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Pullback Diagram of Hilbert Modules over Locally C^*-Algebras

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Abstract. In this paper, we generalize the construction of a pullback diagram in the framework of Hilbert modules over locally C^*-algebras and we explore some conditions under which diagrams of Hilbert modules over the corresponding C^*-algebras are pullback.

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1. Introduction

Pedersen [7] introduced the notion of a pullback diagram in the category of C^*-algebras and investigated some properties of these diagrams. Amyari and Chakoshi [1] generalized the construction of a pullback diagram in the framework of Hilbert C^*-modules [6]. Some properties of pullback diagrams are stable under locally C^*-algebras [1, 2, 7]. In this paper we use these properties to discover new ones for pullback diagrams of locally C^*-algebras. Locally C^*-algebras were systemically studied by Inove, N. C. Phillips (under the name of pro-C^*-algebras [8]), M. Fragoulopoulo and other mathematicians. A locally C^*-algebra is a complete Hausdorff topological involutive algebra, whose topology is determined by a directed family of C^*-seminorms in the sense that

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the net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous C^*-seminorm p. A Frechet locally C^*-algebra is a locally C^*-algebra by countable family of C^*-seminorms. Clearly, each C^*-algebra is a Frechet locally C^*-algebra. Let A be a locally C^*-algebra and $S(A)$ be the set of all continuous C^*-seminorms. For $p \in S(A)$, $A_p = \frac{A}{\ker(p)}$ is a C^*-algebra with the norm induced by p [8]. The canonical map from A onto A_p, $p \in S(A)$, will be denoted by π_p.
Hilbert modules over locally C^*-algebras generalize the notion of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra. Recall that a pre-Hilbert A-module is a complex vector space X which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle .,. \rangle : X \times X \to A$ which is $\mathbb{C}$-linear and A-linear in its second variable and satisfies the following relations
(i) $\langle x, y\rangle^* = \langle y, x\rangle$ for every $x, y \in X$;
(ii) $\langle x, x\rangle \geqslant 0$ for every $x \in X$;
(iii) $\langle x, x\rangle = 0$ if and only if $x = 0$.
We say that X is a Hilbert A-module if X is complete with respect to the topology determined by the family of seminorms $\{\bar{p}\}_{p\in S(A)}$, where $\bar{p}(x) = \sqrt{p(\langle x, x\rangle)}$, $x \in X$, $p \in S(A)$. Let X be a Hilbert A-module, then for $p \in S(A)$, $\ker(\overline{p}) = \{x \in X : p(\langle x, x\rangle) = 0\}$ is a closed submodule of X and $X_p = \frac{X}{\ker(\overline{p})}$ is a Hilbert A_p-module with $(x + \ker(\overline{p}))\pi_p(a) = xa + \ker(\overline{p})$ and $\langle x + \ker(\overline{p}), y + \ker(\overline{p})\rangle = \pi_p(\langle x, y\rangle)$. The canonical map from X onto X_p, $p \in S(A)$, will be denoted by σ_p.
The $* - ideal$ of A generated by $\{\langle x, y\rangle,\ x, y \in X\}$ is denoted by $\langle X, X\rangle$. We say that X is full if the closure of $\langle X, X\rangle$ is the whole of A.
Let $\varphi : A \to B$ be a morphism ($* - homomorphism$) of locally C^*-algebras. A morphism $\Phi : X \to Y$ is said to be a φ-morphism of Hilbert modules over locally C^*-algebras if $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ for all x, y in X. Using polarization identity, one immediately concludes that Φ is a φ-morphism if and only if $\langle \Phi(x), \Phi(x)\rangle = \varphi(\langle x, x\rangle)$ for each $x \in X$. It is easy to see that each φ-morphism is necessarily a linear operator and a module mapping in the sense that $\Phi(xa) = \Phi(x)\varphi(a)$ for all $x \in X, a \in A$. For an account on morphisms the reader is referred to the [3, 4].

We recall from [5] that if A is a locally C^*-algebra, then $b(A)$ (bounded part of A) that is the set of all bounded elements a in A such that $\|a\|_\infty = \sup\{p(a) : p \in S(A)\} < \infty$, together with $\|.\|_\infty$ is a C^*-algebra. Also if φ is a morphism from locally C^*-algebra A into locally C^*-algebra B, then φ maps $b(A)$ into $b(B)$.
In this paper the pullback construction is applied to the class of locally C^*-algebras and to the class of Hilbert modules over locally C^*-algebras. Also consider a pullback diagram of Hilbert modules over locally C^*-algebras. We

explore some conditions under which diagrams of Hilbert modules over corresponding C^*-algebras are pullback.

2. Pullback Construction in Hilbert Modules over Locally C^*-Algebras

In this section we introduce a pullback diagram of locally C^*-algebras and investigate some properties of them. For this we need the following definition.

Definition 2.1. *A commutative diagram of locally C^*-algebras and morphisms*

$$\begin{tikzcd} A_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & B_1 \arrow[d, "\psi_2"] \\ A_2 \arrow[r, "\varphi_2"] & B_2 \end{tikzcd}$$

is pullback if $\ker(\varphi_1) \cap \ker(\psi_1) = \{0\}$ *and for every other pair of morphisms* $\mu_1 : A \to B_1$ *and* $\mu_2 : A \to A_2$*, from a locally C^*-algebra A that satisfy* $\psi_2\mu_1 = \varphi_2\mu_2$ *there is a unique morphism* $\mu : A \to A_1$ *such that* $\mu_1 = \varphi_1\mu$ *and* $\mu_2 = \psi_1\mu$ *.*

$$\begin{tikzcd} A \arrow[drr, bend left, "\mu_1"] \arrow[ddr, bend right, "\mu_2"'] \arrow[dr, dotted, "\mu"] & & \\ & A_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & B_1 \arrow[d, "\psi_2"] \\ & A_2 \arrow[r, "\varphi_2"'] & B_2 \end{tikzcd}$$

The following proposition is proved in framework of C^*-algebras. It is easy to show that this proposition holds in category of locally C^*-algebras.

Proposition 2.2. (see [7, *Proposition* 3.1.]) *A commutative diagram of locally C^*-algebras*

$$\begin{tikzcd} A_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & B_1 \arrow[d, "\psi_2"] \\ A_2 \arrow[r, "\varphi_2"] & B_2 \end{tikzcd} \qquad (1)$$

is pullback if and only if the following conditions hold.
$(i)\ker(\varphi_1)\cap\ker(\psi_1)=\{0\}$,
$(ii){\psi_2}^{-1}(\varphi_2(A_2))=\varphi_1(A_1)$,
$(iii)\psi_1(\ker(\varphi_1))=\ker(\varphi_2)$.

Theorem 2.3. *Let* (1) *be a commutative diagram of locally C^*-algebras and morphisms. Then ψ_1 is surjective and ψ_2 is injective if and only if the following conditions hold.*
$(i){\psi_1}^{-1}(\ker\varphi_2)=\ker\varphi_1$,
$(ii)\psi_1(\ker\varphi_1)=\ker\varphi_2$,
$(iii){\psi_2}^{-1}(\varphi_2(A_2))=\varphi_1(A_1)$.

Proof. Suppose that ψ_1 and ψ_2 are surjective and injective respectively. Let $a_1\in\ker\varphi_1$. Then $\psi_2(\varphi_1(a_1))=0$ and by commutativity of the diagram $a_1\in{\psi_1}^{-1}(\ker\varphi_2)$ which implies that $\ker\varphi_1\subseteq\psi_1^{-1}(\ker\varphi_2)$.
Now let $a_2\in\ker\varphi_2$ and $\psi_1^{-1}(a_2)$ is not in $\ker\varphi_1$. Hence $\varphi_1(\psi_1^{-1}(a_2))\neq 0$ and consequently it is not in $\ker\psi_2$, since ψ_2 is an injection. On the other hand $\psi_2(\varphi_1\psi_1^{-1}(a_2))=\varphi_2\psi_1(\psi_1^{-1}(a_2))=\varphi_2(a_2)=0$, that is a contradiction. This proves (i). We obtain condition (ii) by surjectivity of ψ_1 and part (i).
It remains to verify (iii). For this, let $b_1\in\varphi_1(A_1)$. So $b_1=\varphi_1(a_1)$ for some a_1 in A_1. Put $a_2=\psi_1(a_1)$. By commutativity of the diagram (1) and injectivity of ψ_2 we have $\psi_2^{-1}\varphi_2(a_2)=\psi_2^{-1}\varphi_2(\psi_1(a_1))=\psi_2^{-1}\psi_2(\varphi_1(a_1))=\varphi_1(a_1)=b_1$. Therefore $b_1\in\psi_2^{-1}\varphi_2(A_2)$. Now suppose that $b_1\in\psi_2^{-1}\varphi_2(A_2)$. Then there exists $a_2\in A_2$ such that $b_1=\psi_2^{-1}\varphi_2(a_2)$ and by surjectivity of ψ_1 there exists $a_1\in A_1$ such that $\psi_1(a_1)=a_2$. Thus $b_1=\psi_2^{-1}\varphi_2(\psi_1(a_1))=\psi_2^{-1}\psi_2(\varphi_1(a_1))=\varphi_1(a_1)$ and therefore $b_1\in\varphi_1(A_1)$. Conversely suppose that (i), (ii) and (iii) hold. First we show that ψ_1 is surjective. If on the contrary, there exists $a_2\in A_2$ in which $a_2\neq\psi_1(a_1)$ for each a_1 in A_1, then by (ii), a_2 is not in $\ker\varphi_2$. By (iii) there exists $\acute{a_1}\in A_1$ such that $\psi_2^{-1}\varphi_2(a_2)=\varphi_1(\acute{a_1})$ and $\psi_2\psi_2^{-1}(\varphi_2(a_2))=\psi_2\varphi_1(\acute{a_1})=\varphi_2\psi_1(\acute{a_1})$. Therefore $\varphi_2(a_2)=\varphi_2(\psi_1(\acute{a_1}))$ and $a_2-\psi_1(\acute{a_1})\in\ker\varphi_2$. By (ii) there exists a_1'' in $\ker\varphi_1$ such that $a_2-\psi_1(\acute{a_1})=\psi_1(a_1'')$ and so $a_2=\psi_1(\acute{a_1}+a_1'')$. This contradicts initial assumption. Hence ψ_1 is surjective.
To prove injectivity of ψ_2 , suppose that, $\psi_2(b_1)=0$ for some nonzero $b_1\in B_1$. Then $b_1\in\psi_2^{-1}(\{0\})$ and by (iii) there exist $a_2\in\ker\varphi_2$ and $a_1\in A_1$ such that $b_1=\psi_2^{-1}(\varphi_2(a_2))=\varphi_1(a_1)$. Hence a_1 is not in $\ker\varphi_1$ $(*)$. On the other hand, $\varphi_2\psi_1(a_1)=\psi_2\varphi_1(a_1)=\psi_2(b_1)=0$. Hence $\psi_1(a_1)\in\ker\varphi_2$. Now since $a_1\in\psi_1^{-1}\psi_1(\{a_1\})$, then it is in $\psi_1^{-1}(\ker\varphi_2)$, that is equal to $\ker\varphi_1$ by (i). Thus $a_1\in\ker\varphi_1$. That is in contradiction with relation $(*)$. Then $\ker\psi_2=\{0\}$. $\square$
As an immediate consequence of the above theorem we have the following corollary, which is a new criterion for the pullbackness of a commutative diagram of locally C^*-algebras.

Corollary 2.4. *Let* (1) *be a commutative diagram of locally C^*-algebras and mophisms. If* $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$ *and* ψ_1, ψ_2 *are surjective and injective respectively, then the diagram is pullback and condition* (i) *of Theorem* 2.3. *holds. Conversely, if the diagram is pullback, then* ψ_1 *is surjective and moreover if condition* (i) *holds, then* ψ_2 *is injective.*

Example 2.5. The $*$-algebra $c(X)$ of all complex valued continuous functions on a non empty set X, together with C^*-seminorms, $p_n(f) = \sup\{|f(x)| : x \in k_n\}$ ($n \in \mathbb{N}$, $f \in c(X)$ and $\{k_n\}_{n\in\mathbb{N}}$ is a countable compact subsets of X) is a locally C^*-algebra.
Now let $\{k'_n\}_{n\in\mathbb{N}}$, be a countable compact subintervals of $[0,1]$. Then $c([0,1])$, $c([0,2])$ and $c([0,6])$ are locally C^*-algebras, where $\{k'_n\}_{n\in\mathbb{N}}$, $\{2k'_n\}_{n\in\mathbb{N}}$ and $\{6k'_n\}_{n\in\mathbb{N}}$ are the countable compact subintervals of $[0,1]$, $[0,2]$ and $[0,6]$ respectively. Consider the following commutative diagram of locally C^*-algebras and morphisms

$$\begin{array}{ccc}
c([0,1]) & \xrightarrow{\varphi_1} & c([0,1]) \\
\Big\downarrow{\psi_1} & & \Big\downarrow{\psi_2} \\
c([0,2]) & \xrightarrow{\varphi_2} & c([0,6])
\end{array}$$

where $\varphi_1 = I$ (identity operator), $\varphi_2(f)(x) = f(3x)$, $\psi_1(g)(x) = g(2x)$ and $\psi_2(h)(x) = h(6x)$ for all $f \in c([0,2])$ and for all $g, h \in c([0,1])$. Clearly, ψ_1 and ψ_2 are surjective and injective respectively and $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$. Using Corollary 2.4. the diagram is pullback.

Example 2.6. Let $\{(X_\alpha, \|.\|_\alpha)\}_{\alpha\in\Lambda}$ be a family of C^*-algebras. Then $\Pi_{\alpha\in\Lambda}X_\alpha$ is a locally C^*-algebra, where for each $\alpha'_0 \in \Lambda$, seminorm $p_{\alpha'_0} \in S(\Pi_{\alpha\in\Lambda}X_\alpha)$ is defined by $p_{\alpha'_0}(\{x_\alpha\}_{\alpha\in\Lambda}) = \|x_{\alpha'_0}\|_{\alpha'_0}$. Now let $\Lambda \subset \Lambda'$ and $\alpha_0 \in \Lambda$ be fixed. Consider the following commutative diagram of the locally C^*-algebras and morphisms

$$\begin{array}{ccc}
\Pi_{\alpha\in\Lambda}X_\alpha & \xrightarrow{\varphi_1} & \Pi_{\alpha\in\Lambda'}X_\alpha \\
\Big\downarrow{\psi_1} & & \Big\downarrow{\psi_2} \\
\Pi_{\alpha\in\Lambda'}X_\alpha & \xrightarrow{\varphi_2} & \Pi_{\alpha\in\Lambda'}X_\alpha
\end{array}$$

where the morphisms $\psi_1, \psi_2, \varphi_1$ and φ_2 are defined as follows.
(i) $\psi_1(\{x_\alpha\}_{\alpha\in\Lambda}) = \{y_\alpha\}_{\alpha\in\Lambda'}$, such that $y_\alpha = x_\alpha$ for each $\alpha \in \Lambda$ and $y_\alpha = 0$ for

each $\alpha \in \Lambda' - \Lambda$.
(ii) $\psi_2 = I$(identity operator).
(iii) $\varphi_1(\{x_\alpha\}_{\alpha\in\Lambda}) = \{z_\alpha\}_{\alpha\in\Lambda'}$, such that $z_\alpha = x_\alpha$ for each $\alpha \in \Lambda - \{\alpha_0\}$ and $z_\alpha = 0$ for each $\alpha \in (\Lambda' - \Lambda) \cup \{\alpha_0\}$.
(iv) $\varphi_2(\{x_\alpha\}_{\alpha\in\Lambda'}) = \{w_\alpha\}_{\alpha\in\Lambda'}$ such that $w_\alpha = x_\alpha$ for each $\alpha \in \Lambda - \{\alpha_0\}$ and $w_\alpha = 0$ for each $\alpha \in (\Lambda' - \Lambda) \cup \{\alpha_0\}$.
Clearly, ψ_1 is not surjective and $\ker\varphi_1 = \{\{x_\alpha\}_{\alpha\in\Lambda} : x_\alpha = 0 \text{ for every } \alpha \in \Lambda - \{\alpha_0\}\}$. Hence $\psi_1(\ker\varphi_1) = \{\{x_\alpha\}_{\alpha\in\Lambda'} : x_\alpha = 0 \text{ for every } \alpha \in \Lambda' - \{\alpha_0\}\}$. On the other hand $\ker\varphi_2 = \{\{x_\alpha\}_{\alpha\in\Lambda'} : x_\alpha = 0 \text{ for every } \alpha \in \Lambda - \{\alpha_0\}\}$. Then $\psi_1(\ker\varphi_1) \neq \ker\varphi_2$ and so the above diagram is not pullback by Proposition 2.2. More precisely, we show that if ψ_1 is not surjective, then the diagram is not pullback.
In the following we obtain some conditions under which the diagram of bounded part of locally C^*-algebras of a pullback diagram, is pullback.

Corollary 2.7. *Suppose that the following left diagram is a pullback diagram of locally C^*-algebras.*

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array} \qquad\qquad \begin{array}{ccc} b(A_1) & \xrightarrow{\acute{\varphi_1}} & b(B_1) \\ \downarrow{\scriptstyle \acute{\psi_1}} & & \downarrow{\scriptstyle \acute{\psi_2}} \\ b(A_2) & \xrightarrow{\acute{\varphi_2}} & b(B_2) \end{array}$$

If ψ_2 is injective, then the right diagram is pullback, where for $i = 1, 2$ $\acute{\varphi_i} = \varphi_i|_{b(A_i)}$ and $\acute{\psi_1} = \psi_1|_{b(A_1)}$ and $\acute{\psi_2} = \psi_2|_{b(B_1)}$.

Proof. By the preceding corollary, ψ_1 is surjective. Also ψ_1 is injective. Indeed if $\psi_1(a_1) = 0$, then by commutativity of the diagram, $\psi_2\varphi_1(a_1) = \varphi_2\psi_1(a_1) = 0$ and by injectivity of ψ_2 we have $\varphi_1(a_1) = 0$. This implies that $a_1 \in \ker\varphi_1 \cap \ker\psi_1$ and since the left diagram is pullback so $a_1 = 0$. From injectivity of ψ_1 and ψ_2 we conclude that ψ_1' and ψ_2' are injective and $\ker\varphi_1' \cap \ker\psi_1' = \{0\}$.
Using the preceding corollary, to prove pullbackness of the right diagram it is enough to show that ψ_1' is surjective. Suppose that a_2 is an arbitrary element in $b(A_2)$. By surjectivity of ψ_1 there exists $a_1 \in A_1$ such that $\psi_1(a_1) = a_2$. Furthermore, the norm reducing of ψ_1^{-1} implies that $\|a_1\|_\infty = \|\psi_1^{-1}\psi_1(a_1)\|_\infty \leqslant \|\psi_1(a_1)\|_\infty = \|a_2\|_\infty < \infty$. It implies that, $a_1 \in b(A_1)$ and ψ_1' is surjective.
Our aim now is to generalize the pullback construction in the category of Hilbert modules over locally C^*-algebras. For this we need the following definition. □

Definition 2.8. *A commutative diagram of Hilbert modules over locally C^*-*

algebras

$$\begin{tikzcd} X_1 \arrow[r, "\Phi_1"] \arrow[d, "\Psi_1"] & Y_1 \arrow[d, "\Psi_2"] \\ X_2 \arrow[r, "\Phi_2"] & Y_2 \end{tikzcd}$$

is pullback if $\ker(\Phi_1) \cap \ker(\Psi_1) = \{0\}$ *and for every other pair of morphisms* $\mu_1 : X \to Y_1$ *and* $\mu_2 : X \to X_2$ *from a full Hilbert module over locally* C^*-*algebra* X*, that satisfy condition* $\Psi_2\mu_1 = \Phi_2\mu_2$*, there exists a unique morphism* $\mu : X \to X_1$ *such that* $\mu_1 = \Phi_1\mu$ *and* $\mu_2 = \Psi_1\mu$*.*

$$\begin{tikzcd} X \arrow[drr, bend left, "\mu_1"] \arrow[ddr, bend right, "\mu_2"'] \arrow[dr, dotted, "\mu"] & & \\ & X_1 \arrow[r, "\Phi_1"] \arrow[d, "\Psi_1"] & Y_1 \arrow[d, "\Psi_2"] \\ & X_2 \arrow[r, "\Phi_2"'] & Y_2 \end{tikzcd}$$

The following proposition is proved in the category of Hilbert C^**-modules. It is easy to show that this proposition holds in framework of Hilbert modules over locally* C^**-algebras.*

Proposition 2.9. (see [1, Theorem 3]). *A commutative diagram of full Hilbert modules* X_1 *and* X_2 *and arbitrary Hilbert modules* Y_1 *and* Y_2 *over locally* C^*-*algebras* A_1, A_2, B_1 *and* B_2 *respectively, in which the corresponding map* φ_1 *to* Φ_1 *is surjective,*

$$\begin{tikzcd} X_1 \arrow[r, "\Phi_1"] \arrow[d, "\Psi_1"] & Y_1 \arrow[d, "\Psi_2"] \\ X_2 \arrow[r, "\Phi_2"] & Y_2 \end{tikzcd} \qquad (2)$$

is pullback if and only if the following conditions hold.
(*i*) $\ker(\Phi_1) \cap \ker(\Psi_1) = \{0\}$,
(*ii*) ${\Psi_2}^{-1}(\Phi_2(X_2)) = \Phi_1(X_1)$,
(*iii*) $\Psi_1(\ker(\Phi_1)) = \ker(\Phi_2)$.

Applying Proposition 2.9. we conclude that Theorem 2.3. holds in the category of Hilbert modules over locally C^*-algebras and so we have the following corollary, which is a new criterion for the pullbackness of a commutative diagram of the Hilbert modules over locally C^*-algebras.

Corollary 2.10. *Consider the commutative diagram* (2) *in Proposition 2.9. in which the corresponding map* φ_1 *of* Φ_1 *is surjective. If* $\ker\Phi_1 \cap \ker\Psi_1 = \{0\}$ *and* Ψ_1, Ψ_2 *are surjective and injective respectively, then the diagram is pullback and* $\Psi_2^{-1}(\ker\Phi_2) = \ker\Phi_1$ $(*)$. *Conversely, if the diagram is pullback, then* Ψ_1 *is surjective. Moreover, if the condition* $(*)$ *holds, then* Ψ_2 *is injective.*
We recall from [5] that if X is a (full) Hilbert module over locally C^*-algebra A, then $b(X)$ that is the set of all x in X such that $\|x\|_\infty = \sup\{\overline{p}_X(x) : p \in S(A)\} < \infty$, is a (full) $b(A) - Hilbert$ module. Now suppose that X_1 and X_2 are Hilbert modules over locally C^*-algebras A and B, respectively and Φ is a φ-morphism from X_1 into X_2. Then Φ maps $b(X_1)$ into $b(X_2)$. Indeed, let $x \in b(X_1)$. So $\langle x, x\rangle \in b(A)$ and by norm-reducing of φ on $b(A)$ we have

$$\begin{aligned}
\|\Phi(x)\|_\infty &= \sup\{\overline{p}_{b(X_2)}(\Phi(x)) \ : p \in S(B)\} \\
&= \sup\{\sqrt{p(\varphi(\langle x, x\rangle))} \ : p \in S(B)\} \\
&= \sqrt{\|\varphi(\langle x, x\rangle)\|_\infty} \\
&\leqslant \|x\|_\infty .
\end{aligned}$$

Replace Φ by Φ'. One can verify that, Φ' is a φ'-morphism, where φ' is the restriction of φ to $b(A)$.

Corollary 2.11. *Suppose that* X_1 *and* X_2 *are full Hilbert modules and* Y_1 *and* Y_2 *are arbitrary Hilbert modules over Frechet locally* C^**-algebras* A_1, A_2, B_1 *and* B_2, *respectively. Also suppose that the following left diagram (and hence its corresponding diagram of underlying locally* C^**-algebras (right)) is pullback.*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array} \qquad\qquad \begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array} \qquad (3)$$

If φ_1 *is bijective and* ψ_2 *is injective, then the following left diagram (and hence its corresponding diagram of underlying* C^**-algebras (right)) is pullback.*

$$\begin{array}{ccc} b(X_1) & \xrightarrow{\acute{\Phi}_1} & b(Y_1) \\ \downarrow{\scriptstyle \acute{\Psi}_1} & & \downarrow{\scriptstyle \acute{\Psi}_2} \\ b(X_2) & \xrightarrow{\acute{\Phi}_2} & b(Y_2) \end{array} \qquad\qquad \begin{array}{ccc} b(A_1) & \xrightarrow{\acute{\varphi}_1} & b(B_1) \\ \downarrow{\scriptstyle \acute{\psi}_1} & & \downarrow{\scriptstyle \acute{\psi}_2} \\ b(A_2) & \xrightarrow{\acute{\varphi}_2} & b(B_2) \end{array}$$

Proof. Let $b_1 \in b(B_1)$ be arbitrary. Since φ_1 is surjective, then $b_1 = \varphi_1(a_1)$ for some $a_1 \in A_1$. We have $\|a_1\|_\infty = \|{\varphi_1}^{-1}(\varphi_1(a_1))\|_\infty \leqslant \|b_1\|_\infty < \infty$. Hence $\acute{\varphi}_1$ is surjective. Clearly, the Hilbert C^*-modules $b(X_1)$ and $b(X_2)$ are full, since X_1 and X_2 are full. Using the same technique as Corollary 2.7. and the preceding corollary we get pullbackness of the left diagram.
In the following we explore some conditions under which diagrams of Hilbert modules over the corresponding C^*-algebras of the pullback diagram (3), are pullback. $\square$

Theorem 2.12. *Let* (3) *be a pullback diagram. If for each C^*-seminorm $p_{A_1} \in S(A_1)$ there exist C^*-seminorms $p_{B_1} \in S(B_1)$, $p_{A_2} \in S(A_2)$ and $p_{B_2} \in S(B_2)$ in which satisfy the following conditions*
(i) ${\varphi_1}^{-1}(\ker p_{B_1}) \cap {\psi_1}^{-1}(\ker p_{A_2}) = \ker p_{A_1}$,
(ii) $\varphi_2(\ker p_{A_2}) \subseteq \ker p_{B_2}$,
(iii) $\psi_2(\ker p_{B_1}) \subseteq \ker p_{B_2}$,
and $\acute{\Psi}_2 : \frac{Y_1}{\ker \overline{p}_{B_1}} \to \frac{Y_2}{\ker \overline{p}_{B_2}}$ *is defined by* $\acute{\Psi}_2(y_1 + \ker \overline{p}_{B_1}) = \Psi_2(y_1) + \ker \overline{p}_{B_2}$ *is injective, then the following diagram of Hilbert C^*-modules is pullback, where* $\acute{\Phi}_1(x_1 + \ker \overline{p}_{A_1}) = \Phi_1(x_1) + \ker \overline{p}_{B_1}$ *and* $\acute{\Phi}_2, \acute{\Psi}_1$ *are defined in a similar way.*

$$\begin{array}{ccc} \frac{X_1}{\ker \overline{p}_{A_1}} & \xrightarrow{\acute{\Phi}_1} & \frac{Y_1}{\ker \overline{p}_{B_1}} \\ \Big\downarrow \acute{\Psi}_1 & & \Big\downarrow \acute{\Psi}_2 \\ \frac{X_2}{\ker \overline{p}_{A_2}} & \xrightarrow{\acute{\Phi}_2} & \frac{Y_2}{\ker \overline{p}_{B_2}} \end{array} \qquad (4)$$

Proof. By the assumption $\acute{\Phi}_1$, $\acute{\Phi}_2$, $\acute{\Psi}_1$ and $\acute{\Psi}_2$ are well defined morphisms. For this, assume that $x_1 + \ker \overline{p}_{A_1} = \acute{x}_1 + \ker \overline{p}_{A_1}$. Then $\langle x_1 - \acute{x}_1, x_1 - \acute{x}_1\rangle \in \ker p_{A_1}$ and by (i), $\varphi_1(\langle x_1 - \acute{x}_1, x_1 - \acute{x}_1\rangle) \in \varphi_1(\ker p_{A_1}) \subseteq \ker p_{B_1}$. Hence $\overline{p}_{B_1}(\Phi_1(x_1) - \Phi_1(\acute{x}_1)) = 0$. So $\acute{\Phi}_1(x_1 + \ker \overline{p}_{A_1}) = \acute{\Phi}_1(\acute{x}_1 + \ker \overline{p}_{A_1})$.
Similarly, other morphisms are well defined. One should note that $\frac{X_1}{\ker \overline{p}_{A_1}}, \frac{X_2}{\ker \overline{p}_{A_2}}$ are full Hilbert C^*-modules, by fullness of X_1 and X_2 [4]. Also the diagram of the Hilbert C^*-modules (4) is commutative, since the initial diagram of Hilbert modules over locally C^*-algebras is commutative. Since the left diagram is pullback, so it satisfies in the conditions of Proposition 2.9. i.e.
(a) $\ker(\Phi_1) \cap \ker(\Psi_1) = \{0\}$,
(b) ${\Psi_2}^{-1}(\Phi_2(X_2)) = \Phi_1(X_1)$,
(c) $\Psi_1(\ker(\Phi_1)) = \ker \Phi_2$.
We prove that (4) satisfies in the conditions of Theorem 3 of [1].
(**I**) Let $x_1 + \ker \overline{p}_{A_1} \in \ker \acute{\Phi}_1 \cap \ker \acute{\Psi}_1$.
Then $\Phi_1(x_1) \in \ker \overline{p}_{B_1}$ and $\Psi_1(x_1) \in \ker \overline{p}_{A_2}$. It implies that $p_{B_1}(\varphi_1\langle x_1, x_1\rangle) =$

0 and $p_{A_2}(\psi_1\langle x_1, x_1\rangle) = 0$. Hence by (i), $\langle x_1, x_1\rangle \in \ker p_{A_1}$, whence $\overline{p}_{A_1}(x_1) = 0$ and $x_1 + \ker \overline{p}_{A_1} = 0$.

(**II**) Let $x_2 + \ker \overline{p}_{A_2} \in \frac{X_2}{\ker \overline{p}_{A_2}}$ be arbitrary. Since $\acute{\Psi}_2$ is injective and by (b) there exists $x_1 \in X_1$, in which

$$\begin{aligned}
\acute{\Psi}_2^{-1}(\acute{\Phi}_2(x_2 + \ker \overline{p}_{A_2})) &= \acute{\Psi}_2^{-1}(\Phi_2(x_2) + \ker \overline{p}_{B_2}) \\
&= \Psi_2^{-1}(\Phi_2(x_2)) + \ker \overline{p}_{B_1} \\
&= \acute{\Phi}_1(x_1 + \ker \overline{p}_{A_1}).
\end{aligned}$$

We obtain $\acute{\Psi}_2^{-1}(\acute{\Phi}_2(\frac{X_2}{\ker \overline{p}_{A_2}})) \subseteq \acute{\Phi}_1(\frac{X_1}{\ker \overline{p}_{A_1}})$. A similar argument proves the converse.

(**III**) Let $x_1 + \ker \overline{p}_{A_1}$ be an arbitrary element in $\ker \acute{\Phi}_1$. By the definition of $\acute{\Phi}_1$ we have $\Phi_1(x_1) \in \ker \overline{p}_{B_1}$, whence $\varphi_1(\langle x_1, x_1\rangle) \in \ker p_{B_1}$ and by (iii) $\psi_2\varphi_1(\langle x_1, x_1\rangle) \in \ker p_{B_2}$. Now by commutativity of the diagram we have $p_{B_2}(\varphi_2\psi_1(\langle x_1, x_1\rangle)) = 0$. Then $p_{B_2}(\langle \Phi_2\Psi_1(x_1), \Phi_2\Psi_1(x_1)\rangle) = 0$. We conclude that $\Phi_2\Psi_1(x_1) \in \ker \overline{p}_{B_2}$. Therefore $\acute{\Phi}_2(\acute{\Psi}_1(x_1 + \ker \overline{p}_{A_1})) = 0$ and $\acute{\Psi}_1(x_1 + \ker \overline{p}_{A_1}) \in \ker \acute{\Phi}_2$. Thus $\acute{\Psi}_1(\ker \acute{\Phi}_1) \subseteq \ker \acute{\Phi}_2$. Using the same technique as Theorem 2.3. one can verify that Ψ_1 is surjective. Now suppose that $x_2 + \ker \overline{p}_{A_2} \in \ker \acute{\Phi}_2$, then there exists $x_1 \in X_1$ such that $\Psi_1(x_1) = x_2$. Put $S := \{x_1 \in X_1 : \Psi_1(x_1) = x_2\}$. If for some $x_1 \in S$, $\Phi_1(x_1) \in \ker \overline{p}_{B_1}$, then $\acute{\Phi}_1(x_1 + \ker \overline{p}_{A_2}) = 0$. Hence $\acute{\Psi}_1(x_1 + \ker \overline{p}_{A_1}) = x_2 + \ker \overline{p}_{A_2}$ and $x_2 + \ker \overline{p}_{A_2} \in \acute{\Psi}_1(\ker \acute{\Phi}_1)$. Then there exists such an x_1 in S. On the contrary suppose that for each $x_1 \in S$, $\Phi_1(x_1)$ is not in $\ker \overline{p}_{B_1}$. Then by injectivity of $\acute{\Psi}_2$, we have $\acute{\Psi}_2\acute{\Phi}_1(x_1 + \ker \overline{p}_{A_1}) \neq 0$ for each $x_1 \in S$. Hence by commutativity of diagram $\acute{\Phi}_2\acute{\Psi}_1(x_1 + \ker \overline{p}_{A_1})$ is not equal to zero. On the other hand, $\acute{\Phi}_2\acute{\Psi}_1(x_1 + \ker \overline{p}_{A_1}) = \acute{\Phi}_2(x_2 + \ker \overline{p}_{A_2}) = 0$. That is a contradiction. Hence $\ker \acute{\Phi}_2 \subseteq \acute{\Psi}_1(\ker \acute{\Phi}_1)$. Now by [1, Theorem 3], the diagram is pullback. □

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