitely generated abelian group. Every non-compact Lie group is homeomorphic to the product of a compact Lie group and some Euclidean space by a theorem of Malcev and Iwasawa, so the fundamental group would be finitely generated in this case too.

No countable abelian group, such as, $\mathbb{Q}$, $\mathbb{Z}_{P^\infty}$ and $\mathbb{Q}/_{\mathbb{Z}}$ are fundamental groups of finite dimensional Lie groups: because by the above remark, such fundamental groups are isomorphic to the fundamental groups of compact Lie groups. But by a result due to S. Shelah [12], no compact metric space which is path connected, has a countable, non-finitely generated fundamental group. So it is almost impossible to find, a finite-dimensional Lie group with $\mathbb{Q}$, $\mathbb{Z}_{P^\infty}$ and $\mathbb{Q}/_{\mathbb{Z}}$ as its fundamental group.

We need the following proposition of Hatcher, to tackle the "realization problem of finitely presented group" for smooth manifolds.

Proposition 3.3. *For every group G, there is a 2-dimensional cell complex X_G with $\pi_1(X_G) = G$.*

Proof. See Hatcher [5, Corollary 1.28]. □

By using "handles" instead of "cells" in the construction of $K(G, 1)$, we obtain the following theorem.

Theorem 3.4. *Every finitely presented group is the fundamental group of a closed orientable smooth manifold of dimension n, where n is any given number greater than 3.*

Proof. Let G be a group which has a presentation consisting of k generators and l relations. In dimension $n > 3$ one starts with an n-dimensional ball D^n, thought of as a thickening of a 0-cell. Then attach 1-handles $D^1 \times D^{n-1}$ to the boundary sphere S^{n-1} so as to produce an orientable n-manifold with boundary, homotopy equivalent to a wedge of k circles. The 1-handles can be viewed as thickenings of 1-cells. Next, attach 2-handles $D^2 \times D^{n-2}$ to the boundary of the previously constructed manifold, attaching the 2-handles along $S^1 \times D^{n-2}$ via embeddings of $S^1 \times D^{n-1}$ into the boundary representing relations defining the group G, as in the CW-Complex construction. This is where the assumption $n > 3$ comes in to guarantee that the desired embeddings exist. After these steps one obtains a compact manifold N with boundary and with fundamental group G because the inclusion $\vee S^1 \to N$ leads to an isomorphism $\pi_1(N) = \pi_1(\vee S^1)/K$, where the subgroup K generated by the loops is the same as subgroup generated by the words. The last step is to double this manifold, taking two copies of N and identifying their boundaries to get a closed manifold M. M is therefore obtained from N by attaching $(n-2)$-handles, $(n-1)$-handles, and one n-handle. If $n > 3$ these attached handles do not affect the fundamental group. $\square$

Remark 3.5. When $n = 3$ it is known that not all finitely presented groups occur as the fundamental groups of 3-manifolds. (See [7].)

4 Curvature and Homotopy Groups

One of the most important topological invariants, called the homotopy group, which generalizes the fundamental group.

Definition 4.1. [4, 5, 7]. The n-th homotopy group $\pi_n(M)$ of the manifold M can be defined as homotopy classes of maps, $f : S^n \to M$, of spheres S^n of dimension n into M.

In [10], Samelson studied the second homotopy group of a Lie group and got the following result.

Proposition 4.2. *The second homotopy group $\pi_2(G)$ of a Lie group G is zero.*

Proof. See Samelson [10, page 29]. $\square$

In the following, we see that a complete Riemannian manifold with negative sectional curvature has trivial higher order homotopy groups.

Proposition 4.3. *Let M be a complete Riemannian manifold with sectional curvature $K < 0$, then the homotopy groups of higher order are trivial.*

Proof. By Hadamard's theorem the universal covering of M is diffeomorphic to $\mathbb{R}^n$, then every $f : S^m \to M$ is homotopic to a constant map if $m \geq 2$, see Greenberg [4, page 32]. $\square$

Remark 4.4. By Preissman's theorem [2, Theorem 3.8], the compact Lie group G does not admit a metric of negative curvature, since its fundamental group is abelian.

Example 4.5. Using Preissman theorem [2, Theorem 3.8], it is shown that m-torus $T^m = S^1 \times \cdots \times S^1$ does not admit metrics of negative curvature. In addition, the Synge theorem [2, Corollary 3.10], shows that for even m's this m-torus does not carry a metric of positive sectional curvature, since its fundamental group is $Z \oplus \cdots \oplus Z$ (m times).

References

[1] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1924.

[2] M. do. Carmo, *Riemannian Geometry*, Birkhäuser, 1992.

[3] S. Deng and Z. Hou, *Homogeneous Finsler spaces of negative curvature*, Journal of Geometry and Physics, **57** (2007), 657-664.

[4] M. Greenberg, *Lectures on Algebraic Topology*, W. A. Benjamin, Inc. New york, 1967.

[5] A. Hatcher, *Algebraic Topology*, Cambridge university press, 2002.

[6] O. Loos, *Symmetric Spaces*, W. A. Benjamin. Inc. II: Compact spaces and classification, 1969.

[7] W. S. Massey, *A Basic Course in Algebraic Topology*, Springer-Verlag, 1991.

[8] H. B. Rademacher, *Nonreversible Finsler metrics of positive flag curvature,* Riemann-Finsler geometry, Math. Sci. Res. Inst. Publ. **50** (2004), 261-302.

[9] H. Samelson, *A note on Lie groups*, Bull. Amer. Math. Soc. **52** (1946), 870-873.

[10] H. Samelson, *Topology of Lie groups*, Bull. Amer. Math. Soc. **58** (1952), 2-37.

[11] J. Sekiguchi, *Fundamental groups of semisimple symmetric spaces, Representation of Lie Groups*, Kyoto, Hiroshima, (1986), 519-529.

[12] S. Shelah, *Can the fundamental group of a space be the rationals?*, Proc. of the Amer. Math. Soc. **103** (1988), 627-632.

Gholam Hossein Arab
Assistant Professor of Mathematics
Department of Mathematics
Karaj Branch, Islamic Azad University
Karaj, Iran
E-mail: GH.Arab1352@iau.ac.ir

Megerdich Toomanian
Professor of Mathematics
Department of Mathematics
Karaj Branch, Islamic Azad University
Karaj, Iran
E-mail: toomanian@tabrizu.ac.ir