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A Study on Numerical Algorithms for Differential Equations in Two Cases q-Calculus and (p,q)-Calculus

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Abstract. We investigate the existence and uniqueness of the solution and also the rate of convergence of a numerical method for a fractional differential equation in both q-calculus and (p, q)-calculus versions. We use the Banach and Schauder fixed point theorems in this study. We provide two examples, one by definition of the q-derivative and the other by (p, q)-derivative. We compare the rate of convergence of the numerical method. We like to clear some facts on (p, q)-calculus. The data from our numerical calculations show well that q-calculus works better than (p, q)-calculus in each case.

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1 Introduction

Quantum calculus has received more and more attention in recent decades. The subject of q -difference equations was first introduced by Jackson in 1908 ([19], [20]). Other researchers have also studied the issue of q-difference equations ([1], [2], [3], [4], [5], [11], [13], [14], [23], [25], [30], [34], [38], [41]). This topic has been developed by many researchers in recent years and many new results have been obtained ([8], [12], [15], [22], [26], [40]). Later, the study of quantum calculus was generalized by researchers from one parameter to two parameters (p, q). The (p, q)-calculus was presented by Chakrabarti and Jagannathan ([9]). The extension of studies of (p, q)-calculus was given in ([6], [7], [10], [18], [21], [24], [28], [35], [36]). Recently, Soontharanon and Sitthiwirattham have acquired fractional operators (p, q)-difference and their properties ([37]). One can find some recent studies on the boundary value problem for (p, q)-difference equations in ([27], [29], [31], [32], [33], [39]).

In this article, motivated by several articles about the q-difference and (p, q)-difference, we examine the rate of convergence of both cases. In this way, we review the (p, q)-intgro problem

$$\begin{cases} {}^cD^{\alpha}_{p,q}y(t) = f(t, y(t), \Phi^{\varrho}_{p,q}y(t)), \quad t \in \mathbb{T}_{t_0(p,q)}, \\ y(0) = 1, \\ \kappa y(\frac{t_0}{p}) + (\kappa+1){}^cD^{\sigma}_{p,q}y(\frac{t_0}{p}) + (\kappa+2)\mathcal{I}^{\zeta}_{p,q}y(\frac{t_0}{p}) = 0, \\ \kappa y(\rho) + (\kappa+1){}^cD^{\sigma}_{p,q}y(\rho) + (\kappa+2)\mathcal{I}^{\zeta}_{p,q}y(\rho) = 0, \end{cases} \tag{1}$$

where ${}^cD^{\alpha}_{p,q}$,${}^cD^{\sigma}_{p,q}$ are the Caputo fractional $(p, q)-$derivative of order α, σ and $0 < q < p \leq 1$, $\mathbb{T}_{t_0(p,q)} = \{\frac{q^k}{p^{k+1}}t_0 : k \in \mathbb{N}_0\} \cup \{0\}$, α belongs to $(2, 3]$, $\sigma \in (1, 2]$, $\varrho, \zeta \in (0, 1]$ and $\mathcal{I}^{\zeta}_{p,q}$ denotes the Riemann-Liouville fractional (p, q)-integral of order ζ, κ is a positive real number and $\Phi^{\varrho}_{p,q}y(t) = \left(\mathcal{I}^{\varrho}_{p,q}\xi y\right)(t) = \dfrac{1}{p^{\binom{\varrho}{2}}\Gamma_{p,q}(\varrho)} \int_0^t (t-qs)^{\underline{\varrho-1}}_{p,q}\xi(t,s)y(\frac{s}{p^{\varrho-1}})d_{p,q}s.$

Also, we investigate the q-version of the problem, that is,

$$\begin{cases} {}^cD_q^{\alpha} y(t) = f\left(t, y(t), \Phi_q^{\gamma} y(t)\right), \quad t \in \mathbb{T}_{t_0(q)}, \\ y(0) = 1, \\ \kappa y(t_0) + (\kappa+1) {}^cD_q^{\sigma} y(t_0) + (\kappa+2) \mathcal{I}_q^{\zeta} y(t_0) = 0, \\ \kappa y(\rho) + (\kappa+1) {}^cD_q^{\sigma} y(\rho) + (\kappa+2) \mathcal{I}_q^{\zeta} y(\rho) = 0, \end{cases} \tag{2}$$

where ${}^cD_q^{\alpha}$, ${}^cD_q^{\sigma}$ are the Caputo fractional q-derivative of order α, σ, $0 < q < 1$, $t \in \mathbb{T}_{t_0(q)} = \{t : t_0 q^k, k \in \mathbb{N}\} \cup \{0\}$, $\alpha \in (2,3]$, $\sigma \in (1,2]$, ζ, ϱ belong to $(0,1]$, $\mathcal{I}_q^{\zeta}$ denotes the Riemann-Liouville fractional $q-$integral of order ζ and $\Phi_q^{\varrho} y(t) = (\mathcal{I}_q^{\varrho} \xi y)(t) = \dfrac{1}{\Gamma_q(\varrho)} \int_0^t (t - qs)_q^{\varrho - 1} \xi(t,s) y(s) d_q s$.

2 Preliminaries

In this section, we recall some basic definitions, notations and results.

Definition 2.1. [37] Let $0 < q < p \leq 1$. The (p,q)-analogue of the power function $(c_1 - c_2)_{p,q}^n$ with $n \in \mathbb{N}_0$ is defined by

$$\begin{cases} (c_1 - c_2)_{p,q}^{(n)} = \prod_{j=0}^{n-1} (c_1 p^j - c_2 q^j), \\ (c_1 - c_2)_{p,q}^{(0)} = 1, \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. For a real number ϑ, we define

$$(c_1 - c_2)^{(\vartheta)} = c_1^{\vartheta} \prod_{j=0}^{\infty} \frac{1}{p^{\vartheta}} \frac{1 - (\frac{c_2}{c_1})(\frac{q}{p})^j}{1 - (\frac{c_2}{c_1})(\frac{q}{p})^{\vartheta + j}}, \quad c_1 \neq 0.$$

Definition 2.2. [37] For $\vartheta \in \mathbb{N}$, we put $[\vartheta]_{p,q} = \dfrac{p^{\vartheta} - q^{\vartheta}}{p - q} = p^{\vartheta - 1} [\vartheta]_{\frac{q}{p}}$.

Definition 2.3. [37] We define (p,q)-Gamma and (p,q)-Beta functions by $\Gamma_{p,q}(\vartheta) = \frac{(1 - \frac{q}{p})^{(\vartheta - 1)}}{(1 - \frac{q}{p})^{\vartheta - 1}}$ and $B_{p,q}(b_1, b_2) = p^{\frac{(b_2 - 1)(2b_1 + b_2 - 2)}{2}} \dfrac{\Gamma_{p,q}(b_1) \Gamma_{p,q}(b_2)}{\Gamma_{p,q}(b_1 + b_2)}$, where $\vartheta \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$.

We proposed the algorithm 1 for calculating the (p,q)-Gamma function. Also by definition of $[\vartheta]_{p,q}$, the property $\Gamma_{p,q}(\vartheta+1)=[\vartheta]_{p,q}\Gamma_{p,q}(\vartheta)$ holds [37].

Definition 2.4. [37] The (p,q)- derivative of a function $y:[0,t_0]\to\mathbb{R}$ is defined by

$$(\mathcal{D}_{p,q}y)(t):=\begin{cases}\dfrac{y(pt)-y(qt)}{(p-q)t}, & t\neq 0,\\ y^{'}(0) & t=0.\end{cases}$$

provided that y is differentiable at 0. A map y is called (p,q)-differentiable on $\mathbb{T}_{t_0(p,q)}$ whenever $\mathcal{D}_{p,q}y(t)$ exists for all $t\in\mathbb{T}_{t_0(p,q)}$. The (p,q)-derivative of higher order of a function y, for all $n\geq 1$, is given by

$$\begin{cases}(\mathcal{D}^n_{p,q}y)(t)=\mathcal{D}_{p,q}(\mathcal{D}^{n-1}_{p,q}y)(t),\\ (\mathcal{D}^0_{p,q}y)(t)=y(t).\end{cases}$$

Definition 2.5. [37] The (p,q)-integral of a function y defined on $\mathbb{T}_{t_0(p,q)}$ is given by

$$(\mathcal{J}_{p,q}y)(t)=\int_0^t y(s)\,\mathrm{d}_{p,q}s=t(p-q)\sum_{j=0}^{\infty}y(t\frac{q^j}{p^{j+1}})\frac{q^j}{p^{j+1}},\ (t\in\mathbb{T}_{t_0(p,q)}),$$

whenever the sum is absolutely convergent. Similar to (p,q)-derivatives, we define the operator $\mathcal{J}^n_{p,q}$ for all $n\geq 1$ by

$$\begin{cases}(\mathcal{J}^n_{p,q}y)(t)=\mathcal{J}_{p,q}(\mathcal{J}^{n-1}_{p,q}y)(t),\\ (\mathcal{J}^0_{p,q}y)(t)=y(t),\end{cases}$$

Remark 2.6. Note that, $(\mathcal{D}_{p,q}\mathcal{J}_{p,q}y)(t)=y(t)$, and if y, is continuous at $t=0$, then $(\mathcal{J}_{p,q}\mathcal{D}_{p,q}y)(t)=y(t)-y(0)$.

Definition 2.7. [37] Let $\vartheta\in\mathbb{R}^+$ with $n=[\vartheta]+1$. The Riemann-Liouville (p,q)-integral of a function y defined on $\mathbb{T}_{t_0(p,q)}$ is given by

$$\mathcal{J}^{\vartheta}_{p,q}y(t)=\frac{1}{p^{\binom{\vartheta}{2}}\Gamma_{p,q}(\vartheta)}\int_0^t(t-qs)^{(\vartheta-1)}_{p,q}y(\frac{s}{p^{\vartheta-1}})\,\mathrm{d}_{p,q}s$$

whenever the integral exists.

Definition 2.8. [37] The Caputo (p, q)-derivative of a function y defined on $\mathbb{T}_{t_0(p,q)}$ is given by

$$^{c}\mathcal{D}_{p,q}^{\vartheta}y(t) = \frac{1}{p^{\binom{-\vartheta}{2}}\Gamma_{p,q}(n-\vartheta)}\int_0^t (t-qs)_{p,q}^{(n-\vartheta-1)}\mathcal{D}_{p,q}^{(n)}y(\frac{s}{p^{-\vartheta-1}})\,\mathrm{d}_{p,q}s.$$

We need next results.

Lemma 2.9. *[37] Let $0 < q < p \leq 1$ and $y : \mathbb{T}_{t_0(p,q)} \to \mathbb{R}$ be a map. If $n \geq 1$ and $\vartheta \in (n-1, n)$, then*

$$(\mathcal{J}_{p,q}^{\vartheta}\, {}^{c}\mathcal{D}_{p,q}^{\vartheta})y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}.$$

Theorem 2.10. *[16] Let $\{f_j\}_{j\in J}$ be a collection in $C[a, b]$ by sup norm, then $\{f_j\}_{j\in J}$ is relatively compact iff it is uniformly bounded and equicontinuous on $[a, b]$.*

Theorem 2.11. *[16] Suppose that a set $\mathcal{C}$ be closed and relatively compact, then $\mathcal{C}$ is compact.*

Theorem 2.12. *[17] Let $(X, \|\ .\ \|)$ be a Banach space and $S \subset X$ be closed and convex. Then, any relatively compact operator $A : X \to X$ has at least one fixed point $s^* \in S$, that is, $As^* = s^*$.*

3 Main Results

Now, we are ready to start providing our main results.

Lemma 3.1. *Let $g \in C(\mathbb{T}_{t_0(p,q)}, \mathbb{R})$ and $\Delta_1, \Omega \neq 0$. Then the following (p, q)-fractional boundary value problem*

$$\begin{cases} {}^{c}D_{p,q}^{\alpha}y(t) = g(t), \quad t \in \mathbb{T}_{t_0(p,q)}, \\ y(0) = 0, \\ \kappa y(\frac{t_0}{p}) + (\kappa+1)^{c}D_{p,q}^{\sigma}y(\frac{t_0}{p}) + (\kappa+2)\mathcal{I}_{p,q}^{\zeta}y(\frac{t_0}{p}) = 0, \\ \kappa y(\rho) + (\kappa+1)^{c}D_{p,q}^{\sigma}y(\rho) + (\kappa+2)\mathcal{I}_{p,q}^{\zeta}y(\rho) = 0, \quad \rho \in \mathbb{T}_{t_0(p,q)} - \{0, \frac{t_0}{p}\}, \end{cases} \tag{3}$$

has the unique solution

$$y(t) = -\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} g(\frac{s}{p^{\alpha-1}})d_{p,q}s \\ -\frac{t}{\Delta_1}\left[\frac{(-\Theta_2\Xi_1+\Delta_1\Xi_2)}{\Omega}+\Xi_1\right]+\frac{t^2}{\Omega}(-\Theta_2\Xi_1+\Delta_1\Xi_2), \tag{4}$$

where

$$\begin{cases} \Delta_1 = \left(\kappa(\frac{t_0}{p}) + (\kappa+2)\frac{\Gamma_{p,q}(2)(\frac{t_0}{p})^{\zeta+1}}{\Gamma_{p,q}(\zeta+2)}\right) \\ \Delta_2 = \left(\kappa\rho^2 + (\kappa+1)\frac{\Gamma_{p,q}(3)(\rho)^{2-\sigma}}{\Gamma_{p,q}(3-\sigma)} + (\kappa+2)\frac{\Gamma_{p,q}(3)\rho^{\zeta+2}}{\Gamma_{p,q}(\zeta+3)}\right), \\ \Theta_1 = \left(\kappa(\frac{t_0}{p})^2 + (\kappa+1)\frac{\Gamma_{p,q}(3)(\frac{t_0}{p})^{2-\sigma}}{\Gamma_{p,q}(3-\sigma)} + (\kappa+2)\frac{\Gamma_{p,q}(3)(\frac{t_0}{p})^{\zeta+2}}{\Gamma_{p,q}(\zeta+3)}\right), \\ \Theta_2 = \frac{1}{\Delta_1}\left(\kappa\rho + (\kappa+2)\frac{\Gamma_{p,q}(2)\rho^{\zeta+1}}{\Gamma_{p,q}(\zeta+2)}\right), \\ \Omega = \Theta_1\Theta_2 - \Delta_2\Delta_1, \end{cases} \tag{5}$$

$$\Xi_1 = -\frac{\kappa}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{\frac{t_0}{p}} (\frac{t_0}{p}-qs)_{p,q}^{\underline{\alpha-1}} g(\frac{s}{p^{\alpha-1}})d_{p,q}s \\ + (\kappa+1)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{-\sigma}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\sigma)}\int_0^{\frac{t_0}{p}}\int_0^{\frac{x}{p^{-\sigma-1}}}(\frac{t_0}{p}-qx)_{p,q}^{\underline{-\sigma-1}} \\ \times (\frac{x}{p^{-\sigma-1}}-qs)_{p,q}^{\underline{\alpha-1}} g(\frac{s}{p^{\alpha-1}})d_{p,q}s d_{p,q}x) \\ + (\kappa+2)(-\frac{1}{p^{\binom{\alpha+\zeta}{2}}\Gamma_{p,q}(\alpha+\zeta)}\int_0^{\frac{t_0}{p}}(\frac{t_0}{p}-qs)_{p,q}^{\underline{\alpha+\zeta-1}} g(\frac{s}{p^{\alpha+\zeta-1}})d_{p,q}s), \tag{6}$$

and

$$\Xi_2 = -\frac{\kappa}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{\rho}(\rho - qs)_{p,q}^{\underline{\alpha-1}}g(\frac{s}{p^{\alpha-1}})d_{p,q}s \tag{7}$$
$$+ (\kappa+1)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{-\sigma}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\sigma)}\int_0^{\rho}\int_0^{\frac{x}{p^{-\sigma-1}}}(\rho - qx)_{p,q}^{\underline{-\sigma-1}}$$
$$\times (\frac{x}{p^{-\sigma-1}} - qs)_{p,q}^{\underline{\alpha-1}}g(\frac{s}{p^{\alpha-1}})d_{p,q}sd_{p,q}x)$$
$$+ (\kappa+2)(-\frac{1}{p^{\binom{\alpha+\zeta}{2}}\Gamma_{p,q}(\alpha+\zeta)}\int_0^{\rho}(\rho - qs)_{p,q}^{\underline{\alpha+\zeta-1}}g(\frac{s}{p^{\alpha+\zeta-1}})d_{p,q}s.$$

Proof. To achieve the desired solution by using Lemma 2.9, at first we take $\mathcal{I}_{p,q}^{\alpha}$ from (3), then there exist constants $a_0, a_1, a_2 \in \mathbb{R}$ such that

$$y(t) = -\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{t}(t - qs)_{p,q}^{\underline{\alpha-1}}g(\frac{s}{p^{\alpha-1}})d_{p,q}s + a_0 + a_1 t + a_2 t^2. \tag{8}$$

To apply boundary condition, we take the ${}^{c}D_{p,q}^{\sigma}$ and $\mathcal{I}_{p,q}^{\zeta}$ of y. Thus,

$${}^{c}D_{p,q}^{\sigma}y(t) = -\frac{1}{p^{\binom{\alpha}{2}+\binom{-\sigma}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\sigma)}\int_0^{t}\int_0^{\frac{x}{p^{-\sigma-1}}}(t - qx)_{p,q}^{\underline{-\sigma-1}}$$
$$(\frac{x}{p^{-\sigma-1}} - qs)_{p,q}^{\underline{\alpha-1}}g(\frac{s}{p^{\alpha-1}})d_{p,q}sd_{p,q}x + a_2\frac{\Gamma_{p,q}(3)t^{-\sigma+2}}{\Gamma_{p,q}(3-\sigma)},$$

and

$$\mathcal{I}_{p,q}^{\zeta}y(t) = -\frac{1}{p^{\binom{\alpha+\zeta}{2}}\Gamma_{p,q}(\alpha+\zeta)}\int_0^{t}(t - qs)_{p,q}^{\underline{\alpha+\zeta-1}}g(\frac{s}{p^{\alpha+\zeta-1}})d_{p,q}s$$
$$+ a_0\frac{t^{\zeta}}{\Gamma_{p,q}(\zeta+1)} + a_1\frac{\Gamma_{p,q}(2)t^{\zeta+1}}{p^{\binom{\zeta}{2}}\Gamma_{p,q}(\zeta+2)} + a_2\frac{\Gamma_{p,q}(3)t^{\zeta+2}}{p^{\binom{\zeta}{2}}\Gamma_{p,q}(\zeta+3)}.$$

By using the boundary value condition, we get

$$\begin{cases} a_0 = 0, \\ a_1 = \dfrac{-1}{\Delta_1}\left[\dfrac{(-\Theta_2\Xi_1 + \Delta_1\Xi_2)}{\Omega} + \Xi_1\right] \\ a_2 = \dfrac{1}{\Omega}(\Delta_1\Xi_2 - \Theta_2\Xi_1). \end{cases} \tag{9}$$

in which $\Delta_1, \Delta_2, \Theta_1, \Theta_2, \Omega, \Xi_1, \Xi_2$, are defined by (5)- (7). Now by substituting (9) in (8), we obtain (4). the proof is complete. $\square$

Now, we investigate the problem by using the q-derivative.

Lemma 3.2. *Let $\alpha \in (2,3], \sigma \in (1,2), \zeta \in (0,1)$ and κ be a nonzero real positive constant. Then the q-differential fractional problem*

$$\begin{cases} {}^cD_q^{\alpha} y(t) + g(t) = 0, \quad t \in \mathbb{T}_{t_0(q)}, \\ y(0) = 0, \\ \kappa y(t_0) + (\kappa+1) {}^cD_q^{\sigma} y(t_0) + (\kappa+2)\mathcal{I}_q^{\zeta} y(t_0) = 0, \\ \kappa y(\rho) + (\kappa+1) {}^cD_q^{\sigma} y(\rho) + (\kappa+2)\mathcal{I}_q^{\zeta} y(\rho) = 0, \quad \rho \in \mathbb{T}_{t_0(q)} - \{0, t_0\}, \end{cases} \tag{10}$$

has the unique solution

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{\underline{\alpha-1}} g(s) d_q s \\ &- \frac{t}{\Delta_1^*}\left[\frac{(-\Theta_2^*\Xi_1^* + \Delta_1\Xi_2^*)}{\Omega^*} + \Xi_1^*\right] + \frac{t^2}{\Omega^*}(-\Theta_2^*\Xi_1^* + \Delta_1\Xi_2^*), \end{aligned} \tag{11}$$

where

$$\begin{cases} \Delta_1^* = \left(\kappa t_0 + (\kappa+2)\dfrac{\Gamma_q(2) t_0^{\zeta+1}}{\Gamma_q(\zeta+2)}\right), \\ \Delta_2^* = \left(\kappa \rho^2 + (\kappa+1)\dfrac{\Gamma_q(3)\rho^{2-\sigma}}{\Gamma_q(3-\sigma)} + (\kappa+2)\dfrac{\Gamma_q(3)\rho^{2+\zeta}}{\Gamma_q(\zeta+3)}\right), \\ \Theta_1^* = \left(\kappa t_0^2 + (\kappa+1)\dfrac{\Gamma_q(3) t_0^{2-\sigma}}{\Gamma_q(3-\sigma)} + (\kappa+2)\dfrac{\Gamma_q(3) t_0^{2+\zeta}}{\Gamma_q(\zeta+3)}\right), \\ \Theta_2^* = \frac{1}{\Delta_1^*}\left(\kappa \rho + (\kappa+2)\dfrac{\Gamma_q(2)\rho^{\zeta+1}}{\Gamma_q(\zeta+2)}\right), \\ \Omega^* = \Theta_2^*\Theta_1^* - \Delta_2\Delta_1, \end{cases} \tag{12}$$

$$\Xi_1^* = -\frac{\kappa}{\Gamma_q(\alpha)}\int_0^{t_0}(t_0-qs)_q^{\underline{\alpha-1}}g(s)d_qs \qquad (13)$$
$$+(\kappa+1)(-\frac{1}{\Gamma_q(\alpha-\sigma)}\int_0^{t_0}(t_0-qs)_q^{\underline{\alpha-\sigma-1}}g(s)d_qs)$$
$$+(\kappa+2)(-\frac{1}{\Gamma_q(\alpha+\zeta)}\int_0^{t_0}(t_0-qs)_q^{\underline{\alpha+\zeta-1}}g(s)d_qs),$$

and

$$\Xi_2^* = -\frac{\kappa}{\Gamma_q(\alpha)}\int_0^{\rho}(\rho-qs)_q^{\underline{\alpha-1}}g(s)d_qs + a_0 \qquad (14)$$
$$+(\kappa+1)(-\frac{1}{\Gamma_q(\alpha-\sigma)}\int_0^{\rho}(\rho-qs)_q^{\underline{\sigma-1}}g(s)d_qs)$$
$$+(\kappa+2)(-\frac{1}{\Gamma_q(\alpha+\zeta)}\int_0^{\rho}(\rho-qs)_q^{\underline{\alpha+\zeta-1}}g(s)d_q).$$

Proof. The procedure is similar to the previous case. By taking $\mathcal{I}_q^{\alpha}$ from (10), we get

$$y(t) = -\frac{1}{\Gamma_q(\alpha)}\int_0^{t}(t-qs)_q^{\underline{\alpha-1}}g(s)d_qs + a_0 + a_1t + a_2t^2. \qquad (15)$$

To use the boundary condition, we take ${}^cD_{p,q}^{\sigma}$ and $\mathcal{I}_{p,q}^{\zeta}$ from y. Hence,

$${}^cD_q^{\sigma}y(t) = -\frac{1}{\Gamma_q(\alpha-\sigma)}\int_0^{t}(t-qs)_q^{\underline{\alpha-\sigma-1}}g(s)d_qs \quad + a_2\frac{\Gamma_q(3)t^{-\sigma+2}}{\Gamma_q(3-\sigma)},$$

and

$$\mathcal{I}_q^{\zeta}y(t) = -\frac{1}{\Gamma_{p,q}(\alpha+\zeta)}\int_0^{t}(t-qs)_q^{\underline{\alpha+\zeta-1}}g(s)d_qs$$
$$+ a_0\frac{t^{\zeta}}{\Gamma_q(\zeta+1)} + a_1\frac{\Gamma_q(2)t^{\zeta+1}}{\Gamma_q(\zeta+2)} + a_2\frac{\Gamma_q(3)t^{\zeta+2}}{\Gamma_q(\zeta+3)}.$$

Note that,

$$\begin{cases} a_0 = 0, \\ a_1 = \dfrac{-1}{\Delta_1^*}\left[\dfrac{(-\Theta_2^*\Xi_1^* + \Delta_1\Xi_2^*)}{\Omega^*} + \Xi_1^*\right], \\ a_2 = \dfrac{1}{\Omega^*}(\Delta_1\Xi_2^* - \Theta_2\Xi_1^*), \end{cases} \qquad (16)$$

where $\Delta_1^*, \Delta_2^*, \Theta_1^*, \Theta_2^*, \Omega^*, \Xi_1^*, \Xi_2^*$, are defined by (12)- (14). Now by substituting (16) in (15), we obtain (11). This completes the proof. $\square$

Now, we investigate the existence and uniqueness of the solutions of problems (1) and (2) by using the Banach fixed point theorem. Consider the Banach space $\mathcal{X} = \{y : y \in C(\mathbb{T}_{t_0(p,q)})\}$ via the norm

$$\|y\|_{\mathcal{X}} = \max_{t\in\mathbb{T}_{t_0(p,q)}} \{|\ y(t)\ |\}.$$

Let $\alpha \in (2,3]$, $\sigma \in (1,2)$, $\varrho \in (0,1)$, $0 < q < p \leq 1$, $\kappa \in \mathbb{R}^+$, $\mathbb{T}_{t_0(p,q)} := \{\frac{q^k}{p^{k+1}}t_0, k \in \mathbb{N}_0\} \cup \{0\}$ and $\mathcal{X} = C(\mathbb{T}_{t_0(p,q)}, \mathbb{R})$. Define $\mathcal{B} : \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned}(\mathcal{B}y)(t) = &-\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} \\ &\times \mathbb{B}\left[(\frac{s}{p^{\alpha-1}}), y(\frac{s}{p^{\alpha-1}}), \Phi_{p,q}^{\gamma} y(\frac{s}{p^{\alpha-1}}))\right] d_{p,q}s \\ &- \frac{t}{\Delta_1}\left[\frac{(-\Theta_2\Xi_1\mathbf{B}_y + \Delta_1\Xi_2\mathbf{B}_y)}{\Omega} + \Xi_1\mathbf{B}_y\right] + \frac{t^2}{\Omega}(-\Theta_2\Xi_1\mathbf{B}_y + \Delta_1\Xi_2\mathbf{B}_y),\end{aligned}$$

where $\Xi_1\mathbf{B}_y$ and $\Xi_2\mathbf{B}_y$ are defined by

$$\begin{aligned}\Xi_1\mathbf{B}_y = &-\frac{\kappa}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{\frac{t_0}{p}} (\frac{t_0}{p} - qs)_{p,q}^{\underline{\alpha-1}} \\ &\times \mathbb{B}\left[(\frac{s}{p^{\alpha-1}}), y(\frac{s}{p^{\alpha-1}}), \Phi_{p,q}^{\gamma} y(\frac{s}{p^{\alpha-1}})\right] d_{p,q}s) \\ &+ (\kappa+1)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{-\sigma}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\sigma)}\int_0^{\frac{t_0}{p}}\int_0^{\frac{x}{p^{-\sigma-1}}} (\frac{t_0}{p} - qx)_{p,q}^{\underline{-\sigma-1}} (\frac{x}{p^{\alpha-1}} - qs)_{p,q}^{\underline{\alpha-1}} \\ &\times \mathbb{B}\left[(\frac{s}{p^{\alpha-1}}), y(\frac{s}{p^{\alpha-1}}), \Phi_{p,q}^{\gamma} y(\frac{s}{p^{\alpha-1}}))\right] d_{p,q}s d_{p,q}x) \\ &+ (\kappa+2)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{\zeta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\zeta)}\int_0^{\frac{t_0}{p}}\int_0^{\frac{x}{p^{\zeta-1}}} (\frac{t_0}{p} - qx)_{p,q}^{\underline{\zeta-1}} (\frac{x}{p^{\alpha-1}} - qs)_{p,q}^{\underline{\alpha-1}} \\ &\times \mathbb{B}\left[(\frac{s}{p^{\alpha-1}}), y(\frac{s}{p^{\alpha-1}}), \Phi_{p,q}^{\gamma} y(\frac{s}{p^{\alpha-1}})\right] d_{p,q}s),\end{aligned}$$

and

$$\Xi_2 \mathbf{B}_y = -\frac{\kappa}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^{\rho} (\rho - qs)_{p,q}^{\frac{\alpha-1}{p,q}}$$
$$\times \mathbb{B}\left[(\frac{s}{p^{\alpha-1}}), y(\frac{s}{p^{\alpha-1}}), \Phi_{p,q}^{\gamma} y(\frac{s}{p^{\alpha-1}})\right] d_{p,q}s$$
$$+ (\kappa+1)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{-\sigma}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\sigma)} \int_0^{\rho}\int_0^{\frac{x}{p^{-\sigma-1}}} (\rho - qx)_{p,q}^{\frac{-\sigma-1}{p,q}} (\frac{x}{p^{-\sigma-1}} - qs)_{p,q}^{\frac{\alpha-1}{p,q}}$$
$$\times \mathbb{B}\left[(\frac{s}{p^{\alpha-1}}), y(\frac{s}{p^{\alpha-1}}), \Phi_{p,q}^{\gamma} y(\frac{s}{p^{\alpha-1}})\right] d_{p,q}s d_{p,q}x)$$
$$+ (\kappa+2)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{\zeta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\zeta)} \int_0^{\rho}\int_0^{\frac{x}{p^{\zeta-1}}} (\rho - qx)_{p,q}^{\frac{\zeta-1}{p,q}} (\frac{x}{p^{\zeta-1}} - qs)_{p,q}^{\frac{\alpha-1}{p,q}}$$
$$\times \mathbb{B}\left[(\frac{s}{p^{\alpha-1}}), y(\frac{s}{p^{\alpha-1}}), \Phi_{p,q}^{\gamma} y(\frac{s}{p^{\alpha-1}})\right] d_{p,q}s).$$

Also, the constants Δ_1, Δ_2, Θ_1, Θ_2, Ω are defined by (5).

Theorem 3.3. *Let* $\mathbb{B} : \mathbb{T}_{t_0(p,q)} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\xi : \mathbb{T}_{t_0(p,q)} \times \mathbb{T}_{t_0(p,q)} \to [0, \infty)$ *be continuous and* $\xi_0 = \max\{\xi(t,s) : (t,s) \in \mathbb{T}_{t_0(p,q)} \times \mathbb{T}_{t_0(p,q)}\}$. *Assume that the following conditions hold:*

(H_1) There exist constant $\tau_1, \tau_2 > 0$ *such that*

$$\mid \mathbb{B}[t, y_1, y_2] - \mathbb{B}[t, v_1, v_2] \mid \leq \tau_1 \mid y_1 - v_1 \mid + \tau_2 \mid y_2 - v_2 \mid,$$

for all $t \in \mathbb{T}_{t_0(p,q)}$ *and* $y_i, v_i \in \mathcal{X}$ $(i = 1, 2)$,

(H_2) We have

$$\Sigma := \ell \left[\frac{(\frac{t_0}{p})^{\alpha}}{\Gamma_{p,q}(\alpha+1)} - \frac{\frac{t_0}{p}}{\Delta_1}\left[\frac{(-\Theta_2\Upsilon_1 + \Delta_1\Upsilon_2)}{\Omega} + \Upsilon_2\right] + \frac{(\frac{t_0}{p})^2}{\Omega}(-\Theta_2\Upsilon_1 + \Delta_1\Upsilon_2)\right] < 1,$$

where

$$\begin{cases} \ell = \left[\tau_1 - \tau_2 \dfrac{\xi_0(\frac{t_0}{p})^{\varrho}}{\Gamma_{p,q}(\varrho+1)}\right], \\ \Upsilon_1 = -\dfrac{\kappa(\frac{t_0}{p})^{\alpha}}{\Gamma_{p,q}(\alpha+1)} - \dfrac{(\ell+1)(\frac{t_0}{p})^{\alpha-\sigma}}{\Gamma_{p,q}(\alpha-\sigma+1)} - \dfrac{(\ell+2)(\frac{t_0}{p})^{\alpha+\zeta}}{\Gamma_{p,q}(\alpha+\zeta+1)}, \\ \Upsilon_2 = -\dfrac{\ell(\rho)^{\alpha}}{\Gamma_{p,q}(\alpha+1)} - \dfrac{(\ell+1)\rho^{\alpha-\sigma}}{\Gamma_{p,q}(\alpha-\sigma+1)} - \dfrac{(\ell+2)\rho^{\alpha+\zeta}}{\Gamma_{p,q}(\alpha+\zeta+1)}. \end{cases}$$

Then, the problem (1) *has a unique solution in* $\mathbb{T}_{t_0(p,q)}$.

Proof. For each $t \in \mathbb{T}_{t_0(p,q)}$ and $y, v \in \mathcal{X}$, we have

$$
\begin{aligned}
&\mid \Phi^{\varrho}_{p,q} y(t) - \Phi^{\varrho}_{p,q} v(t) \mid \\
&= -\frac{\xi_0}{p^{\binom{\varrho}{2}}\Gamma_{p,q}(\varrho)} \int_0^t (t-qs)_{p,q}^{\underline{\varrho-1}} \mid y(\frac{s}{p^{\varrho-1}}) - v(\frac{s}{p^{\varrho-1}}) \mid d_{p,q}s \\
&\leq -\frac{\xi_0}{p^{\binom{\varrho}{2}}\Gamma_{p,q}(\varrho)} \mid y - v \mid \int_0^{\frac{t_0}{p}} (\frac{t_0}{p} - qs)_{p,q}^{\underline{\varrho-1}} d_{p,q}s = -\frac{\xi_0 (\frac{t_0}{p})^{\varrho}}{\Gamma_{p,q}(\varrho+1)} \mid y - v \mid .
\end{aligned}
$$

Put $\mathcal{K}_y(t) :=\mid \mathbb{B}[t, y(t), \Phi^{\varrho}_{p,q} y(t)] \mid$. Then, we get

$$
\begin{aligned}
&\mid \Xi_1 \mathcal{B}_y - \Xi_1 \mathcal{B}_v \mid \\
&= (-\frac{\ell}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^{\frac{t_0}{p}} (\frac{t_0}{p} - qs)_{p,q}^{\underline{\alpha-1}} \mid \mathcal{K}_y - \mathcal{K}_v \mid (\frac{s}{p^{\alpha-1}}) d_{p,q}s) \\
&+ (\ell+1)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{-\sigma}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\sigma)} \\
&\quad \int_0^{\frac{t_0}{p}} \int_0^{\frac{x}{p^{-\sigma-1}}} (\frac{t_0}{p} - qx)_{p,q}^{\underline{-\sigma-1}} (\frac{x}{p^{\alpha-1}} - qs)_{p,q}^{\underline{\alpha-1}} \\
&\times \mid \mathcal{K}_y - \mathcal{K}_v \mid (\frac{s}{p^{\alpha-1}}) d_{p,q}s d_{p,q}x + (\ell+2)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{\zeta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\zeta)} \\
&\int_0^{\frac{t_0}{p}} \int_0^{\frac{x}{p^{\zeta-1}}} (\frac{t_0}{p} - qx)_{p,q}^{\underline{\zeta-1}} (\frac{x}{p^{\alpha-1}} - qs)_{p,q}^{\underline{\alpha-1}} \mid \mathcal{K}_y - \mathcal{K}_v \mid (\frac{s}{p^{\alpha-1}}) d_{p,q}s)) \\
&\leq \big(\tau_1 \mid y - v \mid + \tau_2 \mid \Phi^{\varrho}_{p,q} y - \Phi^{\varrho}_{p,q} v \mid\big) \\
&\times \mid -\frac{\ell(\frac{t_0}{p})^{\alpha}}{\Gamma_{p,q}(\alpha+1)} - \frac{(\ell+1)(\frac{t_0}{p})^{\alpha-\sigma}}{\Gamma_{p,q}(\alpha-\sigma+1)} - \frac{(\ell+2)(\frac{t_0}{p})^{\alpha+\zeta}}{\Gamma_{p,q}(\alpha+\zeta+1)} \mid \\
&\leq \left(\tau_1 \mid y - v \mid - \tau_2 \frac{\xi_0 (\frac{t_0}{p})^{\varrho}}{\Gamma_{p,q}(\varrho+1)} \mid y - v \mid \right) \Upsilon_1 \\
&\leq \left(\tau_1 - \tau_2 \frac{\xi_0 (\frac{t_0}{p})^{\varrho}}{\Gamma_{p,q}(\varrho+1)}\right) \Upsilon_1 \mid y - v \mid \leq \ell \Upsilon_1 \parallel y - v \parallel_{\mathcal{X}} .
\end{aligned}
$$

Similarly, we have $\mid \Xi_2\mathbf{B}_y - \Xi_2\mathbf{B}_v \mid\leq (\ell)\Upsilon_2 \parallel y-v \parallel_{\mathcal{X}}$. Thus,

$$
\begin{aligned}
&\mid (\mathcal{B}y)(t) - (\mathcal{B}v)(t) \mid \\
&\leq -\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{\frac{t_0}{p}} (\frac{t_0}{p}-qs)_{p,q}^{\frac{\alpha-1}{p,q}} \mid \mathcal{K}_y - \mathcal{K}_v \mid (\frac{s}{p^{\alpha-1}})d_{p,q}s \\
&+ \frac{\frac{t_0}{p}}{\Delta_1}\left(\frac{(-\Theta_2\Xi_1\mathbf{B}_y + \Theta_2\Xi_1\mathbf{B}_v) + (\Delta_1\Xi_2\mathbf{B}_y - \Delta_1\Xi_2\mathbf{B}_v)}{\Omega} + (\Xi_1\mathbf{B}_y - \Xi_1\mathbf{B}_v)\right) \\
&+ \frac{(\frac{t_0}{p})^2}{\Omega}((\Delta_1\Xi_2\mathbf{B}_y - \Delta_1\Xi_2\mathbf{B}_v) + (-\Theta_2\Xi_1\mathbf{B}_y + \Theta_2\Xi_1\mathbf{B}_v)) \\
&\leq (\frac{(\ell)(\frac{t_0}{p})^\alpha}{\Gamma_{p,q}(\alpha+1)} + \frac{(\frac{t_0}{p})}{\Delta_1}\Big[\frac{1}{\Omega}\Big(\mid -\Theta_2\Xi_1\mathbf{B}_y + \Theta_2\Xi_1\mathbf{B}_v \mid \\
&+ \mid \Delta_1\Xi_2\mathbf{B}_y - \Delta_1\Xi_2\mathbf{B}_v \mid \Big) + \mid \Xi_1\mathbf{B}_y - \Xi_1\mathbf{B}_v \mid \Big] \\
&+ \frac{(\frac{t_0}{p})^2}{\Omega}(\mid -\Theta_2\Xi_1\mathbf{B}_y + \Theta_2\Xi_1\mathbf{B}_y \mid + \mid \Delta_1\Xi_2\mathbf{B}_y - \Delta_1\Xi_2\mathbf{B}_y \mid) \\
&\leq (\frac{(\ell)(\frac{t_0}{p})^\alpha}{\Gamma_{p,q}(\alpha+1)} - \frac{(\frac{t_0}{p})}{\Delta_1}\left[\frac{(-\Theta_2(\ell)\Upsilon_1 + \Delta_1(\ell)\Upsilon_2)}{\Omega} + (\ell)\Upsilon_1\right] \\
&+ \frac{(\frac{t_0}{p})^2}{\Omega}\left[-\Theta_2(\ell)\Upsilon_1 + \Delta_1(\ell)\Upsilon_2\right] \parallel y-v \parallel_{\mathcal{X}} \leq \Sigma \parallel y-v \parallel_{\mathcal{X}} .
\end{aligned}
$$

Hence, $\parallel \mathcal{B}y - \mathcal{B}v \parallel < \Sigma \parallel y-v \parallel_{\mathcal{C}}$. By by using (H_2), we deduce that $\mathcal{B}$ is a contraction and so by using the Banach fixed point theorem, $\mathcal{B}$ has a fixed point which is a unique solution for the problem (1) on $\mathbb{T}_{t_0(p,q)}$. □

Now, we prove the existence and uniqueness result for problem(2). Consider the Banach space $\mathcal{X} = \{y : y \in C(\mathbb{T}_{t_0(q)})\}$ via the norm

$$\|y\|_{\mathcal{X}} = \max_{t\in\mathbb{T}_{t_0(q)}} \{\mid y(t) \mid\},$$

where $\alpha \in (2,3]$, $\sigma \in (1,2)$, $\varrho \in (0,1)$, $0 < q < 1$, $\ell \in \mathbb{R}^+$, $t \in \mathbb{T}_{t_0(q)}$,

$\mathcal{X} = C(\mathbb{T}_{t_0(q)}, \mathbb{R})$. Define the operator $\mathcal{B} : \mathcal{X} \to \mathcal{X}$ by

$$(\mathcal{B}y)(t) = -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)_q^{\underline{\alpha-1}} \mathbb{B}\left[(s, y(s), \Phi_q^{\varrho} y(s)\right] d_q s$$
$$- \frac{t}{\Delta_1^*} \left[\frac{(-\Theta_2^* \Xi_1^* + \Delta_1 \Xi_2^*)}{\Omega^*} + \Xi_1^* \right] + \frac{t^2}{\Omega^*} (-\Theta_2^* \Xi_1^* \mathbf{B}_y + \Delta_1 \Xi_2^* \mathbf{B}_y)$$

where $\Xi_1^* \mathbf{B}_y$ and $\Xi_2^* \mathbf{B}_y$ are defined by

$$\Xi_1^* \mathbf{B}_y = (\ell)(-\frac{1}{\Gamma_q(\alpha)} \int_0^{t_0} (t_0 - qs)_q^{\underline{\alpha-1}} \mathbb{B}\left[s, y(s), \Phi_q^{\varrho} y(s)\right] d_q s)$$

$$+ (\ell+1)(-\frac{1}{\Gamma_q(\alpha - \sigma)} \int_0^{t_0} (t_0 - qs)_q^{\underline{\alpha-\sigma-1}} \mathbb{B}\left[s, y(s), \Phi_q^{\varrho} y(s)\right] d_q s)$$

$$+ (\ell+2)(-\frac{1}{\Gamma_q(\alpha + \zeta)} \int_0^{t_0} (t_0 - qs)_q^{\underline{\alpha+\zeta-1}} \mathbb{B}\left[s, y(s), \Phi_q^{\varrho} y(s)\right] d_q s),$$

and

$$\Xi_2^* \mathbf{B}_y = (\ell)(-\frac{1}{\Gamma_q(\alpha)} \int_0^{\rho} (\rho - qs)_q^{\underline{\alpha-1}} \mathbb{B}\left[s, y(s), \Phi_q^{\varrho} y(s)\right] d_q s)$$

$$+ (\ell+1)(-\frac{1}{\Gamma_q(\alpha - \sigma)} \int_0^{\rho} (\rho - qs)_q^{\underline{\alpha-\sigma-1}} \mathbb{B}\left[s, y(s), \Phi_q^{\varrho} y(s)\right] d_q s)$$

$$+ (\ell+2)(-\frac{1}{\Gamma_q(\alpha + \zeta)} \int_0^{\rho} (\rho - qs)_q^{\underline{\alpha+\zeta-1}} d_q s \mathbb{B}\left[s, y(s), \Phi_q^{\varrho} y(s)\right] d_q s).$$

Theorem 3.4. *Let* $\mathbb{B} : \mathbb{T}_{t_0(q)} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\xi^* : \mathbb{T}_{t_0(q)} \times \mathbb{T}_{t_0(q)} \to [0, \infty)$ *be continuous maps and* $\xi_0^* = \max\{\xi(t, s) : (t, s) \in \mathbb{T}_{t_0(q)} \times \mathbb{T}_{t_0(q)}\}$. *Assume that the following conditions hold:*

(H$_1$*) There exist constant* $\tau_1^*, \tau_2^* > 0$ *such that*

$$| \mathbb{B}[t, y_1, y_2] - \mathbb{B}[t, v_1, v_2] | \leq \tau_1^* \mid y_1 - v_1 \mid + \tau_2^* \mid y_2 - v_2 \mid,$$

for all $t \in \mathbb{T}_{t_0(q)}$ *and* $y_i, v_i \in \mathcal{C}$ $(i = 1, 2)$,

(H_2) We have

$$\Sigma^* := (\ell^*)\left[\frac{t_0^{\alpha}}{\Gamma_q^*(\alpha+1)} - \frac{t}{\Delta_1^*}\left[\frac{(-\Theta_2^*\Upsilon_1^* + \Delta_1^*\Upsilon_2^*)}{\Omega^*} + \Upsilon_1^*\right] + \frac{t^2}{\Omega^*}(-\Theta_2^*\Upsilon_1^* + \Delta_1^*\Upsilon_2^*)\right] < 1,$$

where

$$\begin{cases} \ell^* = \left[\tau_1^* - \tau_2^* \dfrac{\xi_0^*(t_0)^{\varrho}}{\Gamma_q(\varrho+1)}\right], \\ \Upsilon_1^* = -\dfrac{\ell t_0^{\alpha}}{\Gamma_q(\alpha+1)} - \dfrac{(\ell+1)t_0^{\alpha-\sigma}}{\Gamma_q(\alpha-\sigma+1)} - \dfrac{(\ell+2)t_0^{\alpha+\zeta}}{\Gamma_q(\alpha+\zeta+1)}, \\ \Upsilon_2^* = -\dfrac{\ell \rho^{\alpha}}{\Gamma_q(\alpha+1)} - \dfrac{(\ell+1)\rho^{\alpha-\sigma}}{\Gamma_q(\alpha-\sigma+1)} - \dfrac{(\ell+2)\rho^{\alpha+\zeta}}{\Gamma_q(\alpha+\zeta+1)}. \end{cases}$$

Then, the problem (2) has a unique solution in $\mathbb{T}_{t_0(q)}$.

Proof. For each $t \in \mathbb{T}_{\approx_{\not\prec}(\shortparallel)}$ and $y, v \in \mathcal{C}$, we have

$$| \Phi_q^{\varrho} y(t) - \Phi_q^{\varrho} v(t) | = -\frac{\xi_0^*}{\Gamma_q(\varrho)} \int_0^t (t-qs)^{\frac{\varrho-1}{q}} | y(s) - v(s) | \, d_q s$$

$$\leq -\frac{\xi_0^*}{\Gamma_q(\varrho)} | y - v | \int_0^{t_0} (t_0 - qs)^{\frac{\varrho-1}{q}} d_q s$$

$$= -\frac{\xi_0^*(t_0)^{\varrho}}{\Gamma_q(\varrho+1)} | y - v | .$$

Put $\mathcal{K}_y(t) := \mathbb{B}[t, y(t), \Phi_q^{\varrho}(t)]$. Then, we have

$$\mid \Xi_1^* \mathcal{B}_y - \Xi_1^* \mathcal{B}_v \mid = (\ell)(-\frac{1}{\Gamma_q(\alpha)} \int_0^{t_0} (t_0 - qs)^{\underline{\alpha-1}}_q \mid \mathcal{K}_y - \mathcal{K}_v \mid d_q s)$$

$$+ (\ell+1)(-\frac{1}{\Gamma_q(\alpha-\sigma)} \int_0^{t_0} (t_0 - qs)^{\underline{\alpha-\sigma-1}}_q \mid \mathcal{K}_y - \mathcal{K}_y \mid (s) d_q s)$$

$$+ (\ell+2)(-\frac{1}{\Gamma_q(\alpha+\zeta)} \int_0^{t_0} (t_0 - qs)^{\underline{\alpha+\zeta-1}}_q \mid \mathcal{K}_y - \mathcal{K}_v \mid (s) d_q s))$$

$$\leq \tau_1^* \mid y - v \mid + \tau_2^* \mid \Phi_q^{\varrho} y - \Phi_q^{\varrho} v \mid$$

$$\times \mid -\frac{\ell t_0^{\alpha}}{\Gamma_q(\alpha+1)} - \frac{(\ell+1) t_0^{\alpha-\sigma}}{\Gamma_q(\alpha-\sigma+1)} - \frac{(\ell+2) t_0^{\alpha+\zeta}}{\Gamma_q(\alpha+\zeta+1)} \mid$$

$$\leq \left(\left[\tau_1^* - \tau_2^* \frac{\xi_0^*(t_0)^{\varrho}}{\Gamma_q(\varrho+1)} \right] \mid y - v \mid \right) \Upsilon_1^* \leq (\ell^*) \Upsilon_1^* \parallel y - v \parallel_{\mathcal{X}} .$$

Similarly, we have $\mid \Xi_2^* \mathbf{B}_y - \Xi_2^* \mathbf{B}_v \mid \leq (\ell^*) \Upsilon_2^* \parallel y - v \parallel_{\mathcal{X}}$. Thus,

$$\mid (\mathcal{B}y)(t) - (\mathcal{B}v)(t) \mid \leq -\frac{1}{\Gamma_q(\alpha)} \int_0^{t_0} (t_0 - qs)^{\underline{\alpha-1}}_q \mid \mathcal{K}_y - \mathcal{K}_v \mid (s) d_q s$$

$$- \frac{t_0}{\Delta_1^*} \left[\frac{(-\Theta_2^* \Xi_1^* \mathbf{B}_y + \Delta_1^* \Xi_2^* \mathbf{B}_u)}{\Omega^*} + \Xi_1^* \mathbf{B}_y \right] + \frac{t_0^2}{\Omega^*} (\Delta_1^* \Xi_2^* \mathbf{B}_y - \Theta_2^* \Xi_1^* \mathbf{B}_y)$$

$$\leq (\frac{(\ell^*) t_0^{\alpha}}{\Gamma_q(\alpha+1)} - \frac{t_0}{\Delta_1^*} \Big[\frac{1}{\Omega^*} \Big(\mid -\Theta_2^* \Xi_1^* \mathbf{B}_y + \Theta_2^* \Xi_1^* \mathbf{B}_v \mid$$

$$+ \mid \Delta_1^* \Xi_2^* \mathbf{B}_y - \Delta_1^* \Xi_2^* \mathbf{B}_v \mid \Big) + \mid \Xi_1^* \mathbf{B}_y - \Xi_1^* \mathbf{B}_v \mid \Big]$$

$$+ \frac{t_0^2}{\Omega^*} (\mid -\Theta_2^* \Xi_1^* \mathbf{B}_y + \Theta_2^* \Xi_1^* \mathbf{B}_y \mid + \mid \Delta_1^* \Xi_2^* \mathbf{B}_y - \Delta_1^* \Xi_2^* \mathbf{B}_y \mid)$$

$$\leq (\frac{(\ell^*) t_0^{\alpha}}{\Gamma_{p,q}(\alpha+1)} - \frac{t_0}{\Delta_1} \left[\frac{(-\Theta_2^* (\ell^*) \Upsilon_1^* + \Delta_1^* (\ell^*) \Upsilon_2^*)}{\Omega^*} + (\ell^* + \tau_3^*) \Upsilon_1^* \right]$$

$$+ \frac{t_0^2}{\Omega^*} (-\Theta_2^* (\ell^*) \Upsilon_1^* + \Delta_1^* (\ell^*) \Upsilon_2^*)) \parallel y - v \parallel_{\mathcal{X}} \leq \Sigma^* \parallel y - v \parallel_{\mathcal{X}} .$$

Hence, $\parallel \mathcal{B}y - \mathcal{B}v \parallel < \Sigma^* \parallel y - v \parallel_{\mathcal{X}}$. By using (H_2), we conclude that $\mathcal{B}$ is a contraction and so by using the Banach fixed point theorem, $\mathcal{B}$ has a fixed point which is a unique solution of problem (2) on $\mathbb{T}_{t_0(q)}$. □

Now we check some problems via at least one solution.

Theorem 3.5. *Assume that (H_1) and (H_2) in theorem 3.3 hold. Then, the problem* (1) *has at least one solution on* $\mathbb{T}_{t_0(p,q)}$.

Proof. We present the proof here in three steps.
1. We first show that the map $\mathcal{B}$ maps bounded sets into bounded sets of $\mathcal{S}_L = \{y \in \mathcal{X} : \parallel y \parallel_{\mathcal{X}}\} \leq L$. Set $\max_{t\in\mathbb{T}_{t_0(p,q)}} \mid \mathcal{B}(t,0,0) \mid = N$. Now, put

$$L \geq \frac{\frac{(\frac{t_0}{p})^{\alpha}}{\Gamma_{p,q}(\alpha+1)}}{(1-(\ell)+N)\left(\frac{\frac{t_0}{p}}{\Delta_1}\left[\frac{(-\Theta_2\Upsilon_1+\Delta_1^*\Upsilon_2)}{\Omega}+\Upsilon_1\right]+\frac{(\frac{t_0}{p})^2}{\Omega}(-\Theta_2\Upsilon_1+\Delta_1^*\Upsilon_2)\right)}.$$

Note that, $\mid W(t,y,0) \mid = \mid \mathbb{B}[t,y(t),\Phi^{\varrho}_{p,q}y(t)] - \mathbb{B}(t,0,0) \mid + \mid \mathbb{B}(t,0,0) \mid$. For each $t \in \mathbb{T}_{t_0(p,q)}$ and $y \in \mathcal{S}_L$, we have

$$\begin{aligned}
\Xi_1\mathbf{B}_y &= (\ell)(-\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{\frac{t_0}{p}}(\frac{t_0}{p}-qs)^{\alpha-1}_{p,q} \mid W(t,y,0) \mid d_{p,q}s) + (\ell+1) \\
&\times (-\frac{1}{p^{\binom{\alpha}{2}+\binom{-\sigma}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\sigma)}\int_0^{\frac{t_0}{p}}\int_0^{\frac{x}{p^{-\sigma-1}}}(\frac{t_0}{p}-qx)^{-\sigma-1}_{p,q}(\frac{x}{p^{\alpha-1}}-qs)^{\alpha-1}_{p,q} \\
&\times \mid W(t,y,0) \mid d_{p,q}sd_{p,q}x) \\
&+ (\ell+2)(-\frac{1}{p^{\binom{\alpha}{2}+\binom{\zeta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\zeta)}\int_0^{\frac{t_0}{p}}\int_0^{\frac{x}{p^{\zeta-1}}}(\frac{t_0}{p}-qx)^{\zeta-1}_{p,q}(\frac{x}{p^{\alpha-1}}-qs)^{\alpha-1}_{p,q} \\
&\times \mid W(t,y,0) \mid d_{p,q}s)) \\
&\leq \left(\left[\tau_1 - \tau_2\frac{\xi_0(\frac{t_0}{p})^{\gamma}}{\Gamma_{p,q}(\gamma+1)}\right] \mid y \mid + N\right)\Upsilon_1 \leq [L\ell+N]\,\Upsilon_1 \parallel y \parallel_{\mathcal{X}} \leq [L\ell+N]\,\Upsilon_1.
\end{aligned} \tag{17}$$

Similarly, we have

$$\Xi_2\mathbf{B}_y \leq [L\ell+N]\,\Upsilon_2. \tag{18}$$

By using the relations (17)-(18), we find

$$(\mathcal{B}y)(t) = -\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{\frac{t_0}{p}}(\frac{t_0}{p}-qs)_{p,q}^{\frac{\alpha-1}{p,q}} \mid W(t,y,0)\mid d_{p,q}s$$

$$-\frac{\frac{t_0}{p}}{\Delta_1}\left[\frac{(-\Theta_2\Xi_1\mathbf{B}_y+\Delta_1\Xi_2\mathbf{B}_y)}{\Omega}+\Xi_1\mathbf{B}_y\right]+\frac{(\frac{t_0}{p})^2}{\Omega}(-\Theta_2\Xi_1\mathbf{B}_y+\Delta_1\Xi_2\mathbf{B}_y)$$

$$\leq (\frac{(\frac{t_0}{p})^\alpha}{\Gamma_{p,q}(\alpha+1)}-\frac{\frac{t_0}{p}}{\Delta_1}\left[\frac{(-\Theta_1\Xi_1\mathbf{B}_y+\Delta_1\Xi_2\mathbf{B}_y)}{\Omega}+\Xi_1\mathbf{B}_y\right]$$

$$+\frac{(\frac{t_0}{p})^2}{\Omega}(-\Theta_2\Xi_1\mathbf{B}_y+\Delta_1\Xi_2\mathbf{B}_y)$$

$$\leq \frac{(\frac{t_0}{p})^\alpha}{\Gamma_{p,q}(\alpha+1)}-\frac{\frac{t_0}{p}}{\Delta_1}\Big[\frac{\mid -\Theta_2\Xi_1\mathbf{B}_y+\Theta_2\Xi_1\mathbf{B}_v\mid}{\Omega}$$

$$+\mid \Delta_1\Xi_2\mathbf{B}_y-\Delta_1\Xi_2\mathbf{B}_v\mid)+\mid \Xi_1\mathbf{B}_y-\Xi_1\mathbf{B}_v\mid\Big]$$

$$+\frac{(\frac{t_0}{p})^2}{\Omega}(\mid -\Theta_2\Xi_1\mathbf{B}_y+\Theta_2\Xi_1\mathbf{B}_y\mid+\mid \Delta_1\Xi_2\mathbf{B}_y-\Delta_1\Xi_2\mathbf{B}_y\mid)$$

$$\leq (\frac{(\frac{t_0}{p})^\alpha}{\Gamma_{p,q}(\alpha+1)}-[L\ell+N](\frac{\frac{t_0}{p}}{\Delta_1}\left[\frac{(-\Theta_2\Upsilon_1+\Delta_1\Upsilon_2)}{\Omega}+\Upsilon_1)\right]$$

$$+\frac{(\frac{t_0}{p})^2}{\Omega}(-\Theta_2\Upsilon_1+\Delta_1\Upsilon_2))\parallel y\parallel_{\mathcal{X}}\leq L.$$

Thus, $\parallel \mathcal{B}y\parallel_{\mathcal{X}}\leq L$, where yield that $\mathcal{B}$ is uniformly bounded.
2. The operator $\mathcal{B}$ is continuous on $\mathcal{S}_L$ because of the continuity of $\mathbb{B}$.
3. We claim that $\mathcal{B}$ is equi-continuous on $\mathcal{S}_L$. For each arbitrary elements $t_1, t_2 \in \mathbb{T}_{t_0(p,q)}$ with $t_1 < t_2$, we can write

$$\begin{aligned}\mid (\mathcal{B}y)(t_2)-(\mathcal{B}y)(t_1)\mid &\leq \frac{\parallel \mathbb{B}\parallel}{\Gamma_{p,q}(\alpha+1)}\mid t_2^\alpha - t_1^\alpha\mid \\ &-\frac{t_2-t_1}{\Delta_1}\left[\frac{-\Theta_2\Xi_1\mathbf{B}_y+\Delta_1\Xi_2\mathbf{B}_y)}{\Omega}+\Xi_1\mathbf{B}_y\right] \\ &+\frac{(t_2^2-t_1^2)}{\Omega}(-\Theta_2\Xi_1\mathbf{B}_y+\Delta_1\Xi_2\mathbf{B}_y). \qquad (19)\end{aligned}$$

Since the right-hand side of (19) tends to be zero when $| t_2 - t_1 | \to 0$, $\mathbb{B}$ is relatively compact on $\mathcal{S}_L$. This yield that $\mathcal{B}(\mathcal{S}_L)$ is an equi-continuous set. By using these steps and the Arzela-Ascoli theorem 2.10, we conclude that $\mathcal{B} : \mathcal{X} \to \mathcal{X}$ is completely continuous. Now by using the Schauder fixed point theorem 2.12, we conclude that the problem (1) has at least one solution. □

Here, we present the existence of a solution to (2).

Theorem 3.6. *Assume that (H_1) and (H_2) in theorem 3.4 hold. Then, the problem* (2) *has at least one solution on* $\mathbb{T}_{t_0(q)}$.

Proof. The process is similar to the previous theorem.
1. We first show that the map $\mathcal{B}$ maps bounded sets into bounded sets of $\mathcal{S}_L^* = \{u \in \mathcal{X} : \| y \|_{\mathcal{X}}\} \leq L^*$. Put $\max_{t \in \mathbb{T}_{t_0(q)}} | \mathcal{B}(t, 0, 0) | = N^*$ and

$$L^* \geq \frac{\frac{t_0^\alpha}{\Gamma_{p,q}(\alpha+1)}}{(1 - \ell^* + N^*)\left(\frac{-t_0}{\Delta_1^*}\left[\left(\frac{(-\Theta_2^*\Upsilon_1^* + \Delta_1^*\Upsilon_2^*)}{\Omega^*}\right) + \Upsilon_1^*\right] + \frac{t_0^2}{\Omega^*}(-\Theta_2^*\Upsilon_1^* + \Delta_1^*\Upsilon_2^*\right)}.$$

Note that, $| W(t, y, 0) | = | \mathbb{B}[t, y(t), \Phi_q^\varrho y(t)] - \mathbb{B}(t, 0, 0) | + | \mathbb{B}(t, 0, 0) |$. For every $t \in \mathbb{T}_{t_0(q)}$ and $y \in \mathcal{S}_L^*$, we have

$$\begin{aligned}
\Xi_1^* \mathbf{B}_y &= (\ell)(-\frac{1}{\Gamma_q(\alpha)} \int_0^{t_0} (t_0 - qs)_q^{\frac{\alpha-1}{q}} | W(t, y, 0) | d_q s) \\
&+ (\ell+1)(-\frac{1}{\Gamma_q(\alpha-\sigma)} \int_0^{t_0} (t_0 - qs)_q^{\frac{-\sigma-1}{q}} | W(t, y, 0) | d_q s) \\
&+ (\ell+2)(-\frac{1}{\Gamma_q(\alpha+\zeta)} \int_0^{t_0} (t_0 - qs)_q^{\frac{\alpha+\zeta-1}{q}} | W(t, y, 0) | d_q s) \\
&\leq \left(\left[\tau_1^* - \tau_2^* \frac{\xi_0^* t_0^\varrho}{\Gamma_q(\varrho+1)} \right] | y | + N^* \right) \Upsilon_1^* \\
&\leq [L^* \ell^* + N^*] \, \Upsilon_1^* \| y \|_{\mathcal{C}} \leq [L^* \ell^* + N^*] \, \Upsilon_1^*. \qquad (20)
\end{aligned}$$

Similarly, we have

$$\Xi_2^* \mathbf{B}_y \leq [L^* \ell^* + N^*] \, \Upsilon_2^*. \qquad (21)$$

From (20)-(21), we find

$$(\mathcal{B}y)(t) = -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{\frac{\alpha-1}{q}} \mid W(t,y,0) \mid d_qs$$

$$-\frac{t}{\Delta_1^*}\left[\frac{-\Theta_2^*\Xi_1^*\mathbf{B}_y + \Delta_1^*\Xi_2^*\mathbf{B}_y}{\Omega^*} + \Xi_2^*\mathbf{B}_y\right] + \frac{t^2}{\Omega^*}(-\Theta_2^*\Xi_2^*\mathbf{B}_y + \Delta_1^*\Xi_1^*\mathbf{B}_y)$$

$$\leq (\frac{t_0^\alpha}{\Gamma_q(\alpha+1)} - \frac{t_0}{\Delta_1^*}\left[\frac{(-\Theta_2^*\Xi_1^*\mathbf{B}_y + \Delta_1^*\Xi_2^*\mathbf{B}_y)}{\Omega^*} + \Xi_2^*\mathbf{B}_y\right]$$

$$+\frac{t_0^2}{\Omega^*}(-\Theta_2^*\Xi_1^*\mathbf{B}_y + \Delta_1^*\Xi_2^*\mathbf{B}_y)$$

$$\leq (\frac{t_0^\alpha}{\Gamma_q(\alpha+1)} + \frac{t_0}{\Delta_1^*}\Big[\frac{1}{\Omega^*}(\mid -\Theta_2^*\Xi_1^*\mathbf{B}_y + \Delta_1^*\Xi_1^*\mathbf{B}_v \mid$$

$$+ \mid \Delta_1^*\Xi_2^*\mathbf{B}_y - \Delta_1^*\Xi_2^*\mathbf{B}_v \mid) + \mid \Xi_1^*\mathbf{B}_y - \Xi_1^*\mathbf{B}_v \mid \Big]$$

$$+\frac{t_0^2}{\Omega^*}(\mid -\Theta_2^*\Xi_1^*\mathbf{B}_y + \Theta_2^*\Xi_1^*\mathbf{B}_y \mid + \mid \Delta_1^*\Xi_2^*\mathbf{B}_y - \Delta_1^*\Xi_2^*\mathbf{B}_y \mid)$$

$$\leq (\frac{(\frac{t_0}{p})^\alpha}{\Gamma_q(\alpha+1)} + \frac{\frac{t_0}{p}}{\Delta_1^*}\left[\frac{(-L^*\Theta_2^*\ell^*\Upsilon_1^*) + (L^*\Delta_1^*\ell^*\Upsilon_2^*)}{\Omega^*} + (L^*\ell^*\Upsilon_1^*)\right]$$

$$+\frac{(\frac{t_0}{p})^2}{\Omega^*}[(-L^*\Theta_2^*\ell^*\Upsilon_1^*) + (L^*\Delta_1^*\ell^*\Upsilon_1^*)] \parallel y \parallel_{\mathcal{C}} \leq L^*.$$

Thus, $\parallel \mathcal{B}y \parallel_{\mathcal{C}} \leq L^*$ and so $\mathcal{B}$ is uniformly bounded.
2. The operator $\mathcal{B}$ is continuous on $\mathcal{S}_L^*$ because of the continuity of $\mathbb{B}$. We show that $\mathcal{B}$ is equi-continuous on $\mathcal{S}_L$. For each $t_1, t_2 \in \mathbb{T}_{t_0(q)}$ with $t_1 < t_2$, we can write

$$\begin{aligned} \mid (\mathcal{B}y)(t_2) - (\mathcal{B}y)(t_1) \mid &\leq \frac{\parallel \mathbb{B} \parallel}{\Gamma_q(\alpha+1)} \mid t_2^\alpha - t_1^\alpha \mid \\ &- \frac{t_2 - t_1}{\Delta_1^*}\left[\frac{(-\Theta_2^*\Xi_1^*\mathbf{B}_y + \Delta_1^*\Xi_2^*\mathbf{B}_y)}{\Omega^*} + \Xi_1^*\mathbf{B}_y\right] \\ &+ \frac{(t_2^2 - t_1^2)}{\Omega^*}(-\Theta_2^*\Xi_1^*\mathbf{B}_y + \Delta_1^*\Xi_2^*\mathbf{B}_y). \end{aligned} \tag{22}$$

Since the right-hand side of (22) tends to be zero when $| t_2 - t_1 | \to 0$, $\mathbb{B}$ is relatively compact on $\mathcal{S}_L^*$. This shows that $\mathcal{B}(\mathcal{S}_L)$ is equi-continuous. By using the steps and the Arzela-Ascoli theorem 2.10, we conclude that $\mathcal{B} : \mathcal{X} \to \mathcal{X}$ is completely continuous. Now by using the Schauder fixed point theorem 2.12, we find that problem (2) has at least one solution. □

4 Examples

In this section, we provide some examples to illustrate our main results.

Example 4.1. Consider the problem defined below

$$^cD_{p,q}^{\frac{11}{4}}y(t) = \frac{e^{-\pi t}}{24\sqrt{\pi} + e^{-\pi t}}\Big[\frac{\sin t + e^t}{1+t^3} + \frac{| y(t) |}{1+ | y(t) |} + \frac{e^{-\pi t-3}(\cos t + 2)}{1+t^2} | \Phi_{p,q}^{\frac{1}{5}}y(t) | \Big],$$

for $t \in \mathbb{T}_{t_0(p,q)}$, with the boundary value conditions

$$\begin{cases} y(0) = 0, \\ \ell y(\frac{t_0}{p}) + (\ell+1)^c D_{p,q}^{\frac{4}{3}} y(\frac{t_0}{p}) + (\ell+2)\mathcal{I}_{p,q}^{\frac{3}{4}} y(\frac{t_0}{p}) = 0, \\ \ell y(\rho) + (\ell+1)^c D_{p,q}^{\frac{4}{3}} y(\rho) + (\ell+2)\mathcal{I}_{p,q}^{\frac{3}{4}} y(\rho) = 0, \end{cases}$$

where $\xi(t,s) = \dfrac{e^{-|s-t|}}{(t+3)^{\frac{3}{2}}}$, $t \in \mathbb{T}_{t_0(p,q)}$, $\alpha = \frac{11}{4}$, $\varrho = \frac{1}{5}$, $\zeta = \frac{3}{4}$, $\sigma = \frac{4}{3}$, $t_0 = 3$, $\ell = 0.01$, $0 < q < p \leq 1$, $\rho = \dfrac{q^2}{p^3}t_0$ and $k = 2$. Note that,

$$\mathbb{B}\left[t, y(t), \Phi_{p,q}^{\varrho}y(t)\right] = \frac{e^{-\pi t}}{24\sqrt{\pi} + e^{-\pi t}}\Big[\frac{\sin t + e^t}{1+t^3} + \frac{| y(t) |}{1+ | y(t) |} + \frac{e^{-\pi t-3}(\cos t + 2)}{1+t^2} | \Phi_{p,q}^{\frac{1}{5}}y(t) | \Big].$$

By using (5)-(7), we get

$$\begin{cases} \Delta_1 = 0.01(\frac{3}{p}) + (0.01+2)\dfrac{\Gamma_{p,q}(2)(\frac{3}{p})^{\frac{3}{4}+1}}{p^{\binom{\frac{3}{4}}{2}}\Gamma_{p,q}(\frac{3}{4}+2)}, \\ \Delta_2 = 0.01(\rho)^2 + (0.01+1)\dfrac{[2]_{p,q}!(\rho)^{2-\frac{4}{3}}}{\Gamma_{p,q}(3-\frac{4}{3})} + (0.01+2)\dfrac{\Gamma_{p,q}(3)(\rho)^{\frac{3}{4}+2}}{p^{\binom{\frac{3}{4}}{2}}\Gamma_{p,q}(\frac{3}{4}+3)}, \\ \Theta_1 = 0.01(\frac{3}{p})^2 + (0.01+1)\dfrac{[2]_{p,q}!(\frac{3}{p})^{2-\frac{4}{3}}}{\Gamma_{p,q}(3-\frac{4}{3})} + (0.01+2)\dfrac{\Gamma_{p,q}(3)(\frac{3}{p})^{\frac{3}{4}+2}}{p^{\binom{\frac{3}{4}}{2}}\Gamma_{p,q}(\frac{3}{4}+3)}, \\ \Theta_2 = \frac{1}{\Delta_1}0.01(\rho) + (0.01+2)\dfrac{\Gamma_{p,q}(2)(\rho)^{\frac{3}{4}+1}}{p^{\binom{\frac{3}{4}}{2}}\Gamma_{p,q}(\frac{3}{4}+2)}, \\ \Omega = \Theta_1\Theta_2 - \Delta_1\Delta_2, \end{cases}$$

and for all $t \in \mathbb{T}_{t_0(p,q)}, y, v \in \mathbb{R}$,

$$\mid \mathbb{B}\left[t, y(t), \Phi^{\varrho}_{p,q}y(t)\right] - \mathbb{B}\left[t, v(t), \Phi^{\varrho}_{p,q}v(t)\right] \mid = \frac{1}{24\sqrt{\pi}} \mid y - v \mid + \frac{e^{-3}}{8\sqrt{\pi}} \mid \Phi^{\varrho}_{p,q}y(t) - \Phi^{\varrho}_{p,q}v(t) \mid .$$

Thus, the condition (H_1) holds with $\tau_1 = 0.0235079$ and $\tau_2 = 0.003511168$ for all $y, v \in \mathcal{X}$. Also, we have

$$\begin{cases} \ell = \left[0.0235079 - 0.003511168\dfrac{0.19245009(\frac{3}{p})^{\frac{1}{5}}}{\Gamma_{p,q}(\frac{1}{5}+1)}\right], \\ \Upsilon_1 = \mid -\dfrac{0.01(\frac{3}{p})^{\frac{11}{4}}}{\Gamma_{p,q}(\frac{11}{4}+1)} - \dfrac{(0.01+1)(\frac{3}{p})^{\frac{11}{4}-\frac{4}{3}}}{\Gamma_{p,q}(\frac{11}{4}-\frac{4}{3}+1)} - \dfrac{(0.01+2)(\frac{3}{p})^{\frac{11}{4}+\frac{3}{4}}}{\Gamma_{p,q}(\frac{11}{4}+\frac{3}{4}+1)} \mid, \\ \Upsilon_2 = \mid -\dfrac{0.01(\rho)^{\frac{11}{4}}}{\Gamma_{p,q}(\frac{11}{4}+1)} - \dfrac{(0.01+1)(\rho)^{\frac{11}{4}-\frac{4}{3}}}{\Gamma_{p,q}(\frac{11}{4}-\frac{4}{3}+1)} - \dfrac{(0.01+2)(\rho)^{\frac{11}{4}+\frac{3}{4}}}{\Gamma_{p,q}(\frac{11}{4}+\frac{3}{4}+1)} \mid . \end{cases}$$

In the last section, the tables show us that

$$\Sigma := \ell\left[\frac{(\frac{3}{p})^{\frac{11}{4}}}{\Gamma_{p,q}(\frac{11}{4}+1)} - \frac{\frac{3}{p}}{\Delta_1}\left[\frac{(-\Theta_2 \mid \Upsilon_1 \mid + \Delta_1 \mid \Upsilon_2 \mid)}{\mid \Omega \mid} + \mid \Upsilon_2 \mid\right] + \frac{(\frac{3}{p})^2}{\mid \Omega \mid}(-\Theta_2 \mid \Upsilon_1 \mid + \Delta_1 \mid \Upsilon_2 \mid)\right] < 1.$$

By using Theorem 3.3, this problem has a unique solution. Some considerable numerical results presented in Tables 4, 5, 6, 8 and 11 about this example. By comparing the Tables, it can be concluded that the q-calculus is better than (p,q)-calculus.

Example 4.2. Consider the q-differential equation

$$ {}^{c}D_q^{\frac{11}{4}} y(t) = \frac{e^{-\pi t}}{24\sqrt{\pi}+e^{-\pi t}}\Big[\frac{\sin t + e^t}{1+t^3} + \frac{\mid y(t) \mid}{1+\mid y(t) \mid} + \frac{e^{-\pi t-3}(\cos t+2)}{1+t^2} \mid \Phi_q^{\frac{1}{5}} y(t) \mid\Big], $$

for $t \in \mathbb{T}_{t_0(q)}$, with the boundary conditions

$$ \begin{cases} y(0) = 0, \\ \ell y(t_0) + (\ell+1)^c D_{p,q}^{\frac{4}{3}} y(t_0) + (\ell+2)\mathcal{I}_q^{\frac{3}{4}} y(t_0) = 0, \\ \ell y(\rho) + (\ell+1)^c D_q^{\frac{4}{3}} y(\rho) + (\ell+2)\mathcal{I}_q^{\frac{3}{4}} y(\rho) = 0, \end{cases} $$

where $\xi^*(t,s) = \dfrac{e^{-|s-t|}}{(t+3)^{\frac{3}{2}}}$, $t \in [0,1]$, $\alpha = \frac{11}{4}$, $\varrho = \frac{1}{5}$, $\zeta = \frac{3}{4}$, $\sigma = \frac{4}{3}$, $\ell^* = 0.01$, $t_0 = 3$, $0 < q < 1$, $\rho = t_0 q^2$, $k = 2$ and $\xi_0^* = 0.19245009$. Note that,

$$ \mathbb{B}\left[t, y(t), \Phi_q^{\gamma} y(t)\right] = \frac{e^{-\pi t}}{24\sqrt{\pi}+e^{-\pi t}}\Big[\frac{\sin t + e^t}{1+t^3} + \frac{\mid y(t) \mid}{1+\mid y(t) \mid} + \frac{e^{-\pi t-3}(\cos t+2)}{1+t^2} \mid \Phi_q^{\frac{1}{5}} y(t) \mid\Big]. $$

For every $t \in \mathbb{T}_{t_0}, y, v \in \mathbb{R}$, we have

$$ \mid \mathbb{B}\left[t, y(t), \Phi_q^{\varrho} y(t)\right] - \mathbb{B}\left[t, v(t), \Phi_q^{\varrho} v(t),\right] \mid = \frac{1}{24\sqrt{\pi}} \mid y - v \mid + \frac{e^{-3}}{8\sqrt{\pi}} \mid \Phi_q^{\varrho} y(t) - \Phi_q^{\varrho} v(t) \mid . $$

Hence, (H_1) holds with $\tau_1^* = 0.0235079$, $\tau_2^* = 0.003511168$ and all $y, v \in \mathcal{X}$. By performing calculations similar to the previous steps, we get $\sum^* < 1$ and so by using Theorem 3.3, this problem has a unique solution. Some numerical results presented in Tables 1, 2, 3, 7 about this example. By comparing the Tables, it can be concluded that the q-calculus is better than (p,q)-calculus.

5 Conclusion

The role and importance of generalization in learning and teaching mathematics are not hidden from anyone. In fact, the history of mathematics is nothing but the recording of successive generalizations in mathematics. Sometimes the problem can be solved more easily by changing the problem to a more general one and by generalizing it. But this is not always the case and there are gaps in some generalizations. For this reason, in this paper, we examined the generalization of q-calculus, ie (p,q)-calculus. To achieve this goal, we examined the fractional differential equations in both modes with the Banach fixed point theorem, and numerical result. By comparing the data from Tables 1–3 with 4–6, 7 with 8, and 9 with 10, it can be concluded that the rate of convergence of the designed algorithms to the desired solution is higher in q-calculus than in (p,q)-calculus. It can be clearly seen that as the value of p parameter gets closer to 1, the convergence rate of the algorithm increases. And if $p=1$, we actually have nothing but the q-calculus.

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Algorithm 1: The proposed procedure To calculate $\Gamma_{p,q}(x)$

```
function g = pqGamma1(p,q,x,n)
d=1;
for k=1:n
g=(d.*(1-(q./p).^(k+1))./(1-(q).^(x+k)))./(p-q).^(x-1);
end
end
```

Table 1: Some numerical results of Coefficients in Example 4.2 for different value of q.

n	$\Delta^*_{(1)}$	$\Delta^*_{(2)}$	$\Theta^*_{(1)}$	$\Theta^*_{(2)}$	Ω^*	$\Upsilon^*_{(1)}$	$\Upsilon^*_{(2)}$	κ^*	Σ^*
					$q = 0.6, p = 1$				
1	7.5498	4.4032	27.9060	0.9551	−2.7243	−12.7158	−0.3997	0.0227	0.0365
2	7.2712	4.6050	27.9290	1.0527	−1.2348	−11.0704	−0.3564	0.0228	0.0014
3	7.1291	4.7154	27.9417	1.1025	−0.5546	−10.4074	−0.3389	0.0228	0.0554
4	7.0513	4.7784	27.9488	1.1298	−0.2250	−10.1053	−0.3309	0.0228	0.1176
5	7.0070	4.8365	27.9530	1.1453	−0.0608	−9.9600	−0.3270	0.0228	0.1778
6	6.9812	4.8493	27.9555	1.1544	0.0221	−9.8883	−0.3251	0.0228	0.2285
7	6.9660	4.8569	27.9569	1.1597	0.0643	−9.8525	−0.3241	0.0228	0.2668
8	6.9570	4.8615	27.9578	1.1628	0.0858	−9.8345	−0.3237	0.0228	0.2935
9	6.9516	4.8642	27.9583	1.1647	0.0967	−9.8255	−0.3234	0.0228	0.3112
10	6.9484	4.8659	27.9586	1.1658	0.1023	−9.8209	−0.3233	0.0228	0.3224
11	6.9465	4.8668	27.9588	1.1669	0.1052	−9.8186	−0.3232	0.0228	0.3294
12	6.9453	4.8674	27.9589	1.1672	0.1066	−9.8174	−0.3232	0.0228	0.3337
13	6.9446	4.8678	27.9590	1.1673	0.1074	−9.8168	−0.3232	0.0228	0.3363
14	6.9442	4.8680	27.9590	1.1674	0.1078	−9.8165	−0.3232	0.0228	0.3379
15	6.9440	4.8681	27.9591	1.1675	0.1080	−9.8164	−0.3232	0.0228	0.3388
16	6.9438	4.8682	27.9591	1.1675	0.1081	−9.8163	−0.3232	0.0228	0.3394
17	6.9437	4.8682	27.9591	1.1675	0.1081	−9.8162	−0.3232	0.0228	0.3398
18	6.9437	4.8683	27.9591	1.1675	0.1081	−9.8162	−0.3232	0.0228	0.3400
19	6.9436	4.8683	27.9591	1.1675	0.1082	−9.8162	−0.3232	0.0228	0.3401
...	...	...	...	...	...	...	...	...	...
22	6.9436	4.8683	27.9591	1.1675	0.1082	−9.8162	−0.3232	0.0228	0.3402
23	6.9436	4.8683	27.9591	1.1675	0.1082	−9.8162	−0.3232	0.0228	0.3403

Table 2: Some numerical results of Coefficients in Example 4.2 for different value of q.

n	$\Delta^*_{(1)}$	$\Delta^*_{(2)}$	$\Theta^*_{(1)}$	$\Theta^*_{(2)}$	Ω^*	$\Upsilon^*_{(1)}$	$\Upsilon^*_{(2)}$	κ^*	Σ^*
					$q = 0.7, p = 1$				
1	6.2846	7.7973	26.4935	1.2559	−15.7296	−3.5559	−0.6320	0.0227	0.0679
2	6.0151	8.1856	26.7723	1.3972	−11.8317	−2.9676	−0.5407	0.0228	0.0806
3	5.8661	8.4196	26.9403	1.4752	−9.6467	−2.6912	−0.4979	0.0228	0.0915
4	5.7762	8.5682	27.0470	1.5223	−8.3179	−2.5384	−0.4742	0.0228	0.0997
5	5.7192	8.6655	27.1169	1.5522	−7.4698	−2.4463	−0.4599	0.0228	0.1056
...	...	...	...	...	...	...	...	...	...
21	5.6022	8.8733	27.2661	1.6135	−5.7166	−2.2681	−0.4323	0.0228	0.1190
22	5.6020	8.8701	27.2656	1.6135	−5.7150	−2.2679	−0.4323	0.0228	0.1191
23	5.6019	8.8712	27.2659	1.6136	−5.7138	−2.2678	−0.4322	0.0228	0.1191
...	...	...	...	...	...	...	...	...	...
26	5.6019	8.8730	27.2665	1.6136	−5.7121	−2.2676	−0.4322	0.0228	0.1191
27	5.6019	8.8733	27.2665	1.6137	−5.7118	−2.2676	−0.4322	0.0228	0.1191
...	...	...	...	...	...	...	...	...	...
31	5.6019	8.8739	27.2665	1.6137	−5.7113	−2.2676	−0.4322	0.0228	0.1191
32	5.6019	8.8739	27.2665	1.6137	−5.7113	−2.2675	−0.4322	0.0228	0.1191
33	5.6019	8.8739	27.2666	1.6137	−5.7113	−2.2675	−0.4322	0.0228	0.1191
34	5.6019	8.8739	27.2666	1.6137	−5.7112	−2.2675	−0.4322	0.0228	0.1191
35	5.6019	8.8739	27.2666	1.6137	−5.7112	−2.2675	−0.4322	0.0228	0.1191
36	5.6019	8.8739	27.2666	1.6137	−5.7112	−2.2675	−0.4322	0.0228	0.1191
37	5.6019	8.8740	27.2666	1.6137	−5.7112	−2.2675	−0.4322	0.0228	0.1191

Table 3: Some numerical results of Coefficients in Example 4.2 for different value of q.

n	$\Delta^*_{(1)}$	$\Delta^*_{(2)}$	$\Theta^*_{(1)}$	$\Theta^*_{(2)}$	Ω^*	$\Upsilon^*_{(1)}$	$\Upsilon^*_{(2)}$	κ^*	Σ^*
					$q = 0.9, p = 1$				
1	2.9739	32.7752	40.7239	1.2095	−48.2147	−0.3615	−0.2499	0.0229	0.0126
2	2.8286	35.3934	43.6447	1.3564	−40.9141	−0.3069	−0.2132	0.0229	0.0155
3	2.7421	37.1229	45.5741	1.4439	−35.9891	−0.2798	−0.1950	0.0230	0.0196
4	2.6850	38.3447	46.9372	1.5017	−32.4667	−0.2636	−0.1842	0.0230	0.0244
5	2.6446	39.2493	47.9464	1.5017	−29.8374	−0.2529	−0.1770	0.0230	0.0297
...	...	...	...	...	...	...	...	...	...
56	2.4748	43.4962	52.6843	1.7143	−17.2848	−0.2137	−0.1506	0.0230	0.1241
57	2.4747	43.4963	52.6843	1.7144	−17.2846	−0.2136	−0.1506	0.0230	0.1242
...	...	...	...	...	...	...	...	...	...
77	2.4744	43.4962	52.6843	1.7147	−17.2848	−0.2136	−0.1506	0.0230	0.1247
78	2.4744	43.4963	52.6843	1.7147	−17.2846	−0.2136	−0.1505	0.0230	0.1247
...	...	...	...	...	...	...	...	...	...
83	2.4744	43.4962	52.6843	1.7147	−17.2848	−0.2136	−0.1505	0.0230	0.1247
84	2.4743	43.4963	52.6843	1.7148	−17.2846	−0.2136	−0.1505	0.0230	0.1247
...	...	...	...	...	...	...	...	...	...
91	2.4743	43.4968	52.6847	1.7148	−17.2831	−0.2136	−0.1505	0.0230	0.1247
92	2.4743	43.4968	52.6848	1.7148	−17.2830	−0.2136	−0.1505	0.0230	0.1248
...	...	...	...	...	...	...	...	...	...
95	2.4743	43.4968	52.6849	1.7148	−17.2830	−0.2136	−0.1505	0.0230	0.1248
96	2.4743	43.4969	52.6850	1.7148	−17.2829	−0.2136	−0.1505	0.0230	0.1248
97	2.4743	43.4969	52.6850	1.7148	−17.2829	−0.2136	−0.1505	0.0230	0.1248
98	2.4743	43.4969	52.6850	1.7148	−17.2828	−0.2136	−0.1505	0.0230	0.1248
...	...	...	...	...	...	...	...	...	...
102	2.4743	43.4969	52.6850	1.7148	−17.2828	−0.2136	−0.1505	0.0230	0.1248
103	2.4743	43.4970	52.6850	1.7148	−17.2828	−0.2136	−0.1505	0.0230	0.1248

Table 4: Some numerical results of Coefficients in Example 4.1 for different value of q.

n	$\Delta_{(1)}$	$\Delta_{(2)}$	$\Theta_{(1)}$	$\Theta_{(2)}$	Ω	$\Upsilon_{(1)}$	$\Upsilon_{(2)}$	κ	Σ
					$q = 0.6, p = 0.99$				
1	7.4737	5.9052	30.2633	1.0629	−7.4545	−7.0497	−0.5664	0.0228	0.0475
2	7.1980	6.1707	30.3748	1.1900	−4.7391	−5.9650	−0.4922	0.0228	0.0209
3	7.0573	6.3162	30.4358	1.2617	−3.3287	−5.4829	−0.4591	0.0228	0.0188
4	6.9803	6.3991	30.4706	1.3051	−2.5393	−5.2349	−0.4420	0.0228	0.0657
5	6.9365	6.4472	30.4908	1.3325	−2.0793	−5.0977	−0.4326	0.0228	0.1124
6	6.9109	6.4756	30.5027	1.3500	−1.8052	−5.0187	−0.4271	0.0228	0.1529
7	6.8959	4.8569	30.5098	1.3614	−1.6397	−4.9722	−0.4239	0.0228	0.1844
8	6.8870	6.4925	30.5140	1.3688	−1.5391	−4.9445	−0.4220	0.0228	0.2072
9	6.8817	6.5025	30.5166	1.3736	−1.4776	−4.9278	−0.4208	0.0228	0.2227
10	6.8785	6.5085	30.5190	1.3767	−1.4400	−4.9178	−0.4201	0.0228	0.2329
11	6.8766	6.5121	30.5195	1.3788	−1.4169	−4.9117	−0.4197	0.0228	0.2395
12	6.8754	6.5143	30.5198	1.3801	−1.4028	−4.9080	−0.4194	0.0228	0.2436
13	6.8748	6.5156	30.5200	1.3809	−1.3941	−4.9058	−0.4192	0.0228	0.2462
14	6.8743	6.5163	30.5201	1.3815	−1.3888	−4.9044	−0.4192	0.0228	0.2478
15	6.8741	6.5171	30.5202	1.3819	−1.3855	−4.9036	−0.4191	0.0228	0.2488
16	6.8740	6.5172	30.5203	1.3821	−1.3835	−4.9031	−0.4191	0.0228	0.2494
17	6.8739	6.5173	30.5203	1.3822	−1.3822	−4.9028	−0.4190	0.0228	0.2498
18	6.8738	6.5174	30.5203	1.3823	−1.3815	−4.9026	−0.4190	0.0228	0.2500
19	6.8738	6.5174	30.5203	1.3824	−1.3810	−4.9025	−0.4190	0.0228	0.2502
20	6.8738	6.5175	30.5203	1.3824	−1.3807	−4.9024	−0.4190	0.0228	0.2502
21	6.8738	6.5175	30.5203	1.3825	−1.3806	−4.9024	−0.4190	0.0228	0.2503
22	6.8737	6.5175	30.5203	1.3825	−1.3805	−4.9024	−0.4190	0.0228	0.2503
23	6.8737	6.5175	30.5203	1.3825	−1.3804	−4.9023	−0.4190	0.0228	0.2503
24	6.8737	6.5175	30.5203	1.3825	−1.3804	−4.9023	−0.4190	0.0228	0.2504
25	6.8737	6.5175	30.5203	1.3825	−1.3803	−4.9023	−0.4190	0.0228	0.2504
26	6.8737	6.5175	30.5203	1.3825	−1.3803	−4.9023	−0.4190	0.0228	0.2504

Table 5: Some numerical results of Coefficients in Example 4.1 for different value of q.

n	$\Delta_{(1)}$	$\Delta_{(2)}$	$\Theta_{(1)}$	$\Theta_{(2)}$	Ω	$\Upsilon_{(1)}$	$\Upsilon_{(2)}$	κ	Σ
					$q = 0.7, p = 0.99$				
1	6.2361	8.3024	26.9852	1.2659	−17.6146	−3.3938	−0.6396	0.0228	0.0693
2	5.9686	8.7174	27.3016	1.4132	−13.4479	−2.8244	−0.5454	0.0228	0.0841
3	5.8208	8.9676	27.4924	1.4967	−11.0501	−2.5548	−0.5007	0.0228	0.0972
4	5.7317	9.1263	27.6134	1.5484	−9.5521	−2.4043	−0.4756	0.0228	0.1075
5	5.6751	9.2304	27.6928	1.5821	−8.5693	−2.3126	−0.4603	0.0228	0.1152
...	...	...	...	...	...	...	...	...	...
22	5.5589	9.4527	27.8623	1.6582	−6.3454	−2.1273	−0.4291	0.0228	0.1341
23	5.5588	9.4528	27.8624	1.6583	−6.3433	−2.1272	−0.4291	0.0228	0.1342
24	5.5588	9.4529	27.8625	1.6583	−6.3418	−2.1271	−0.4291	0.0228	0.1342
25	5.5587	9.4530	27.8626	1.6584	−6.3408	−2.1270	−0.4291	0.0228	0.1342
26	5.5587	9.4530	27.8626	1.6584	−6.3400	−2.1270	−0.4290	0.0228	0.1342
...	...	...	...	...	...	...	...	...	...
28	5.5587	9.4531	27.8626	1.6584	−6.3391	−2.1269	−0.4290	0.0228	0.1342
29	5.5587	9.4531	27.8627	1.6584	−6.3388	−2.1269	−0.4290	0.0228	0.1342
30	5.5587	9.4531	27.8627	1.6584	−6.3386	−2.1269	−0.4290	0.0228	0.1342
31	5.5587	9.4531	27.8627	1.6584	−6.3385	−2.1268	−0.4290	0.0228	0.1342
32	5.5587	9.4531	27.8627	1.6584	−6.3384	−2.1268	−0.4290	0.0228	0.1342
33	5.5587	9.4532	27.8627	1.6584	−6.3383	−2.1268	−0.4290	0.0228	0.1342
...	...	...	...	...	...	...	...	...	...
36	5.5587	9.4532	27.8627	1.6584	−6.3382	−2.1268	−0.4290	0.0228	0.1342
37	5.5586	9.4532	27.8627	1.6584	−6.3381	−2.1268	−0.4290	0.0228	0.1342

Table 6: Some numerical results of Coefficients in Example 4.1 for different value of q.

n	$\Delta_{(1)}$	$\Delta_{(2)}$	$\Theta_{(1)}$	$\Theta_{(2)}$	Ω	$\Upsilon_{(1)}$	$\Upsilon_{(2)}$	κ	Σ
	$q = 0.9, p = 0.99$								
1	2.7988	37.6769	45.4170	1.0796	−17.6146	−0.3338	−0.2412	0.0229	0.0122
2	2.6622	40.7759	48.8551	1.2159	−13.4479	−0.2828	−0.2052	0.0229	0.0157
3	2.5808	42.8230	51.1262	1.2997	−11.0501	−0.2571	−0.1870	0.0229	0.0210
4	2.5271	44.2692	52.7307	1.3573	−9.5521	−0.2414	−0.1759	0.0229	0.0280
5	2.4891	45.3400	53.9187	1.3998	−8.5693	−0.2309	−0.1684	0.0229	0.0365
...	...	...	...	...	...	...	...	...	...
9	2.4089	47.7496	56.5919	1.6584	−6.3382	−0.2092	−0.1530	0.0229	0.0826
10	2.3977	48.1006	56.9814	1.6584	−6.3381	−0.2061	−0.1508	0.0230	0.0964
...	...	...	...	...	...	...	...	...	...
69	2.3291	50.3647	59.4929	1.6706	−6.3381	−0.1820	−0.1336	0.0230	0.4581
70	2.3291	50.3650	59.4933	1.6707	−6.3381	−0.1820	−0.1335	0.0230	0.4584
...	...	...	...	...	...	...	...	...	...
72	2.3291	50.3656	59.4939	1.6709	−6.3381	−0.1820	−0.1335	0.0230	0.4590
73	2.3290	50.3658	59.4942	1.6710	−6.3381	−0.1820	−0.1335	0.0230	0.4593
...	...	...	...	...	...	...	...	...	...
75	2.3290	50.3662	59.4947	1.6711	−6.3381	−0.1820	−0.1335	0.0230	0.4597
76	2.3290	50.3663	59.4949	1.6712	−6.3381	−0.1819	−0.1335	0.0230	0.4599
...	...	...	...	...	...	...	...	...	...
93	2.3290	50.3675	59.4963	1.6716	−6.3381	−0.1819	−0.1335	0.0230	0.4615
94	2.3290	50.3675	59.4964	1.6717	−6.3381	−0.1819	−0.1335	0.0230	0.4616
...	...	...	...	...	...	...	...	...	...
97	2.3290	50.3676	59.4964	1.6717	−6.3381	−0.1819	−0.1335	0.0230	0.4617
98	2.3290	50.3676	59.4965	1.6717	−6.3381	−0.1819	−0.1335	0.0230	0.4617
99	2.3290	50.3676	59.4965	1.6717	−6.3381	−0.1819	−0.1335	0.0230	0.4617
100	2.3290	50.3677	59.4965	1.6717	−6.3381	−0.1819	−0.1335	0.0230	0.4617
...	...	...	...	...	...	...	...	...	...
107	2.3290	50.3677	59.4964	1.6717	−6.3381	−0.1819	−0.1335	0.0230	0.4618
108	2.3290	50.3677	59.4964	1.6717	−6.3381	−0.1819	−0.1335	0.0230	0.4619

Table 7: Some numerical results of Coefficients in Example 4.2 for different value of q.

n	$\Delta^*_{(1)}$	$\Delta^*_{(2)}$	$\Theta^*_{(1)}$	$\Theta^*_{(2)}$	Ω^*	$\Upsilon^*_{(1)}$	$\Upsilon^*_{(2)}$	κ^*	Σ^*
	$q = 0.19, p = 1$								
1	11.8238	0.3046	38.0837	0.0351	−2.2634	−50.4305	−0.0337	0.0227	0.2129
2	11.7770	0.3072	38.0812	0.0360	−2.2482	−48.9596	−0.0328	0.0227	0.2144
3	11.7681	0.3077	38.0807	0.0361	−2.2453	−48.6899	−0.0326	0.0227	0.2146
4	11.7664	0.3078	38.0806	0.0362	−2.2448	−48.6390	−0.0326	0.0227	0.2147
5	11.7661	0.3078	38.0806	0.0362	−2.2446	−48.6293	−0.0326	0.0227	0.2147
6	11.7661	0.3078	38.0806	0.0362	−2.2446	−48.6275	−0.0326	0.0227	0.2147
7	11.7661	0.3078	38.0806	0.0362	−2.2446	−48.6272	−0.0326	0.0227	0.2147
8	11.7661	0.3078	38.0806	0.0362	−2.2446	−48.6271	−0.0326	0.0227	0.2147
9	11.7661	0.3078	38.0806	0.0362	−2.2446	−48.6271	−0.0326	0.0227	0.2147
10	11.7661	0.3078	38.0806	0.0362	−2.2446	−48.6271	−0.0326	0.0227	0.2147
	$q = 0.2, p = 1$								
1	11.7229	0.3320	37.8055	0.0415	−2.3248	−48.5707	−0.0385	0.0227	0.1941
2	11.6703	0.3352	37.8015	0.0425	−2.3042	−47.0204	−0.0374	0.0227	0.1955
3	11.6598	0.3358	37.8007	0.0427	−2.3001	−46.7223	−0.0371	0.0227	0.1958
4	11.6577	0.3360	37.8006	0.0428	−2.2993	−46.6631	−0.0371	0.0227	0.1958
5	11.6573	0.3360	37.8005	0.0428	−2.2991	−46.6513	−0.0371	0.0227	0.1958
6	11.6572	0.3360	37.8005	0.0428	−2.2991	−46.6489	−0.0371	0.0227	0.1958
7	11.6572	0.3360	37.8005	0.0428	−2.2991	−46.6485	−0.0371	0.0227	0.1958
8	11.6572	0.3360	37.8005	0.0428	−2.2991	−46.6484	−0.0371	0.0227	0.1958
9	11.6572	0.3360	37.8005	0.0428	−2.2991	−46.6484	−0.0371	0.0227	0.1958
10	11.6572	0.3360	37.8005	0.0428	−2.2991	−46.6484	−0.0371	0.0227	0.1958
	$q = 0.4, p = 1$								
1	9.7192	1.3333	32.4596	0.3452	−1.7541	−21.3048	−0.2280	0.0227	0.3601
2	9.5231	1.3785	32.3953	0.3692	−1.1691	−19.2504	−0.2096	0.0227	0.6311
3	9.4489	1.3962	32.3701	0.3782	−0.9500	−18.5420	−0.2032	0.0227	0.8209
4	9.4198	1.4032	32.3602	0.3818	−0.8645	−18.2741	−0.2008	0.0227	0.9214
5	9.4083	1.4060	32.3562	0.3832	−0.8307	−18.1692	−0.1999	0.0227	0.9670
6	9.4037	1.4072	32.3546	0.3837	−0.8172	−18.1276	−0.1995	0.0227	0.9862
7	9.4019	1.4076	32.3540	0.3839	−0.8118	−18.1110	−0.1994	0.0227	0.9941
8	9.4012	1.4078	32.3537	0.3840	−0.8097	−18.1044	−0.1993	0.0227	0.9973
9	9.4009	1.4078	32.3536	0.3841	−0.8088	−18.1018	−0.1993	0.0227	0.9986
10	9.4008	1.4079	32.3536	0.3841	−0.8085	−18.1007	−0.1993	0.0227	0.9991
11	9.4007	1.4079	32.3535	0.3841	−0.8083	−18.1003	−0.1993	0.0227	0.9993
12	9.4007	1.4079	32.3535	0.3841	−0.8083	−18.1001	−0.1993	0.0227	0.9993
13	9.4007	1.4079	32.3535	0.3841	−0.8082	−18.1000	−0.1993	0.0227	0.9994
14	9.4007	1.4079	32.3535	0.3841	−0.8082	−18.1000	−0.1993	0.0227	0.9994
15	9.4007	1.4079	32.3535	0.3841	−0.8082	−18.1000	−0.1993	0.0227	0.9994

Table 8: Some numerical results of Coefficients in Example 4.1 for different value of q.

n	$\Delta_{(1)}$	$\Delta_{(2)}$	$\Theta_{(1)}$	$\Theta_{(2)}$	Ω	$\Upsilon_{(1)}$	$\Upsilon_{(2)}$	κ	Σ
					$q = 0.19, p = 0.67$				
1	16.0990	0.5740	79.0227	0.0200	−7.6605	−35.1351	−0.0148	0.0227	0.2544
2	16.0353	0.5790	79.0318	0.0211	−7.6184	−33.0747	−0.0140	0.0227	0.2607
3	16.0232	0.5799	79.0335	0.0214	−7.6018	−32.5329	−0.0138	0.0227	0.2625
4	16.0210	0.5801	79.0338	0.0215	−7.5961	−32.3823	−0.0137	0.0227	0.2630
5	16.0205	0.5801	79.0339	0.0215	−7.5943	−32.3398	−0.0137	0.0227	0.2631
6	16.0204	0.5801	79.0339	0.0215	−7.5938	−32.3278	−0.0137	0.0227	0.2631
7	16.0204	0.5802	79.0339	0.0215	−7.5936	−32.3244	−0.0137	0.0227	0.2632
8	16.0204	0.5802	79.0339	0.0215	−7.5936	−32.3234	−0.0137	0.0227	0.2632
9	16.0204	0.5802	79.0339	0.0215	−7.5936	−32.3231	−0.0137	0.0227	0.2632
10	16.0204	0.5802	79.0339	0.0215	−7.5936	−32.3230	−0.0137	0.0227	0.2632
					$q = 0.2, p = 0.67$				
1	15.8590	1.5727	78.1094	0.2138	−33.0847	−35.1351	−0.1095	0.0227	0.0158
2	15.7878	1.5871	78.1188	0.2268	−30.9731	−33.0747	−0.1028	0.0227	0.0479
3	15.7737	1.5900	78.1206	0.2308	−30.3931	−32.5329	−0.1009	0.0227	0.0605
4	15.7708	1.5906	78.1210	0.2320	−30.2239	−32.3823	−0.1004	0.0227	0.0648
5	15.7703	1.5907	78.1211	0.2324	−30.1738	−32.3398	−0.1002	0.0227	0.0661
6	15.7702	1.5907	78.1211	0.2325	−30.1588	−32.3278	−0.1002	0.0227	0.0665
7	15.7701	1.5907	78.1211	0.2325	−30.1543	−32.3244	−0.1001	0.0227	0.0667
8	15.7701	1.5907	78.1211	0.2325	−30.1530	−32.3234	−0.1001	0.0227	0.0667
9	15.7701	1.5907	78.1211	0.2325	−30.1526	−32.3231	−0.1001	0.0227	0.0667
10	15.7701	1.5907	78.1211	0.2325	−30.1525	−32.3230	−0.1001	0.0227	0.0667
11	15.7701	1.5907	78.1211	0.2325	−30.1524	−32.3230	−0.1001	0.0227	0.0667
					$q = 0.4, p = 0.67$				
1	10.7739	10.1909	62.0035	1.1890	−36.0724	−8.0181	−0.6199	0.0227	0.1107
2	10.5568	10.4706	62.2349	1.3943	−23.7625	−6.6185	−0.5175	0.0227	0.1122
3	10.4746	10.5801	62.3255	1.5207	−16.0433	−5.9913	−0.4705	0.0228	0.0921
4	10.4425	10.6235	62.3615	1.5991	−11.2115	−5.6680	−0.4459	0.0228	0.0494
5	10.4297	10.6408	62.3758	1.6475	−8.2188	−5.4899	−0.4322	0.0228	0.0111
...	...	...	...	...	...	...	...	...	...
9	10.4215	10.6520	62.3851	1.7123	−4.1879	−5.2737	−0.4154	0.0228	0.2525
10	10.4213	10.6522	62.3852	1.7162	−3.9441	−5.2615	−0.4144	0.0228	0.2839
11	10.4213	10.6522	62.3853	1.7185	−3.7979	−5.2542	−0.4139	0.0228	0.3046
12	10.4213	10.6523	62.3853	1.7200	−3.7104	−5.2499	−0.4135	0.0228	0.3179
13	10.4213	10.6523	62.3853	1.7208	−3.6581	−5.2473	−0.4133	0.0228	0.3261
...	...	...	...	...	...	...	...	...	...
16	10.4213	10.6523	62.3853	1.7218	−3.5970	−5.2443	−0.4131	0.0228	0.3360
17	10.4213	10.6523	62.3853	1.7219	−3.5903	−5.2439	−0.4130	0.0228	0.3371
18	10.4213	10.6523	62.3853	1.7219	−3.5864	−5.2437	−0.4130	0.0228	0.3381
19	10.4213	10.6523	62.3853	1.7220	−3.5840	−5.2436	−0.4130	0.0228	0.3384
20	10.4213	10.6523	62.3853	1.7220	−3.5826	−5.2436	−0.4130	0.0228	0.3385
21	10.4213	10.6523	62.3853	1.7220	−3.5817	−5.2435	−0.4130	0.0228	0.3386
22	10.4213	10.6523	62.3853	1.7220	−3.5826	−5.2435	−0.4130	0.0228	0.3387
...	...	...	...	...	...	...	...	...	...
26	10.4213	10.6523	62.3853	1.7220	−3.5806	−5.2435	−0.4130	0.0228	0.3387
27	10.4213	10.6523	62.3853	1.7220	−3.5805	−5.2435	−0.4130	0.0228	0.3387

Table 9: Some numerical results for calculation of $\Gamma_{p,q}$ with $p = 1$ and $q = 0.2, 0.5, 0.6, 0.7, 0.8, 0.9$ that is constant $x = 2.5$ and $n = 1, 2, ..., 92$.

n	$q = 0.2$	$q = 0.5$	$q = 0.6$	$q = 0.7$	$q = 0.8$	$q = 0.9$
			$p = 1, x = 2.5$			
1	1.3465	2.3270	3.0381	4.3529	7.4253	19.4816
2	1.3874	2.5893	3.4449	5.0035	8.6105	22.6972
3	1.3955	2.7116	3.6611	5.3813	9.3376	24.7266
4	1.3971	2.7707	3.7822	5.6157	9.8190	26.1175
5	1.3975	2.7997	3.8519	5.7672	10.1541	27.1253
...	...	...	...	...	...	...
12	1.3975	2.8282	3.9501	6.0612	10.9983	30.1220
13	1.3975	2.8284	3.9512	6.0686	11.0359	30.3085
...	...	...	...	...	...	...
18	1.3975	2.8284	3.9527	6.0829	11.1340	30.9169
19	1.3975	2.8284	3.9528	6.0838	11.1434	30.9957
...	...	...	...	...	...	...
28	1.3975	2.8284	3.9528	6.0857	11.1754	31.3959
29	1.3975	2.8284	3.9528	6.0858	11.1764	31.4195
...	...	...	...	...	...	...
44	1.3975	2.8284	3.9528	6.0858	11.1802	31.5821
45	1.3975	2.8284	3.9528	6.0858	11.1803	31.5862
...	...	...	...	...	...	...
91	1.3975	2.8284	3.9528	6.0858	11.1803	31.6224
92	1.3975	2.8284	3.9528	6.0858	11.1803	31.6225

Table 10: Some numerical results for calculation of $\Gamma_{p,q}$ with $p = 0.95$ and $q = 0.2, 0.5, 0.6, 0.7, 0.8, 0.9$ that is constant $x = 2.5$ and $n = 1, 2, ..., 250$.

n	$q = 0.2$	$q = 0.5$	$q = 0.6$	$q = 0.7$	$q = 0.8$	$q = 0.9$
	$p = 0.95, x = 2.5$					
1	1.4766	2.6273	3.4863	5.1282	9.2364	29.7243
2	1.5263	2.9606	4.0159	6.0060	10.9431	35.4698
3	1.5368	3.1276	4.3213	6.5649	12.1048	39.5508
4	1.5390	3.2144	4.5070	6.9461	12.9633	42.7302
5	1.5395	3.2603	4.6232	7.2169	13.6317	45.3628
6	1.5396	3.2847	4.6970	7.4142	14.1704	47.6362
...	...	...	...	...	...	...
15	1.5396	3.3126	4.8270	7.9551	16.4436	61.5187
16	1.5396	3.3127	4.8279	7.9663	16.5529	62.6948
...	...	...	...	...	...	...
21	1.5396	3.3127	4.8293	7.9922	16.9100	67.9298
22	1.5396	3.3127	4.8295	7.9942	16.9543	68.8620
...	...	...	...	...	...	...
36	1.5396	3.3127	4.8295	7.9999	17.1866	78.7064
37	1.5396	3.3127	4.8295	8.0000	17.1907	79.2148
..	...	...	...	...	...	...
73	1.5396	3.3127	4.8295	8.0000	17.2131	80.1671
74	1.5396	3.3127	4.8295	8.0000	17.2132	80.6125
...	...	...	...	...	...	...
249	1.5396	3.3127	4.8295	8.0000	17.2132	89.4425
250	1.5396	3.3127	4.8295	8.0000	17.2132	89.4426

Table 11: numerical results for calculation of $\rho_{(p,q)}$.

	ρ		
	$p = 1$	$p = 0.99$	$p = 0.67$
$q = 0.19$	0.1083	–	0.36008419
$q = 0.2$	0.12	–	0.39898525
$q = 0.4$	0.48	–	1.59594099
$q = 0.6$	1.08	1.11305896	–
$q = 0.7$	1.47	1.51499692	–
$q = 0.8$	1.92	1.97877149	–
$q = 0.9$	2.43	2.50438267	–