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Weakly Hypercyclic Composition Operators on some Hilbert Spaces of Analytic Functions

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Abstract. In this paper, weakly supercyclicity and weakly hypercyclicity of composition operators on some Hilbert spaces of analytic functions, especially on some weighted Hardy spaces are investigated.

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1. Introduction

Let $\mathbb{D}$ denotes the open unit disk in the complex plane, and H be a Hilbert space of analytic functions on $\mathbb{D}$ such that $1, z \in H$ and for each $\lambda \in \mathbb{D}$, the linear functional of point evaluation at λ, e_λ, where $e_\lambda(f) = f(\lambda)$, is bounded. The Riesz representation Theorem implies that $e_\lambda(f) =< f, K_\lambda >$ for some $K_\lambda \in H$. The weighted Hardy spaces are examples of such spaces. Recall that a weighted Hardy space with weight sequence $\beta(j) =\| z^j \|$, for all $j \in \mathbb{N} \cup \{0\}$, will be denoted by $H^2(\beta)$ and is defined as the Hilbert space of functions $f(z) = \Sigma_{j=0}^{\infty} a_j z^j$ analytic on $\mathbb{D}$ for which

$$\| f \|_\beta^2 = \Sigma_{j=0}^{\infty} |a_j|^2 \beta_j^2 < \infty.$$

It is useful to note that this norm is induced by the following inner product:

$$< \Sigma_{j=0}^{\infty} a_j z^j, \Sigma_{j=0}^{\infty} b_j z^j >= \Sigma_{j=0}^{\infty} a_j \bar{b_j} \beta_j^2.$$

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The generating function for $H^2(\beta)$ is the function $k(z) = \sum_{j=0}^{\infty} \frac{z^j}{\beta(j)^2}$, which is analytic on $\mathbb{D}$ and $K_\lambda(z) = k(\bar{\lambda} z)$ for all λ and z in $\mathbb{D}$.
If φ is a holomorphic self map of the unit disk, then the composition operator C_φ is defined by $C_\varphi f = f \circ \varphi$ for $f \in H$. The holomorphic self maps of the unit disk are divided in two classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. The non-elliptic one has a unique fixed point $p \in \bar{\mathbb{D}}$, called the Denjoy-Wolff point of φ, so that the sequence of iterates of φ, $\{\varphi_n\}$ converges to p uniformly on compact subsets of $\mathbb{D}$, see [2] and [4] for more details.
Suppose that $x \in H$, and T is an operator on H. The set $\{x, Tx, T^2x, \cdots\}$, denoted by $\mathrm{Orb}\{T, x\}$, is called the orbit of x under T. If there is a vector $x \in H$ whose orbit is (weakly) dense in H, then T is called a (weakly) hypercyclic operator and x is called a (weakly) hypercyclic vector for T. The operator T is called (weakly) supercyclic, if the set of scalar multiples of the elements of $\mathrm{Orb}\{T, x\}$ is (weakly) dense. In this case the vector x is called (weakly) supercyclic vector for T. The classes of weakly hypercyclic and weakly supercyclic operators are more general than hypercyclic and supercyclic operators. In fact every hypercyclic (supercyclic) operator is a weakly hypercyclic (weakly supercyclic) operator but not viceversa. There exists a weakly hypercyclic (weakly supercyclic) operator that is not a norm hypercyclic (supercyclic), (see [3,7]). Bourdon and Shapiro gave characterization of the hypercyclic composition operators on Hardy space H^2 in ([2]). Also, Rezaei in [8] obtained that every linear fractional composition operators on H^2 is hypercyclic if and only if it is weakly hypercyclic.

2. Main Results

The following proposition characterizes the kinds of maps that can produce weakly supercyclic composition operators on H.

Proposition 2.1. *Suppose that C_φ is weakly supercyclic, then φ must be univalent.*

Proof. Let f be a weakly supercyclic vector for C_φ. Suppose $\varphi(a) = \varphi(b)$ and put $g = K_a - K_b$. For any $\epsilon > 0$, the set

$$V = \{h \in H : | < h - g, g > | < \epsilon\}$$

is a weak neighborhood of g, so there exist $n \geqslant 1$ and scalar $\lambda \in \mathbb{C}$ such that $\lambda C_{\varphi_n} f \in V$. Hence

$$| < \lambda C_{\varphi_n} f - g, g > | = | < \lambda f, C^*_{\varphi_n} g > - \parallel g \parallel |^2 = \parallel g \parallel^2 < \epsilon,$$

thus $g \equiv 0$. Since $z \in H$, we get $a = b$.
In Theorem 1, we state the sufficient condition under which a composition operator on H is not weakly supercyclic. $\square$

Theorem 2.2. *Let φ be a self map of $\mathbb{D}$ with an interior fixed point p. If C_φ is a bounded composition operator on H, then C_φ is not weakly supercyclic.*

Proof. To reach a contradiction assume that f is a weakly supercyclic vector for C_φ. Therefore for $\epsilon > 0$ and $g \in H$, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that

$$|K_p(\lambda f \circ \varphi_n - g)| = |\lambda f(p) - g(p)| < \epsilon.$$

Now, if $f(p) = 0$, then $g(p) = 0$ for every g in H. But $1 \in H$, implies that $f(p)$ is non-zero. Without loss of generality, we may assume that $f(p) = 1$. Let $z \in \mathbb{D}$. First, suppose φ is not elliptic automorphism. Put $g_1(z) = g(z) - g(p)$. Since the set

$$U = \{h \in H : | < h - g_1, K_z > | < \varepsilon\} \cap \{h \in H : | < h - g_1, K_p > | < \varepsilon\}$$

is a non-empty weak neighborhood of g_1 (consider that $g_1 \in U$) and the set $\{\lambda f o \varphi_n : \lambda \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}\}$ is weakly dense in H, there exist $\lambda \in \mathbb{C}$ and $n \geqslant 1$ such that $\lambda f o \varphi_n \in U$. So

$$|\lambda f o \varphi_n(z) - g_1(z)| = |< \lambda f o \varphi_n - g_1, K_z > | < \varepsilon$$

and

$$|\lambda| = |\lambda f o \varphi_n(p) - g_1(p)| = | < \lambda f o \varphi_n - g_1, K_p > | < \varepsilon.$$

On the other hand the sequence of iterates of φ tends uniformly on compact subsets of $\mathbb{D}$ to its Denjoy-Wolff point, p, (see [2, page 5]). Consequently, $\lim_{n\to\infty} f(\varphi_n(z)) = 1$ and this sequence is bounded, so $|f(\varphi_n(z))| \leqslant C$ for some constant C. Thus

$$|g_1(z)| = |g_1(z) - \lambda f o \varphi_n(z) + \lambda f o \varphi_n(z)| \leqslant \varepsilon + \varepsilon C,$$

hence $g \equiv g(p)$, but $z \in H$, and C_φ cannot be weakly supercyclic. Now, Suppose φ is an elliptic automorphism. We can assume $p = 0$, then $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Thus, $\{\varphi_n(z) : n \in \mathbb{N} \cup \{0\}\} \subseteq z\partial\mathbb{D}$. $z\partial\mathbb{D}$ is a compact subset of $\mathbb{D}$ and continuity of f implies that, $\{f(\varphi_n(z))\}$ is a bounded sequence. Since $z \in H, C_\varphi z = \varphi \in H$. Similarly, since $U_{n,\varphi}$ is a weak neighborhood of φ, we can choose subsequence $\{n_k\}$ and $\{\lambda_k\}$ such that $\lim_{k\to\infty} \lambda_k f(\varphi_{n_k}(z) = \varphi(z)$ and $\lim_{k\to\infty} \lambda_k = 0$. So we must have $\varphi \equiv 0$, which is a contradiction. $\square$

If the generating function in a weighted Hardy space is continuous on the closed unit disk, all functions in $\mathcal{H}^2(\beta)$ can be extended continuously to the closed disk, (see [4, page 28]). For example if $\Sigma_{j=0}^{\infty}\frac{1}{\beta(j)^2}<\infty$, all functions in $\mathcal{H}^2(\beta)$ are continuous on the closed unit disk. In addition, if $\Sigma_{j=0}^{\infty}\frac{1}{\beta(j)^2}<\infty$, then for all $p\in\partial\mathbb{D}$, the function $K_p(z)=k(\bar{p}z)=\Sigma_{j=0}^{\infty}\frac{(\bar{p}z)^j}{\beta(j)^2}$ is in $\mathcal{H}^2(\beta)$ and for all $f\in\mathcal{H}^2(\beta)$, $<f,K_p>=f(p)$.

Theorem 2.3. *Let $\mathcal{H}^2(\beta)$ be a weighted Hardy space with weight sequence $\{\beta_j\}_j$ such that $\sum_{j=1}^{\infty}\frac{1}{\beta(j)^2}<\infty$. If φ is a self map of $\mathbb{D}$ with boundary fixed point p, and C_φ is a bounded composition operator on $\mathcal{H}^2(\beta)$, then λC_φ is not weakly hypercyclic for every $\lambda\in\mathbb{C}$.*

Proof. Since $\sum_{j=1}^{\infty}\frac{1}{\beta(j)^2}<\infty$, the reproducing kernel at p, $K_p(z)=\sum_{j=1}^{\infty}\frac{\bar{p}^j z^j}{\beta(j)^2}$, is in $\mathcal{H}^2(\beta)$. For any $f\in\mathcal{H}^2(\beta)$ we have:

$$<\bar{\lambda}C_\varphi{}^*K_p,f>=\bar{\lambda}<K_p,f\circ\varphi>=\bar{\lambda}f\circ\varphi(p)=\bar{\lambda}f(p)=<\bar{\lambda}K_p,f>.$$

Thus K_p is an eigenvector of $\bar{\lambda}C_\varphi{}^*$. Since the adjoint of weakly hypercyclic operator has empty point spectrum, it follows that λC_φ cannot be weakly hypercyclic for any λ. □

The following theorem is a consequence of two preceding theorems:

Theorem 2.4. *Let $\mathcal{H}^2(\beta)$ be a weighted Hardy space with weight sequence $\{\beta_j\}_j$ such that $\sum_{j=1}^{\infty}\frac{1}{\beta(j)^2}<\infty$. If C_φ is bounded composition operator on $\mathcal{H}^2(\beta)$, then λC_φ is never weakly hypercyclic for every λ and φ.*

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