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Original Research Paper

Study of Hyers-Ulam Stability for a Class of Multi-Singular Fractional Integro-Differential Equation with Boundary Conditions

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Abstract. One of the considerable strategies for the investigation of integro-differential equation is stability. The notion of this strategy shows us that we can rest assured of the numerical results obtained from the computer software. Since there are usually large errors in the numerical results of singular differential equations, this strategy will help us to be able to examine singular equations more easily with computer software. In this work, we study the stability of a multi-singular fractional boundary value problem in the sense of Hyers-Ulam stability.We also present three examples and three figures to illustrate our main result.

AMS Subject Classification: 34B16; 34K20.
Keywords and Phrases: The fractional Caputo derivative, Hyers-Ulam stability, multi singular equation, fractional integro-differential equation.

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1 Introduction

The issue of stability has a long history and was studied by scientists more than 120 years ago, although at that time, information processing was not a concept for them. In fact, this was natural because from their point of view, physical phenomena had to be somehow sustainable [8, 14, 28, 29, 39, 51, 52]. After the second world war, which gave mathematicians more leisure time, some research groups flourished in European countries, Russia, and the United States, and talented young people became active.

One of these young people was an individual named Stanislaw Marcin Ulam who played an effective role in creating the concept of stability and significant progress in achieving various results in this field [17]. A few years later, Hyers became interested in this field. They were able to publish several joint articles on stability [18, 19, 20, 21, 59, 60]. In the last years of the last century, Rassias joined Hyers for extending the stability theory [22, 23]. Rassias was introduced as a considerable researcher before joining Ulam [24, 25, 40, 41, 42, 43, 44, 45, 46, 47, 48].

With the advent of computers and spread of numerical computations, numerous software applications in mathematics have emerged which became an effective factor for researchers. These calculations show that there are often errors in calculating numerical solutions of differential equations, and these errors sometimes get out of control, especially when we want to find a numerical solution of a singular differential equation. It was here when the concept of stability came to its true application. In fact, by using this old notion introduced by Ulam, we can rest assured of the numerical results obtained from the computer software. This was one of the main reasons why many researchers worked on the stability of differential equations in recent decades [1, 2, 3, 5, 6, 7, 9, 11, 12, 15, 16, 26, 30, 31, 32, 33, 36, 37, 49, 57, 62, 63, 64].

It is commonly known that in mathematical modeling of some phenomena, we encounter differential equations that have singularities ([27]). How can we be sure that the obtained errors in numerical calculations for finding the solution of a singular differential equation are related to computer software or not? The answer is easy. If we can prove that the singular differential equation is stable, then we can be sure that the numerical sequence which we obtain in our calculations, will be convergent to the solution. A researcher naturally tends to study complicated differential equations, and this sense has led many researchers to investigate singular equations because those are one of the certain complicated cases (see, for example, [4, 50, 53, 54, 55]). Our aim in this work is to study the stability of a multi-singular fractional integro-differential equation which the existence of its solutions has been studied by Shabibi et al.

In 2015, Muniyappan and Rajan [34] investigated stability of the following BVP;

$$\begin{cases} D^{\kappa}v(t) = f(t, v(t)), \\ av(0) + bv(t) = c. \end{cases}$$

Also, Rezapour and Shabibi [50] studied the existence of solutions for the following singular BVP;

$$\begin{cases} D^{\kappa}v(t) = f(t, v(t)), \\ v(0) = 0, \\ av(1) = I^{p}v(1). \end{cases}$$

where $a \geq 1$, $\kappa \geq 3$, $p \geq 1$, $t \in (0, 1)$ and f is singular at $t = 0$. In 2017, Haq et al. [15] reviewed the stability of the following problem;

$$\begin{cases} D^{\kappa}u(t) = f(t, u(t), D^{\kappa-1}u(t)), \\ u(0) = \delta u(1), \\ D^{p}u(1) = \gamma D^{p}u(\xi), \end{cases}$$

where $1 < \kappa \leq 2$, $t \in J = [0, 1]$, $0 < p < 1$ and $\xi \in (0, 1)$. Also, Shabibi et al. [54] investigated the following singular problem;

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), x'(t), D^{\beta}x(t), \int_0^1 h(\xi)x(\xi)d\xi), \\ x(0) = 0, \\ x(1) = D^{\gamma}x(\mu), \end{cases}$$

where $x \in c^1[0,1]$, $\alpha \geq 2$, $\beta, \gamma, \mu \in (0,1)$, $0 < t < 1$, $h \in L^1[0,1]$ is nonnegative with $\|h\|_1 = m$ and f is singular. Also, they introduced the multi-singular notion [56] and studied the complicated problem

$$\begin{cases} D^{\mu}x(t) = f(t, x(t), x'(t), D^{\beta}x(t), I^{p}x(t)), \\ x'(0) = x(\xi), \\ x(1) = \int_0^{\eta} x(s)ds, \end{cases}$$

under some different conditions. In fact, f is said to be multi-singular whenever it is singular at more than one point t.

We must try to show that not only well-known differential equations are stable, but also some complicated equations are stable. This issue will show us that if we can not obtain numerical solutions of multi-singular differential equations which are stable, we should be sure that the existence of computer software is incomplete, and those need new versions by using modern mathematical methods. Thus, we study only the stability of some special cases of the multi-singular problem which has been studied in [56]. Note that, studying each type of multi-singular equation is important in a sense. We choose the problem because it is one of the most complicated cases for studying stability.

Here, $\|.\|_1$ is the norm of $L^1[0,1]$, $\|.\|$ is the norm of $Y = C[0,1]$ and $\|x\|_* = \max\{\|x\|, \|x\|'\}$ is the norm of $X = C^1[0,1]$. We need next result.

Lemma 1.1. *([38]) Let $n-1 \leq \alpha < n$ and $u \in C(0,1) \cap L^1(0,1)$. Then, we have $I^{\alpha}({}^{c}D^{\alpha}u(t)) = u(t) + \sum_{i=0}^{n-1} c_i t^i$, where $c_1, \ldots, c_{n-1}$ are some real numbers.*

In next section, we provide our man results on the existence and the Hyers-Ulam stability of the above complicated problem. In last section, we give thee examples to illustrate our main result. In this way, we provide some figures for more illustration of readers on special cases of the examples.

2 Main Results

Here, we are going to investigate the Hyers-Ulam stability of the multi-singular problem

$$D^{\mu}z(s) = f(s, z(s), z'(s), D^{\beta}z(s), I^{p}z(s)), \tag{1}$$

via the following conditions

$$\begin{cases} z'(0) = z(\xi), \\ z(1) = \int_0^{\eta} z(s)ds, \end{cases}$$

where $\mu \in [2,3)$ and $z^{(j)}(0) = 0$ for $j = 2, ..., [\mu] - 1$. Also, we check it for the case $\mu \in [3,\infty)$, where $t \in J = [0,1]$, $z \in C^1[0,1]$, $\mu \in [2,+\infty)$, $\beta, \xi, \eta \in (0,1)$, $p > 1$, D^{μ} is the Caputo fractional derivative of order μ and $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a function such that $f(s, ., ., ., .)$ is singular at some points $s \in [0,1]$ (see [56]). This problem has been studied in [56] and one can find the following results in the work.

Lemma 2.1. *[56] Let $\eta \geq 2$, $\mu, \xi \in (0,1)$ and $y_0 \in L^1[0,1]$. Then, $z(s) = \int_0^1 H_1(s,r)y_0(r)dr$ is a solution for the pointwise defined problem $D^{\mu}x(s) + y_0(s) = 0$ via the conditions $z'(0) = z(\mu)$ and $z(1) = \int_0^{\eta} x(s)ds$ whenever $\mu \in [2,3)$, and also $x'(0) = x(\xi)$, $x(1) = \int_0^{\xi} z(s)ds$ and $z^{(j)}(0) = 0$ for $j = 2, ..., [\eta] - 1$ whenever $\eta \in [3,\infty)$. Here,*

$$H(s,r) = H_1(s,r) + \frac{1}{1-\xi}\int_0^{\xi} H_1(s,r)dt,$$

and the mapping H_1 *is defined by*

$$H_1(s,r)=\begin{cases}\frac{-(s-r)^{\eta-1}-s(\mu-r)^{\eta-1}+(1-r)^{\eta-1}+(\mu-r)^{\eta-1}}{\Gamma(\eta)}, & 0\le r\le s\le 1,\ r\le\mu,\\ \frac{-(s-r)^{\eta-1}+(1-r)^{\eta-1}}{\Gamma(\eta)}, & 0\le\mu\le r\le s\le 1,\\ \frac{-s(\xi-r)^{\eta-1}+(1-r)^{\eta-1}+(\xi-r)^{\eta-1}}{\Gamma(\eta)}, & 0\le s\le r\le\mu\le 1,\\ \frac{(1-r)^{\eta-1}}{\Gamma(\eta)}, & 0\le s\le r\le 1,\quad \mu\le r.\end{cases}$$

One can check that $H^*=sup_{s\in J}|\int_0^1 H(s,r)ds|=1+\frac{\eta}{1-\eta}<\infty$.

Theorem 2.2. *[56] Assume that* $f:[0,1]\times(C[0,1])^4\to\mathbb{R}$ *is a singular function at some points* $s\in[0,1]$, $b_1,...,b_4\in L^1[0,1]$ *are some nonnegative real valued maps that*

$$|f(s,x_1,...,x_4)-f(s,y_1,...,y_4)|\le\sum_{i=1}^{4}b_i(s)\|x_i-y_i\|,$$

for almost all $s\in[0,1]$ *and all* $x_1,...,x_4,y_1,...,y_4\in X$. *Suppose that there exists a natural number* k_0, $M_1,...,M_{k_0}\in L^1[0,1]$ *and* $\Lambda_1,...,\Lambda_{k_0}:\mathbb{R}^4\to[0,\infty)$ *such that* $M_1,...,M_{k_0}$ *are nonnegative,* $\Lambda_1,...,\Lambda_{k_0}$ *are nonnegative and nondecreasing functions in their components,*

$$|f(s,x_1,...,x_4)|\le\sum_{i=1}^{k_0}M_i(s)\Lambda_i(x_1,...,x_4),$$

for all $(x_1,...,x_4)\in X^4$, *almost all* $s\in[0,1]$ *and* $\lim_{z\to\infty}\frac{\Lambda_i(z,z,z,z)}{z}=\theta_0$, *where* θ_0 *is a nonnegative real number with* $0\le\theta_0\le\frac{\gamma_0}{C_{\mu,\eta}\sum_{i=1}^{k_0}\|M_i\|+\delta_0}$ *for some* $\delta_0>0$, $\gamma_0=min\{1,\Gamma(p+1),\Gamma(2-\beta)\}$ *and*

$$C_{\mu,\eta}=max\{A_{\mu,\eta},B_{\mu,\eta}\},$$

such that $A_{\mu,\eta}=\frac{3}{(1-\eta)\Gamma(\mu)}$, $B_{\mu,\eta}=\frac{2}{(1-\eta)\Gamma(\mu-1)}$. *Assume that* $[\hat{b}_1+\hat{b}_2+\frac{\hat{b}_3}{\Gamma(2-\beta)}+\frac{\hat{b}_4}{\Gamma(p+1)}]C_{\mu,\eta}<1$. *Then the pointwise defined problem (1) via the both cases of conditions has a solution. Here, we considered* $\hat{b}_i=\int_0^1(1-s)^{\mu-1}b_i(s)ds$.

Consider the map $H : X \to X$ defined by

$$H_z(t) = \int_0^1 H(t,s) f(s, z(s), z'(s), D^\beta z(s), I^p z(s)) ds,$$

where $H(t,s)$ was introduced in Lemma 2.1. By using Lemma 2.1, one can check that the fractional problem (1) has a solution z^* if and only if z^* is a fixed point of H. We say that the problem (1) is Ulam-Hyers stable whenever there is $c_f > 0$ so that for every $\varepsilon > 0$ and $\phi \in C(J,\mathbb{R})$ satisfying the inequality

$$|{}^c D^\mu \phi(s) - f(s, \phi(s), \phi'(s), D^\beta \phi(s), I^p \phi(s))| \leq \varepsilon, \quad (s \in J)$$

there is a solution $y \in C(J,\mathbb{R})$ of the problem (1) such that $|\phi(s) - y(s)| \leq c_f \varepsilon$ for all $s \in J$. Consider the Banach space $X = C^1[0,1]$ via the norm $\|z\| = \sup_{s\in J} |z(s)| + \sup_{s\in J} |z'(s)|$.

In below theorem, $m(t) = \sum_{i=1}^4 b_i(t)$ and the other condition satisfied. Thus, we provide our main result.

Theorem 2.3. *Let $f : J \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function and $\psi : [0,\infty) \to [0,\infty)$ a nondecreasing upper semi-continuous map with $\psi(t) < t$ for all $t > 0$. Assume that there exists $m \in L^1[0,1]$ such that $m^* = sup_{t\in J} |\frac{1}{m(t)}| < \infty$ and*

$$\Big|f(s,x_1,x_2,x_3,x_4) - f(s,y_1,y_2,y_3,y_4)\Big| \leq m(t)\psi\Big(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|+|x_4-y_4|\Big),$$

for all $s \in J$ and $x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4 \in \mathbb{R}$. If $\frac{m^*}{H^*}(2 + \frac{1}{\Gamma(p+1)} + \frac{1}{\Gamma(2-\beta)}) < 1$, then the problem (1) is Ulam-Hyers stable, where $H^* = sup_{t\in J} |\int_0^1 H(t,s)ds| < \infty$.

Proof. Put $c_f = \frac{1}{\Gamma(\mu+1)\Big(1 - \frac{m^*}{H^*}(2 + \frac{1}{\Gamma(p+1)} + \frac{1}{\Gamma(2-\beta)})\Big)}$. Since

$$\frac{m^*}{H^*}(2 + \frac{1}{\Gamma(p+1)} + \frac{1}{\Gamma(2-\beta)}) < 1,$$

and $\Gamma(\mu+1) > 0$, we get $c_f > 0$. Let $\varepsilon > 0$ be given. Assume that $\phi \in C(J,\mathbb{R})$ satisfying the inequality

$$|D^\mu \phi(s) - f(s, \phi(s), \phi'(s), I^p \phi(s), D^\beta \phi(s))| \leq \varepsilon,$$

for all $s \in J$. Then, we have

$$-\varepsilon \leq D^{\mu}\phi(s) - f(s, \phi(s), \phi'(s), D^{\beta}\phi(s), I^{p}\phi(s)) \leq \varepsilon$$

for all $s \in J$ and so

$$\Big| I^{\mu}D^{\mu}\phi(s) - I^{\mu}\left(f(s, \phi(s), \phi'(s), D^{\beta}\phi(s)), I^{p}\phi(s)\right) \Big| \leq I^{\mu}\varepsilon = \frac{s^{\mu}}{\Gamma(\mu+1)}\varepsilon.$$

Thus, $\left|\phi(s) - \int_0^1 H(s,r) f(t, \phi(t), \phi'(s), D^{\beta}\phi(s), I^{p}\phi(s)) dr\right| \leq \frac{s^{\mu}}{\Gamma(\mu+1)}\varepsilon$. We have to show that the inequality $|\phi(t) - x^*(t)| \leq c_f \varepsilon$ holds. Note that,

$$\begin{aligned}
&|\phi(t) - x^*(t)| = |\phi(t) - F_{x^*}(t)| \\
&\leq \Big|\phi(t) - \int_0^1 H(s,r) f(s, \phi(s), \phi'(s), I^{p}\phi(s), D^{\beta}\phi(s)) dr\Big| \\
&+ \Big| \int_0^1 H(s,r)\Big[f(s, \phi(s), \phi'(s), I^{p}\phi(s), D^{\beta}\phi(s)) \\
&- f(s, x^*(s), x^{*\prime}(s), I^{p}x^*(s), D^{\beta}x^*(s))\Big] dr\Big| \\
&\leq \frac{t^{\mu}}{\Gamma(\mu+1)}\varepsilon + \int_0^1 H(s,r) m(t) \psi\Big[|\phi(s) - x^*(s)| + |\phi'(s) - x^{*\prime}(s)| \\
&+ |I^{p}\phi(s) - I^{p}x^*(s)| + |D^{\beta}\phi(s) - D^{\beta}x^*(s)|\Big] dr.
\end{aligned}$$

Since $|I^{p}x_1 - I^{p}x_2| \leq \dfrac{\|x_1 - x_2\|}{\Gamma(p+1)}$ and $|D^{\beta}x_1 - D^{\beta}x_2| \leq \dfrac{\|x_1' - x_2'\|}{\Gamma(2-\beta)}$. Also, we assume that $H^* = sup_{t\in J}\int_0^1 H(s,r)dr$ and $m^* = sup_{t\in J}|\frac{1}{m(t)}|$. Thus,

$$
\begin{aligned}
|\phi(t) - x^*(t)| &\le \frac{t^\mu}{\Gamma(\mu+1)}\varepsilon + \frac{1}{H^*}\int_0^1 m(t)\psi\Big[|\phi(s) - x^*(s)| \\
&+ |\phi'(s) - x^{*\prime}(s)| + \frac{\|\phi - x^*\|}{\Gamma(p+1)} + \frac{\|\phi' - x^{*\prime}\|}{\Gamma(2-\beta)}\Big]dr \\
&\le \frac{t^\mu}{\Gamma(\mu+1)}\varepsilon + \frac{1}{H^*}\int_0^1 m(t)\psi\left[\|\phi - x^*\| + \|\phi' - x^{*\prime}\| + \frac{\|\phi - x^*\|}{\Gamma(p+1)} + \frac{\|\phi' - x^{*\prime}\|}{\Gamma(2-\beta)}\right]dr \\
&\le \frac{t^\mu}{\Gamma(\mu+1)}\varepsilon + \frac{1}{H^*}\int_0^1 m(t)\psi\left[(1 + \frac{1}{\Gamma(p+1)})\|\phi - x^*\| + (1 + \frac{1}{\Gamma(2-\beta)})\|\phi' - x^{*\prime}\|\right]dr \\
&\le \frac{t^\mu}{\Gamma(\mu+1)}\varepsilon + \frac{1}{H^*}\int_0^1 m(t)\psi\left[(2 + \frac{1}{\Gamma(p+1)} + \frac{1}{\Gamma(2-\beta)})\|\phi - x^*\|_*\right]dr \\
&\le \frac{t^\mu}{\Gamma(\mu+1)}\varepsilon + \frac{1}{H^*}\int_0^1 m(t)\left[2 + \frac{1}{\Gamma(p+1)} + \frac{1}{\Gamma(2-\beta)}\right]\|\phi - x^*\|_* dr \\
&\le \frac{t^\mu}{\Gamma(\mu+1)}\varepsilon + \frac{m^*}{H^*}\left[2 + \frac{1}{\Gamma(p+1)} + \frac{1}{\Gamma(2-\beta)}\right]\|\phi - x^*\|_*.
\end{aligned}
$$

Hence, we get $\|\phi - x^*\| \le \frac{1}{\Gamma(\mu+1)\left(1 - \frac{m^*}{H^*}\left(2 + \frac{1}{\Gamma(p+1)} + \frac{1}{\Gamma(2-\beta)}\right)\right)}\varepsilon$. Thus, we obtain $\|\phi - x^*\| \le c_f\varepsilon$. Therefore (1) is Hyers-Ulam stable. □

3 Examples

Example 3.1. Consider the multi singular fractional integro-differential equation $D^{\frac{5}{2}}x(t) - \frac{3}{(t-\frac{1}{10})} \times \frac{1}{2}\Big(\frac{|x(t)|}{1+|x(t)|} + x'(t) + \frac{1}{D^\beta x(t)} + I^p x(t)\Big) = 0$ for almost all $t \in [0,1]$, with boundary condition $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$ when $\mu \in [2,3)$, $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$, $x^{(j)}(0) = 0$, $j = 2, ..., [\mu] - 1$ when $\mu \in [3,\infty)$ and $x \in C^1[0,1]$, $\beta, \xi, \eta \in (0,1)$ and $p > 1$. Put $\beta = \eta = \frac{1}{2}$, $p = 2$, $\xi = \frac{1}{7}$, $\mu = \frac{5}{2}$ and

$$F(t, x(t), x'(t), D^\beta x(t), I^p x(t)) = \frac{3}{(t-\frac{1}{10})} \times \frac{1}{2}\Big(\frac{|x(t)|}{1+|x(t)|} + x'(t) + \frac{1}{D^\beta x(t)} + I^p x(t)\Big).$$

For $x, y \in C^3[0,1]$ and $t \in J$, clearly we have

$$
\begin{aligned}
&|F(t, x(t), x'(t), D^\beta x(t), I^p x(t)) - F(t, y(t), y'(t), D^\beta y(t), I^p y(t))| \\
&\le m(t)\psi(|x(t) - y(t)| + |x'(t) - y'(t)| + |D^\beta x(t) - D^\beta y(t)| + |I^p x(t) - I^p y(t)|)
\end{aligned}
$$

with $m(t) = \frac{3}{(t-\frac{1}{10})}$, $m^* = \frac{9}{20} < \infty$, $H^* = 1 + \frac{1}{1-\eta} = 3 < \infty$, $\psi(t) = \frac{t}{2}$. Since all the assumptions in Theorem (2.3) are satisfied, this fractional integro-differential equation is Ulam-Hyers stable.

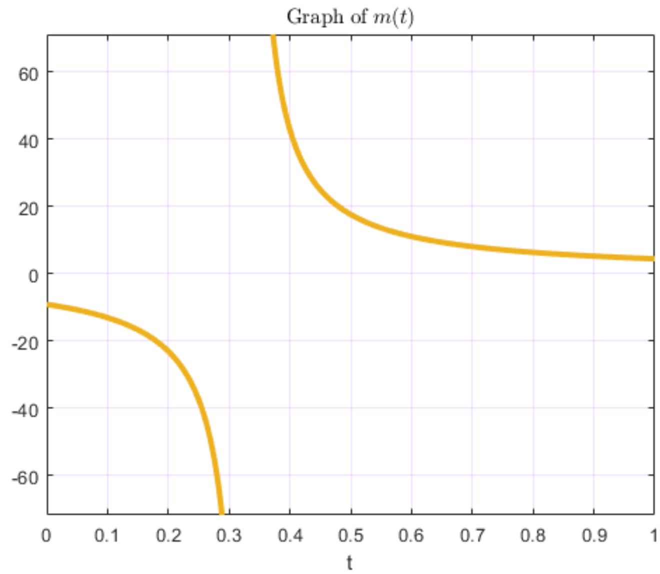

Figure 1: The graph of $m(t)$ in Example 3.1.

Example 3.2. Consider the multi singular fractional integro-differential equation $D^{\frac{9}{2}}x(t) - \frac{1}{t(t-\frac{1}{3})} \times \frac{2}{3}\Big(\frac{1}{2+|x(t)|} + x'(t) + \frac{2}{3D^{\beta}x(t)} + \frac{1}{I^{p}x(t)}\Big) = 0$ for almost all $t \in [0,1]$, with boundary condition $x'(0) = x(\xi)$, $x(1) = \int_0^{\eta} x(s)ds$ when $\mu \in [2,3)$ and $x'(0) = x(\xi)$, $x(1) = \int_0^{\eta} x(s)ds$, $x^{(j)}(0) = 0$, $j = 2, ..., [\mu] - 1$ when $\mu \in [3,\infty)$ and $x \in C^1[0,1]$, $\beta, \xi, \eta \in (0,1)$ and $p > 1$. Put $\eta = \beta = \frac{1}{2}$, $p = 3$, $\xi = \frac{1}{9}$, $\mu = \frac{9}{2}$ and

$$F(t, x(t), x'(t), D^{\beta}x(t), I^{p}x(t)) = \frac{1}{t(t-\frac{1}{3})} \times \frac{2}{3}\Big(\frac{1}{2+|x(t)|} + x'(t) + \frac{2}{3D^{\beta}x(t)} + \frac{1}{I^{p}x(t)}\Big).$$

For $x, y \in C^3[0,1]$ and $t \in J$, we have

$$|F(t, x(t), x'(t), D^{\beta}x(t), I^{p}x(t)) - F(t, y(t), y'(t), D^{\beta}y(t), I^{p}y(t))|$$

$$\leq m(t)\psi(|x(t)-y(t)| + |x'(t)-y'(t)| + |D^{\beta}x(t)-D^{\beta}y(t)| + |I^{p}x(t)-I^{p}y(t)|)$$

with $m(t) = \frac{1}{t(t-\frac{1}{3})}$, $m^* = \frac{1}{3} < \infty$, $H^* = 1 + \frac{1}{1-\eta} = 3 < \infty$, $\psi(t) = \frac{2}{3}t$. Since all the assumptions in Theorem (2.3) are satisfied, this fractional integro-differential equation is Ulam-Hyers stable.

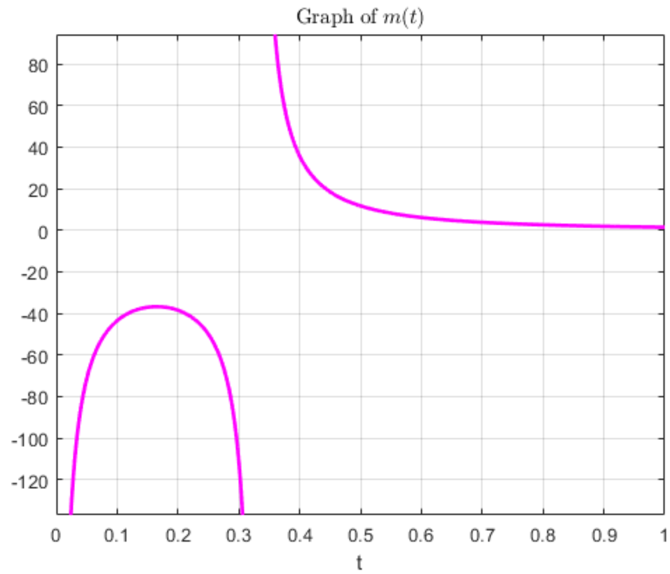

Figure 2: The graph of $m(t)$ in Example 3.2.

Example 3.3. Consider the multi-singular fractional differential equation $D^{\frac{9}{2}}x(t) - \frac{1}{(t-\frac{1}{2})^2(t-\frac{1}{3})}.\frac{1}{2}(\sum_{i=1}^{4}\frac{\|x_i\|^2}{1+\|x_i\|}) = 0$ for almost all $t \in [0,1]$, with boundary condition $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$ when $\mu \in [2,3)$ and $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$, $x^{(j)}(0) = 0$, $j = 2, ..., [\mu] - 1$ when $\mu \in [3,\infty)$ and $x \in C^1[0,1]$, $\beta, \xi, \eta \in (0,1)$ and $p > 1$. Put $\eta = \beta = \frac{1}{2}$, $p = 4$, $\xi = \frac{7}{8}$, $\mu = \frac{9}{2}$ and

$$F(t, x(t), x'(t), D^\beta x(t), I^p x(t)) = \frac{1}{(t-\frac{1}{2})^2(t-\frac{1}{3})} \times \frac{1}{2}\Big(\frac{||x(t)||^2}{1+||x(t)||} + \frac{||x'(t)||^2}{1+||x'(t)||}$$
$$+ \frac{||D^\beta x(t)||^2}{1+||D^\beta x(t)||} + \frac{||I^p x(t)||^2}{1+||I^p x(t)||}\Big).$$

For $x, y \in C^3[0,1]$ and $t \in J$, we have

$$|F(t, x(t), x'(t), D^\beta x(t), I^p x(t)) - F(t, y(t), y'(t), D^\beta y(t), I^p y(t))|$$
$$\leq m(t)\psi(|x(t) - y(t)| + |x'(t) - y'(t)| + |D^\beta x(t) - D^\beta y(t)| + |I^p x(t) - I^p y(t)|)$$

with $m(t) = \frac{1}{(t-\frac{1}{2})^2(t-\frac{1}{3})}$, $m^* = \frac{1}{6} < \infty$, $G^* = 1 + \frac{1}{1-\eta} = 3 < \infty$, $\psi(t) = \frac{t}{2}$. Since all the assumptions in Theorem 2.3 are satisfied, this fractional integro-differential equation is Ulam-Hyers stable.

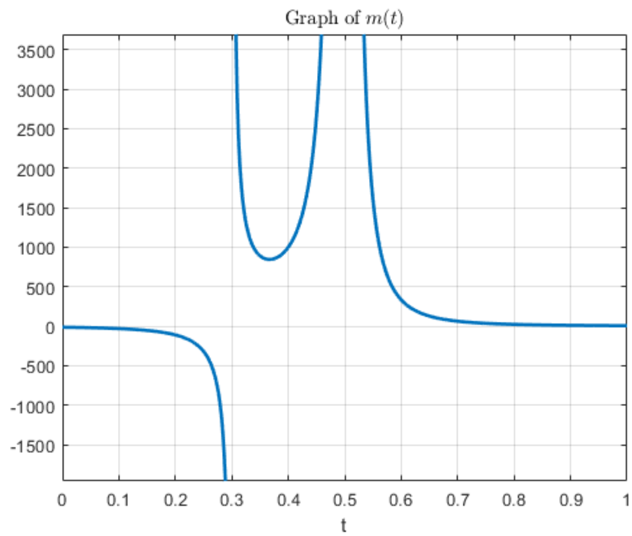

Figure 3: The graph of $m(t)$ in Example 3.3.

4 Conclusion

One of the most important issues that researchers always consider in computer calculations is to reduce computational error. Therefore, finding numerical methods with high accuracy is always one of the hot topics in the field of differential equations. Increasing this accuracy is especially important when the equation under study has single points. In this study, we took a step in this direction by examining the issue of stability under different boundary conditions. We also provided examples for our main result to better understand the subject. Young researchers can use different ideas for continuing the results o this work. One of best ideas is numerical approach. As you know, there are lot of published numerical papers (see for example, [10, 13, 35, 58]). Now the key note is that which numerical techniques can be used for finding the solution of such complicate multi-singular fractional integro-differential equations? Which numerical technique is better? This research path can be useful for completing mathematical softwares in future.

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