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Original Research Paper

On a Self-Adjoint Fractional Nabla Finite Difference Equation with Initial Value Condition

Sh. Dehghani
Azarbaijan Shahid Madani University

Sh. Rezapour*
Azarbaijan Shahid Madani University

Abstract. By using discrete fractional calculus, we investigate the existence of solutions for a self-adjoint finite Nabla fractional difference equation on the time scale $\mathbb{N}_{a+1}^{b}$ via initial boundary conditions. Also, we check some conditions for uniqueness of solution of the problem. For finding the solution, we use the Green function which defined by using the Cauchy function. The principle of contraction mapping also plays an essential role in the existence of the solution. We provided two examples, a figure and numerical results to illustrate our main result.

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1 Introduction

Today, the totalitarian of continuous fractional calculus is undeniable for almost all mathematics researchers due to its unparalleled capabilities in

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*Corresponding Author

various fields such as engineering, interpretation of physical phenomena, biological mathematics and modeling [1, 7, 9, 10, 12, 17, 18, 20, 21, 23, 24, 25, 26]. The history of this subject goes back to the logical dialectic between L'Hopital and Leibniz in 1695 on the half-order derivative. The reader can refer to [19] for more information on the history of fractional calculus. On the other hand, using the fractional calculus was not limited to the continuous case and some researchers have been studied discrete version. In 2020, Atici et al. modeled tumor growth by using fractional difference equations [8]. In this research, we intend to focus on discrete version.

There are generally two approaches on discrete fraction calculus; the Delta fractional calculus by using the forward difference operator and te Nabla fractional calculus by using the backward difference operator [6, 16]. In 1988, Gray et al. introduced a new definition of the fractional difference [15]. Later Atichi et al. [5] and Anastassiou [4] investigated some Nabla operators and inequalities. Over the past decade, by hard working of researchers in this field, fractional calculus found a relatively strong foundation. One can find basic notions and results of this field in the book of Goodrich and Peterson [13].

Recently, some researcher investigated the initial and boundary value problems by using the Nabla operators. In 2020, Ahrendt et al. [2] reviewed the self-adjoint fractional nabla difference equation

$$\nabla\Big[q(t+1)\nabla_a^{\mu} y(t+1)\Big] + p(t)y(t) = f(t), \quad t \in \mathbb{N}_{a+1}.$$

In 2021 Cabada et al. [11] investigated the nontrivial solutions for the following nabla fractional difference BVP

$$\begin{cases} -\Big(\nabla_a^{\mu-1}(\nabla v)\Big)(t) + f(t)v(t) = g(t, v(t)), \quad t \in \mathbb{N}_{a+2}^{b}, \\ \alpha v(a+1) - \beta(\nabla v)(a+1) = 0, \\ \lambda v(b) + \delta(\nabla v)(b) = 0. \end{cases}$$

Motivated by these works, we investigate the existence and uniqueness of the solution of the self-adjoint fractional nabla finite difference equation

$$\begin{cases} -\nabla_a^\ell[q\overline{\nabla}y + e^t\mathfrak{U}](t) = h(t+\ell-a+1), \quad t \in \mathbb{N}_{a+1}^b, \\ y(a) = 0, \\ y(b) = \frac{e^b}{2}\sum_{s=a+1}^b h(t+\ell-a+1)\mathcal{CF}(b,\rho(s)), \end{cases}$$

where $0 < \ell < 1$, $b - a \in \mathbb{N}_1$, $q(t) > 0$ and $\mathfrak{U}, q, h : \mathbb{N}_{a+1}^b \to \mathbb{R}$.

To achieve our goal, we use the Green function and the principle of contraction mapping in complete metric space. The continuation of the work is as follows: In the Section 2, we remind the concepts that we need from discrete calculus related to the Nabla operator. In the Section 3, we will prove our main results and, in the Section 4 we will provide two examples with numerical results to confirm our results.

2 Preliminaries

First, we need next notation to get started.

Remark 2.1. Let $a, b \in \mathbb{R}$ with $a < b$ and $b - a \in \mathbb{N}$. We consider the notations $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$ and $\mathbb{N}_a^b := \{a, a+1, a+2, \dots, b\}$.

Definition 2.2. [14] Let $\rho : \mathbb{N}_a \to \mathbb{N}_a$. The backward jump operator is defined by $\rho(t) = \max\{a, t-1\}$.

Definition 2.3. [14] The nabla difference operator $\overline{\nabla}$ on a function $\mathfrak{f} : \mathbb{N}_a \to \mathbb{R}$ is defined by $\overline{\nabla}\mathfrak{f}(s) = \mathfrak{f}(s) - \mathfrak{f}(s-1)$ for $s \in \mathbb{N}_{a+1}$.

Remark 2.4. In view of definition 2.3, we can consider the higher order difference $\overline{\nabla}^n\mathfrak{f}(s) := \overline{\nabla}(\overline{\nabla}^{n-1}\mathfrak{f}(s))$ for $s \in \mathbb{N}_{a+n}$. We accept that $\overline{\nabla}^0$ is the identity operator.

Definition 2.5. [14] For every real number k and natural number n, the rising function expressed by the formula

$$k^{\overline{n}} := k(k+1)(k+2)\dots(k+n-1).$$

Also by Nabla power rule, we have $\overline{\nabla}(k+\lambda)^{\overline{n}} = n(k+\lambda)^{\overline{n-1}}$.

Definition 2.6. [15] By using the Gamma function, we can generalize definition 2.5 for any real number r as $k^{\overline{r}} = \dfrac{\Gamma(k+r)}{\Gamma(k)}$, where we supposed that $k, k+r \notin \{0, -1, -2, \dots\}$.

Definition 2.7. Let $c, d \in \mathbb{N}_a$ and $\mathfrak{f} : \mathbb{N}_a \to \mathbb{R}$ be a function. Then the Nabla definite integral is defined by

$$\int_c^d \mathfrak{f}(t)\overline{\nabla}(t) := \begin{cases} \sum_{t=c+1}^{d} \mathfrak{f}(t), & c < d \\ 0, & c \geq d. \end{cases}$$

Definition 2.8. [3] Let $\ell > 0$ and $\mathfrak{f} : \mathbb{N}_{a+1} \to \mathbb{R}$ be a function. Then the Nabla ℓ-th order fractional sum (Nabla fractional sum) is defined by

$$\overline{\nabla}_a^{-\ell}\mathfrak{f}(x) = \int_a^x \frac{(x-\rho(s))^{\overline{\ell-1}}}{\Gamma(\ell)}\mathfrak{f}(s)\overline{\nabla}s, \quad (x \in \mathbb{N}_a).$$

Definition 2.9. [3] Let $\ell > 0$, $\mathfrak{f} : \mathbb{N}_{a+1} \to \mathbb{R}$ be a function and $M < \ell < M - 1$. Then the Nabla ℓ-th order fractional difference is defined by

$$\overline{\nabla}_a^{\ell}\mathfrak{f}(t) = \overline{\nabla}^M \overline{\nabla}^{-(M-\ell)}\mathfrak{f}(t), \quad (t \in \mathbb{N}_{a+M}).$$

Lemma 2.10. *[13] Let $\mathfrak{f}, \mathfrak{g} : \mathbb{N}_a \to \mathbb{R}$ be functions and $\ell \in (0,1)$. Then the fractional initial value problem*

$$\begin{cases} \overline{\nabla}_a^{\ell}\mathfrak{f}(t) = \mathfrak{g}(t), & t \in \mathbb{N}_{a+1}, \\ \mathfrak{f}(a+1) = \mathcal{A}, & \mathcal{A} \in \mathbb{R}, \end{cases}$$

has a unique solution expressed by

$$\mathfrak{f}(t) = \nabla_a^{-\ell}\mathfrak{g}(t) + \big(\mathcal{A} - \mathfrak{g}(a+1)\big)\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)}.$$

Definition 2.11. [13] Consider the following homogeneous fractional equation

$$\overline{\nabla}_a^{\ell}(q\overline{\nabla}y)(t) = 0$$

and a function $\mathcal{CF} : \mathbb{N}_{a+1} \times \mathbb{N}_{a+1} \longrightarrow \mathbb{R}$ which for any fixed $u \in \mathbb{N}_{a+1}$, we have $\mathcal{CF}(\cdot, \rho(u))$ be the unique solution of the following IVP

$$\begin{cases} \overline{\nabla}^{\ell}_{\rho(u)}(q\overline{\nabla}y)(t) = 0, \quad t \in \mathbb{N}_{u+1}, \\ \mathcal{CF}(\rho(u)) = 0, \ \overline{\nabla}\,\mathcal{CF}(u) = \frac{1}{\rho(u)}. \end{cases}$$

Then we called $\mathcal{CF}(t, \rho(u))$ the Cauchy function and it is expressed by

$$\mathcal{CF}(t, \rho(u)) = \sum_{\tau=u}^{t} \frac{(\tau - \rho(u))^{\overline{\ell-1}}}{\Gamma(\ell)\rho(\tau)}, \quad t \in \mathbb{N}_{a+1}. \tag{1}$$

Theorem 2.12. *[22] Assume that $(\aleph, \|\cdot\|)$ is a Banach space and $\mathsf{T} : \aleph \to \aleph$ a contraction map, that is, there is a$0 < \Theta < 1$ such that $\|\mathsf{T}(m) - \mathsf{T}(n)\| \le \Theta\|m - n\|$ for all $m, n \in \aleph$. Then, T has a unique fixed point n_* in $\aleph$.*

Remark 2.13. For $\kappa \in \mathbb{R}$, the space

$$\Psi := \{y : \mathbb{N}_a \to \mathbb{R} | \lim_{t\to\infty} y(t) = \kappa\}$$

with the supremum norm $\|y\| = \sup_{t\in\mathbb{N}_a} |y(t)|$ is a complete metric space.

3 Main Results

Theorem 3.1. *Consider the following fractional boundary value problem*

$$\begin{cases} -\nabla^{\ell}_{a}[q\overline{\nabla}y + e^t\mathfrak{U}](t) = h(t + \ell - a + 1), \quad t \in \mathbb{N}^b_{a+1}, \\ y(a) = 0, \\ y(b) = \frac{e^b}{2}\sum_{s=a+1}^{b} h(t + \ell - a + 1)\mathcal{CF}(b, \rho(s)), \end{cases}$$

where $0 < \ell < 1$, $b - a \in \mathbb{N}_1$, $q(t) > 0$ and $\mathfrak{U}, q, h : \mathbb{N}^b_{a+1} \to \mathbb{R}$ has the unique solution

$$y(t) = \int_a^b G(t, s)h(s + \ell - a + 1)\overline{\nabla}s + \int_a^b H(\tau)Z(t, \tau)\overline{\nabla}\tau$$

where

$$G(t,s)=\begin{cases}\frac{\mathcal{CF}(b,\rho(s))(1+\frac{e^b}{2})}{\mathcal{CF}(b,a)}\mathcal{CF}(t,a), & t\le s-1,\\ \frac{\mathcal{CF}(b,\rho(s))(1+\frac{e^b}{2})}{\mathcal{CF}(b,a)}\mathcal{CF}(t,a)-\mathcal{CF}(t,\rho(s)), & t\ge s,\end{cases}$$

and

$$Z(t,\tau)=\begin{cases}\mathcal{CF}(\tau,b), & t\le\tau-1,\\ \mathcal{CF}(\tau,b)-1, & t\ge\tau,\end{cases}$$

which $\mathcal{CF}(t,\rho(s))$ *is the Cauchy function* (1) *in Definition 2.11.*

Proof. Let

$$\mathcal{CF}(t)=q(t)\overline{\nabla}y(t)+e^t\mathfrak{U}(t), \tag{2}$$

and

$$A=q(a+1)\overline{\nabla}y(a+1)+e^{a+1}\mathfrak{U}(a+1).$$

Then $\mathcal{CF}(t)$ solves the following fractional IVP

$$\begin{cases}-\overline{\nabla}_a^{\ell}\mathcal{CF}(t)=h(t),\\ x(a+1)=A,\end{cases}$$

So in view of Lemma 2.10, we obtain

$$\mathcal{CF}(t)=-\overline{\nabla}_a^{-\ell}h(t+\ell-a+1)-(A-h(\ell+2))\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)}.$$

Letting $c_0=(A-h(a+1))$, from (2) we have

$$\overline{\nabla}y(t)=\frac{\mathcal{CF}(t)-e^t\mathfrak{U}(t)}{q(t)}.$$

So

$$\begin{aligned}\overline{\nabla}y(t)&=\frac{1}{q(t)}\left[-\overline{\nabla}_a^{-\ell}h(t+\ell-a+1)-c_0\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^t\mathfrak{U}(t)\right],\\ &=-\frac{1}{q(t)}\left[\frac{1}{\Gamma(\ell)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\ell-1}}h(s+\ell-a+1)+c_0\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)}+e^t\mathfrak{U}(t)\right],\end{aligned}$$

if we summing from $a+1$ to t,then

$$y(t) = -\sum_{\tau=a+1}^{t}\left[\sum_{s=a+1}^{\tau}\frac{(\tau-\rho(s))^{\ell-1}}{\Gamma(\ell)q(\tau)h(s+\ell-a+1)+c_0}\frac{(\tau-a)^{\overline{\ell-1}}}{\Gamma(\ell)q(\tau)}+e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)}\right].$$

by changing the order of sums,yield

$$\begin{aligned}
y(t) &= -\sum_{s=a+1}^{t} h(s+\ell-a+1)\sum_{\tau=s}^{t}\frac{(\tau-\rho(s))^{\overline{\ell-1}}}{\Gamma(\ell)q(\tau)} \\
&- c_0\sum_{\tau=a+1}^{t}\frac{(\tau-a)^{\overline{\ell-1}}}{\Gamma(\ell)q(\tau)} - \sum_{\tau=a+1}^{t} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)} \\
&= -\sum_{s=a+1}^{t} h(s+\ell-a+1)\mathcal{CF}(t,\rho(s)) - c_0\mathcal{CF}(t,a) - \sum_{\tau=a+1}^{t} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)}.
\end{aligned}$$

Putting $t=b$ and compute c_0, we have

$$(1+\frac{e^b}{2})\sum_{s=a+1}^{b} h(s+\ell-a+1)\mathcal{CF}(b,\rho(s)) = -c_0\mathcal{CF}(b,a) - \sum_{\tau=a+1}^{b} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)},$$

and so

$$c_0 = \frac{-(1+\frac{e^b}{2})\sum_{s=a+1}^{b} h(s+\ell-a+1)\mathcal{CF}(b,\rho(s)) - \sum_{\tau=a+1}^{b} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)}}{\mathcal{CF}(b,a)}. \quad (3)$$

With placement (3) in $y(t)$, we get

$$y(t) = -\sum_{s=a+1}^{t} h(s+\ell-a+1)\mathcal{CF}(t,\rho(s))$$
$$+\frac{\mathcal{CF}(t,a)}{\mathcal{CF}(b,a)}(1+\frac{e^b}{2})\sum_{s=a+1}^{b} h(s+\ell-a+1)\mathcal{CF}(b,\rho(s))$$
$$+\mathcal{CF}(t,a)\sum_{\tau=a+1}^{b} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)} - \sum_{\tau=a+1}^{t} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)} = -\sum_{s=a+1}^{t} h(s+\ell-a+1)\mathcal{CF}(t,\rho(s))$$
$$+\mathcal{CF}(t,a)x(b,a)(1+\frac{e^b}{2})\sum_{s=a+1}^{t} h(s+\ell-a+1)\mathcal{CF}(b,\rho(s))$$
$$+\frac{\mathcal{CF}(t,a)}{x(b,a)}(1+\frac{e^b}{2})\sum_{s=t+1}^{b} h(s+\ell-a+1)\mathcal{CF}(b,\rho(s)) + \mathcal{CF}(t,b)\sum_{\tau=a+1}^{t} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)}$$
$$+\mathcal{CF}(t,b)\sum_{\tau=t+1}^{b} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)} - \sum_{\tau=a+1}^{t} e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)}.$$

Therefore

$$y(t) = \sum_{s=a+1}^{t} h(s+\ell-a+1)\left[\frac{\mathcal{CF}(b,\rho(s))}{\mathcal{CF}(b,a)}\mathcal{CF}(t,a)(1+\frac{e^b}{2}) - \mathcal{CF}(t,\rho(s))\right]$$
$$+\sum_{\tau=a+1}^{t}(\mathcal{CF}(t,b)-1)e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)}$$
$$+\sum_{s=t+1}^{b} h(s+\ell-a+1)\left[\frac{\mathcal{CF}(b,\rho(s))}{\mathcal{CF}(b,a)}\mathcal{CF}(t,a)(1+\frac{e^b}{2})\right] + \sum_{\tau=t+1}^{b}\mathcal{CF}(\tau,b)e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)}$$
$$=\sum_{s=a+1}^{b} h(s+\ell-a+1)G(t,s) + \sum_{\tau=a+1}^{b} H(\tau)Z(t,\tau),$$

where $H(\tau) = e^{\tau}\frac{\mathfrak{U}(\tau)}{q(\tau)}$,

$$G(t,s) = \begin{cases} \frac{x(b,\rho(s))(1+\frac{e^b}{2})}{x(b,a)}\mathcal{CF}(t,a), & t \leq s-1, \\ \frac{x(b,\rho(s))(1+\frac{e^b}{2})}{x(b,a)}x(t,a) - x(t,\rho(s)), & t \geq s, \end{cases}$$

and

$$Z(t,\tau) = \begin{cases} \mathcal{CF}(\tau,b), & t \le \tau - 1, \\ \mathcal{CF}(\tau,b) - 1, & t \ge \tau. \end{cases}$$

□

The principle of Banach contraction is the main technique used in our next theorem. But to use it, we need to first prove the following lemma.

Lemma 3.2. *Suppose that $q : \mathbb{N}_{a+1}^{b} \to (0,\infty)$ and $h, \mathfrak{U} : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$. For some $\kappa \ge 0$, define $\Psi = \{y : \mathbb{N}_a \to \mathbb{R} \mid \lim_{t\to\infty} y(t) = \kappa\}$. Let $\forall y \in \Psi$, the series*

$$\sum_{s=a+1}^{\infty} \frac{1}{q(s)} \left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1) + \mathcal{Q}(S) \right],$$

be converges, where

$$\mathcal{Q}(S) = -(q(a+1)\overline{\nabla} y(a+1) + e^{(a+1)}\mathfrak{U}(a+1) - h(\ell+2))) \frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)} - e^{s}\mathfrak{U}(s).$$

Then the fractional finite self-adjoint type difference equation

$$-\overline{\nabla}_a^{\ell} \left[q\overline{\nabla} y + e^{t}\mathfrak{U} \right](t) = h(t+\ell-a+1), \tag{4}$$

has a solution $y \in \Psi$ if and only if the equation

$$y(t) = \kappa + \sum_{s=a+1}^{\infty} \frac{1}{q(s)} \left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1) + \mathcal{Q}(S) \right], \tag{5}$$

has a solution $y(t)$ on $\mathbb{N}_a^b$.

Proof. Let the difference equation (4) has a solution $y \in \Psi$ and $\mathfrak{x}(t) = q\overline{\nabla} y + e^{t}\mathfrak{U}(t)$. Then $\mathfrak{x}(t)$ solves the following fractional IVP

$$\begin{cases} -\overline{\nabla}_a^{\ell}\mathfrak{x}(t) = h(t+\ell-a+1), \\ \mathfrak{x}(a+1) = q(a+1)\overline{\nabla} y(a+1) + e^{(a+1)}\mathfrak{U}(a+1). \end{cases}$$

In view of lemma 2.10, we have

$$\mathfrak{x}(t) = -\overline{\nabla}_a^{-\ell} h(t+\ell-a+1) - (q(a+1)\overline{\nabla} y(a+1) + e^{(a+1)}\mathfrak{U}(a+1) - h(\ell+2))\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)}.$$

For simplification put $\mathfrak{A}^* = q(a+1)\overline{\nabla} y(a+1) + e^{(a+1)}\mathfrak{U}(a+1) - h(\ell+2))$, so by above equation we get that

$$\overline{\nabla} y(t) = \frac{1}{q(t)}\left[-\overline{\nabla}_a^{-\ell} h(t+\ell-a+1) - \mathfrak{A}^*\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)} - e^t\mathfrak{U}(t)\right]$$
$$= -\frac{1}{q(t)}\left[\sum_{\tau=a+1}^{t}\frac{(t-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(t+\ell-a+1) - \mathfrak{A}^*\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)} - e^t\mathfrak{U}(t)\right].$$

Therefore

$$\sum_{s=t+1}^{\infty}\overline{\nabla} y(t) = -\sum_{s=t+1}^{\infty}\frac{1}{q(t)}$$
$$\times\left[\sum_{\tau=a+1}^{s}\frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(s+\ell-a+1) - \mathfrak{A}^*\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)} - e^s\mathfrak{U}(s)\right],$$

by summing from $s = t+1$ to ∞. So

$$\lim_{s\to\infty} y(s) - y(t) = -\sum_{s=t+1}^{\infty}\frac{1}{q(s)}\left[\sum_{\tau=a+1}^{s}\frac{(t-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(s+\ell-a+1) + \mathcal{Q}(S)\right].$$

Then we obtain

$$y(t) = \kappa + \sum_{s=t+1}^{\infty}\frac{1}{q(t)}\left[\sum_{\tau=a+1}^{s}\frac{(t-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(s+\ell-a+1) + \mathcal{Q}(S)\right]. \tag{6}$$

Thus $y(t)$ is also a solution of (5). Conversely, we must show that if $y(t)$ is a solution of (5), then it is also a solution of (4) with $\lim_{t\to\infty} y(t) = \kappa$. For do this, we cam write

$$y(t) = \kappa + \sum_{s=t+1}^{\infty}\frac{1}{q(t)}\left[\sum_{\tau=a+1}^{s}\frac{(t-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(s+\ell-a+1) + \mathcal{Q}(S)\right].$$

By taking the difference with respect to t of both sides, multiplying by q, and then rewriting sums, we simplify to find

$$\nabla y(t) = -\frac{1}{q(t)}\left[\sum_{\tau=a+1}^{s}\frac{(t-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(s+\ell-a+1) - \mathfrak{A}^*\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)} - e^s\mathfrak{U}(s)\right]$$

$$-(q\nabla y)(s) = \frac{1}{\Gamma(\ell)}\sum_{\tau=a+1}^{s}(t-\rho(\tau))^{\overline{\ell-1}}h(s+\ell-a+1) - \mathfrak{A}^*\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)} - e^s\mathfrak{U}(s)$$

$$-(q\nabla y)(s) - e^s\mathfrak{U}(s) = \frac{1}{\Gamma(\ell)}\sum_{\tau=a+1}^{s}(t-\rho(\tau))^{\overline{\ell-1}}h(s+\ell-a+1) - \mathfrak{A}^*\frac{(t-a)^{\overline{\ell-1}}}{\Gamma(\ell)}.$$

So

$$-\overline{\nabla}_a^{\ell}\left[q\overline{\nabla}y + e^s\mathfrak{U})\right](s) = h'(s+\ell-a+1)$$

for $s \in \mathbb{N}_{a+2}^{b}$. Since

$$\sum_{s=a+1}^{\infty}\frac{1}{q(s)}\left[\sum_{\tau=a+1}^{s}\frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1) + \mathcal{Q}(S)\right]. \qquad (7)$$

is converges by take the limit we get $\lim_{t\to\infty} y(t) = \kappa$. □

Now we are ready to prove our next main result.

Theorem 3.3. *Suppose that* $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}, \mathfrak{U} : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, *and assume* $\kappa \in \mathbb{R}$, $0 \le \kappa < \infty$. *Let*

(1) $q(t) > 0$ *for* $t \in \mathbb{N}_a^b$,

(2) $\sum_{s=a+1}^{\infty}\frac{1}{q(s)}\left[\sum_{\tau=a+1}^{s}\frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1)\right] < \infty$,

(3) $\sum_{s=a+1}^{\infty}\frac{1}{q(s)}\left(\mathfrak{A}^*\frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)} - e^s\mathfrak{U}(s)\right) < \infty$.

Then $\exists t_0 \in \mathbb{N}_a$, *which the fractional finite self-adjoint type difference equation*

$$-\overline{\nabla}_{t_0}^{\ell}\left[q\overline{\nabla}y + e^t\mathfrak{U}\right](t) = h(t+\ell-a+1) \qquad (8)$$

has a solution $y \in \mathbb{R}$ *that* $\lim_{t\to\infty} y(t) = \kappa$.

Proof. Since the series

$$\sum_{s=a+1}^{\infty}\frac{1}{q(t)}\left[\sum_{\tau=a+1}^{s}\frac{(t-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1)\right],$$

converges, we can choose $b\in\mathbb{N}_a$ such that

$$\vartheta:=\sum_{s=b+1}^{\infty}\frac{1}{q(t)}\left[\sum_{\tau=b+1}^{s}\frac{(t-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1)\right]<1.$$

Suppose that $\Psi_b=\{y:\mathbb{N}_{t_0}\to\mathbb{R}\mid\lim_{t\to\infty}y(t)=\kappa\}$ and define the supremum norm, $\|\ .\ \|$, on Ψ_b by $\|\ y\ \|=\sup_{t\in\mathbb{N}_b}|y(t)|$. The pair $(\Psi_b,\|\ .\ \|)$ defines a complete metric space. Define the operator T on Ψ_b by

$$Ty(t)=\kappa+\sum_{s=t+1}^{\infty}\frac{y(s)}{q(s)}$$

$$\times\left[\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1)-\mathfrak{A}^*\frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^s\mathfrak{U}(s)\right]$$

As first step, we show that $T:\Psi_b\to\Psi_b$. Let arbitrary $y\in\Psi_b$ be fixed, so that $\lim_{t\to\infty}y(t)=\kappa$. Thus for some $M>0$,$|y(t)|<M$, $\forall t\in\mathbb{N}_b$. So

$$\begin{aligned}
&|\ Ty(t)\ |=|\ \kappa+\sum_{s=t+1}^{\infty}\frac{y(s)}{q(s)}\\
&\times\left[\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1)-\mathfrak{A}^*\frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^s\mathfrak{U}(s)\right]|\leq\kappa+\\
&+|\sum_{s=t+1}^{\infty}\frac{M}{q(s)}\left[\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1)-\mathfrak{A}^*\frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^s\mathfrak{U}(s)\right]|\\
&\leq\kappa+M|\sum_{s=t+1}^{\infty}\frac{1}{q(s)}\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1)\ |\\
&+M\ |\sum_{s=t+1}^{\infty}\frac{1}{q(s)}\left(\mathfrak{A}^*\frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^s\mathfrak{U}(s)\right)|,
\end{aligned}$$

by assumptions in the statement of this theorem, we have

$$
\begin{aligned}
&M \sum_{s=t+1}^{\infty}\left(\frac{1}{q(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1)\right) \\
&\leq M \sum_{s=a+1}^{\infty}\left(\frac{1}{q(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1)\right)<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\sum_{s=t+1}^{\infty} \frac{1}{q(s)}\left(\mathfrak{A}^{*} \frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^{s} \mathfrak{U}(s)\right)\right| \\
&\leq\left|\sum_{s=a+1}^{\infty} \frac{1}{q(s)}\left(\mathfrak{A}^{*} \frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^{s} \mathfrak{U}(s)\right)\right|<\infty .
\end{aligned}
$$

Hence T is well defined. Since the series in the definition of T converges, we can take the limit as $t \rightarrow \infty$ to find that $y: \mathbb{N}_{t_0} \rightarrow \mathbb{R}$ and thus $Ty \in \Psi_b$. Let $x, y \in \mathbb{N}_{t_0}$ and $t \in \mathbb{N}_b$ be fixed and arbitrary. So

$$
\begin{aligned}
&|Tx(t)-Ty(t)| \\
&=\Bigg|\sum_{s=t+1}^{\infty} \frac{x(s)}{q(s)}\left[\sum_{\tau=b+1}^{s} \frac{(s-\rho\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1)-\mathfrak{A}^{*} \frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^{s} \mathfrak{U}(s)\right] \\
&-\sum_{s=t+1}^{\infty} \frac{y(s)}{q(s)}\left[\sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1)-\mathfrak{A}^{*} \frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)}-e^{s} \mathfrak{U}(s)\right]\Bigg| \\
&=\left|\sum_{s=t+1}^{\infty} \frac{x(s)-y(s)}{q(s)}\left[\sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1)\right]\right| \\
&\leq \sum_{s=t+1}^{\infty} \frac{|x(s)-y(s)|}{q(s)}\left[\sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1)\right] \\
&\leq\left(\sum_{s=t+1}^{\infty} \frac{1}{q(s)} \sum_{\tau=b+1}^{s} \frac{(s-q(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)} h(\tau+\ell-a+1)\right)\|x-y\| \\
&=\vartheta\|x-y\| .
\end{aligned}
$$

$\|Tx-Ty\| \leq \vartheta\|x-y\|$, with $\vartheta<1$ for all $x, y \in \Psi_b$ and thus T is a contraction mapping. So T has a unique fixed point in Ψ_b call it y_*.

This fixed point satisfies the equation (5), and so in view of Lemma 3.2, it is also a solution of the fractional difference equation (8) that satisfies $\lim_{t\to\infty} y_* = \kappa$. $\square$

4 Examples

Example 4.1. Consider the following fractional IVP

$$\begin{cases} \overline{\nabla}_0^{\frac{7}{8}}\mathfrak{f}(t) = 7, \quad t \in \mathbb{N}_1, \\ \mathfrak{f}(1) = \pi \end{cases}$$

In view of lemma 2.10, the solution is given by

$$\mathfrak{f}(t) = \nabla_a^{-\frac{7}{8}} 7 + \big(\pi - 7\big)\frac{(t)^{\overline{\frac{-1}{8}}}}{\Gamma(\frac{7}{8})},$$

by definition 2.8, we have

$$\begin{aligned} \nabla_a^{-\frac{7}{8}} 7 &= \int_0^t \frac{(t-\rho(s))^{\overline{\frac{-1}{8}}}}{\Gamma(\frac{7}{8})} 7\overline{\nabla}s \\ &= \frac{7}{\Gamma(\frac{7}{8})} \int_0^t (t-\rho(s))^{\overline{\frac{-1}{8}}}\overline{\nabla}s \\ &= \frac{7}{\Gamma(\frac{7}{8})} \Big[\frac{-8}{7}(t-s)^{\overline{\frac{7}{8}}}\Big]_0^t \\ &= \frac{8}{\Gamma(\frac{7}{8})} t^{\overline{\frac{7}{8}}}. \end{aligned}$$

So the solution of the above fractional IVP is

$$\mathfrak{f}(t) = \frac{8}{\Gamma(\frac{7}{8})} t^{\overline{\frac{7}{8}}} + \frac{(\pi-7)}{\Gamma(\frac{7}{8})} t^{\overline{\frac{-1}{8}}}.$$

In the table 1 we present some numerical result for $\mathfrak{f}(t)$.

t	$\mathfrak{f}(t)$	$\mathfrak{f}(t_{n+1}) - \mathfrak{f}(t_n)$
1	π	***
2	9.7489	6.6073
3	15.7021	5.9532
4	21.3369	5.6348
5	26.7628	5.4257
6	32.0338	5.2710
7	37.1829	5.1491
8	42.2315	5.0486
9	47.1950	4.9635
10	52.0847	4.8897

Table 1: Numerical resual for $\mathfrak{f}(t)$ in example4.1.

Example 4.2. Consider $a > 1$, suppose $\mathfrak{U}(t) = \frac{1}{2}$, $h(t) = \frac{1}{2}e^{a+1} - e^t$ and $q(t) = \frac{(t-a)^{\overline{\ell}}}{\frac{1}{t}\sin(\frac{\pi}{t})}$, $q(t) > 0$, $\mathfrak{U}(t) > 0$ for $t \in \mathbb{N}_a^b$ we have that

$$\sum_{s=a+1}^{\infty} \frac{(t-a)^{\overline{\ell}}}{q(s)} = \sum_{s=a+1}^{\infty} \frac{\sin(\frac{\pi}{s})}{s} < \infty,$$

and, moreover

$$\begin{aligned}\sum_{s=a+1}^{\infty} \frac{1}{q(s)} &\left(\mathfrak{A}^* \frac{(s-a)^{\overline{\ell-1}}}{\Gamma(\ell)} - e^s \mathfrak{U}(s) \right) \\ &= \sum_{s=a+1}^{\infty} \frac{\frac{\sin(\frac{\pi}{s})}{s}}{(s-a)^{\overline{\ell}}} \left(e^{\ell} \frac{(s-a)^{\overline{\ell}}}{\Gamma(\ell+1)} - \frac{e^s}{2} \right) \\ = \sum_{s=a+1}^{\infty} &\frac{e^{\ell+2}}{\Gamma(\ell+1)} \left(\frac{\sin(\frac{\pi}{s})}{s} \right) < \infty,\end{aligned}$$

where $\mathfrak{A}^*(s) = q(a+1)\overline{\nabla} y(a+1) + e^{(a+1)}\mathfrak{U}(a+1) - h(\ell+2) = e^{\ell+2}$. On the other hand

$$
\begin{aligned}
&\sum_{s=a+1}^{\infty}\frac{1}{q(s)}\left[\sum_{\tau=a+1}^{s}\frac{(s-\rho(\tau))^{\overline{\ell-1}}}{\Gamma(\ell)}h(\tau+\ell-a+1)\right]\\
&=\sum_{s=a+1}^{\infty}\frac{(s-a)^{\overline{\ell}}}{q(s)\Gamma(\ell+1)}h(\tau+\ell-a+1)\\
&=\sum_{s=a+1}^{\infty}\frac{1}{\Gamma(\ell+1)}\left(\frac{\sin(\frac{\pi}{s})}{s}\left[\frac{1}{2}e^{a+1}-e^{r}\right]\right)\\
&\leq\sum_{s=a+1}^{\infty}\frac{e^{a+1}}{2\Gamma(\ell+1)}\frac{\sin(\frac{\pi}{s})}{s}<\infty,
\end{aligned}
$$

where $r=\tau+\ell-a+1$. Also there exist κ such that $\lim_{t\to\infty} y_{*}=\kappa$. We plotted part of the $q(t)$ in Figure 1.

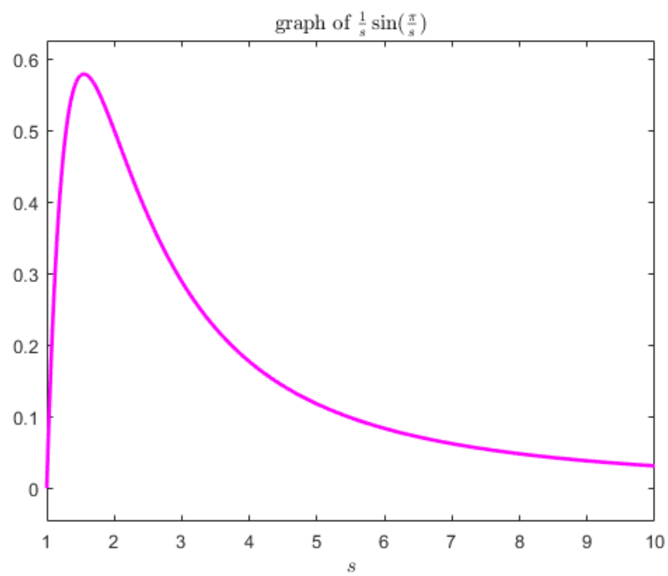

Figure 1: The graph of $\frac{1}{s}\sin(\frac{\pi}{s})$.

5 Conclusion

In this work, we study the existence and uniqueness of solution for a Nabla fractional self-adjoint finite difference problem on the time scale $\mathbb{N}_{a+1}^{b}$ via initial boundary conditions. To achieve our goal of this research, we used the Green function and contraction mapping fo proving our min results. We also used numerical interpretations and some figures for the examples to illustrate more our result.

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Shiva Dehghani
Department of Mathematics
PHD Candidate
Azarbaijan Shahid Madani University
Tabriz, Iran
E-mail: dehghani.sh73@gmail.com

Shahram Rezapour
Department of Mathematics
Professor of Mathematics
Azarbaijan Shahid Madani University
Tabriz, Iran
E-mail: sh.rezapourshahram@azaruniv.ac.ir