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Original Research Paper

Existence And Uniqueness of Solutions of a Terminal Value Problem For Fractional-Order Differential Equations

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Abstract. By using Banach's and Schauder's fixed point theorems, we study the existence and uniqueness of solutions of a terminal value problem for a class of fractional differential equations involving the modified fractional Liouville derivative. We also give some examples illustrating the application of our results.

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1 Introduction

The objective of this article is to give some results about existence and uniqueness of solutions for the following problem

$$\begin{cases} D_{-}^{\alpha}u(x) = f(x, u(x)), \ x \in [1, +\infty), \\ \lim\limits_{x \to +\infty} x^{\alpha-1}u(x) = a, \end{cases} \tag{1}$$

where $D_{-}^{\alpha}u(x) = -\dfrac{x^2}{\Gamma(1-\alpha)}\dfrac{d}{dx}\int\limits_{x}^{+\infty}(xt)^{\alpha}\dfrac{u(t)}{t^2(t-x)^{\alpha}}dt$ is the modified fractional Liouville derivative of order α with $0 < \alpha < 1$, $f : [1, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$.

Fractional differential equations have great application in many scientific areas, such as viscoelasticity, electrical circuits, electroanalytical chemistry, biology, control theory, electromagnetic theory, biomedical problems and so on (see [2], [19], [22], [23], [25] and the references cited in [28]).

Differential equations have been studied by several authors using the Leray-Schauder fixed point theorem, the coincidence degree theory of Mawhin, fixed point theorems, the method of upper and lower solutions, the method of upper and lower solutions coupled with monotone iterative technique and numerical methods (see [1], [3], [5], [6], [11], [12], [13], [15], [16], [20], [21], [24], [29] and [31]).

In [18], A. A. Kilbas and N. V. Kniaziuk studied the problem

$$\begin{cases} D_{-}^{\alpha}u(x) = \widetilde{f}(x, u(x)), \ x \in [1, +\infty), \\ \lim\limits_{x \to +\infty} \left(J_{-}^{1-\alpha}u\right)(x) = \widetilde{a}, \end{cases} \tag{2}$$

where $\left(J_{-}^{1-\alpha}u\right)(x) = \int\limits_{x}^{+\infty}(xt)^{\alpha}\dfrac{u(t)}{t^2(t-x)^{\alpha}}dt$ is the modified Liouville fractional Integral of order $1-\alpha$, $\widetilde{f} : [1, +\infty) \times O \to \mathbb{R}$, with O is an open subset of $\mathbb{R}$ and $\widetilde{a} \in \mathbb{C}$. By using Banach's fixed point theorem, A. A. Kilbas and N. V. Kniaziuk proved the uniqueness of solutions for the problem (2) if the function $\widetilde{f}$ is such that $\widetilde{f}(x, y) \in \pounds(1; +\infty)$ for any $y \in O$, and globally Lipschitz with respect to the second variable. The objective of this paper is to prove the existence of a unique solution for the terminal value problem (1) by using Banach's fixed point theorem, but the method of our proof is different to that used in [18]. Furthermore, under some assumptions on f, we prove the existence of solutions for the problem (1) by using Schauder's fixed-point theorem.

This paper is organized as follows: in Section 2, we give some definitions and preliminary results that will be used in the remainder of this paper. In Section 3, we state and prove our main results. In Section 4, we give two examples and, lastly, in Section 5, we give a conclusion.

2 Preliminaries

2.1 Modified Liouville fractional integrals and derivatives on infinite intervals

Definition 2.1. *([18]). For $\alpha > 0$, the modified Liouville fractional Integral of order α of a function g is defined by*

$$J_{-}^{\alpha} g\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_{x}^{+\infty} (xt)^{1-\alpha} \frac{g\left(t\right)}{t^{2}\left(t-x\right)^{1-\alpha}} dt, \text{ for all } x \in \left[1,+\infty\right).$$

Property 1. For all $\alpha > 0$, $\beta < 1-\alpha$ and $x \geq 1$, we have

$$\begin{aligned} J_{-}^{\alpha} x^{\beta} &= \frac{1}{\Gamma\left(\alpha\right)} \int_{x}^{+\infty} (xt)^{1-\alpha} \frac{t^{\beta}}{t^{2}\left(t-x\right)^{1-\alpha}} dt \\ &= \frac{x^{1-\alpha}}{\Gamma\left(\alpha\right)} \int_{x}^{+\infty} t^{1-\alpha} \frac{t^{\beta}}{t^{2}\left(t-x\right)^{1-\alpha}} dt \\ &= \frac{x^{1-\alpha+\beta}}{\Gamma\left(\alpha\right)} \int_{x}^{+\infty} \left(1-\frac{x}{t}\right)^{\alpha-1} \left(\frac{x}{t}\right)^{-\beta} \frac{dt}{t^{2}} \\ &= \frac{\Gamma\left(-\beta+1\right)}{\Gamma\left(\beta-\alpha+1\right)} x^{\beta-\alpha}. \end{aligned}$$

Notation 1 (See [18, Page 71])**.** For all $c > 0$, we note $\pounds\left(c;+\infty\right)$ the following space

$$\pounds\left(c;+\infty\right)=\left\{u:\ \ \left\|u\right\|_{\pounds\left(c;+\infty\right)}=\int\limits_{c}^{+\infty}\frac{\left|u\left(x\right)\right|}{x^{2}}dx<\infty\right\}.$$

We have the following results.

Lemma 2.2 (See [18, Lemma 2 page 70])**.** *For all $\alpha>0$, $\beta>0$ and $g\in\pounds\left(1;+\infty\right)$, we have*

$$J_{-}^{\alpha}J_{-}^{\beta}g=J_{-}^{\alpha+\beta}g.$$

Lemma 2.3 (See [18, Lemma 1 page 70])**.** *If $\alpha>0$, then the he modified Liouville fractional integral is bounded from $\pounds\left(c;+\infty\right)$ to $\pounds\left(c;+\infty\right)$ and we have*

$$\left\|J_{-}^{\alpha}g\right\|_{\pounds\left(c;+\infty\right)}\leq\frac{1}{a^{\alpha}\Gamma\left(\alpha+1\right)}\left\|g\right\|_{\pounds\left(c;+\infty\right)}.$$

Notation 2. We note $C\left(1;+\infty\right)$ the following space

$$C\left(1;+\infty\right)=\left\{u\in C\left(\left[1,+\infty\right),\mathbb{R}\right),\ \lim_{x\to+\infty}x^{\alpha-1}u\left(x\right)=a\right\}.$$

Note that $\left(C\left(1;+\infty\right),\left\|.\right\|_{0}\right)$ is a Banach space, where

$$\left\|u\right\|_{0}=\sup_{x\in\left[1,+\infty\right)}\left|x^{\alpha-1}u\left(x\right)\right|.$$

Remark 2.4. *It is not difficult to prove that if $u\in C\left(1;+\infty\right)$, then $u\in\pounds\left(1;+\infty\right)$.*

Lemma 2.5. *If $\alpha>0$, then the modified Liouville fractional integral is bounded from $C\left(1;+\infty\right)$ to $C\left(1;+\infty\right)$ and we have*

$$\left\|J_{-}^{\alpha}g\right\|_{0}\leq\frac{\Gamma\left(\alpha\right)}{\Gamma\left(2\alpha\right)}\left\|g\right\|_{0}.$$

Proof. Let $\alpha > 0$ and $g \in C(1;+\infty)$, we have

$$\begin{aligned}
\left|x^{\alpha-1}J_{-}^{\alpha}g(x)\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_{x}^{+\infty} t^{1-\alpha}\frac{g(t)}{t^{2}(t-x)^{1-\alpha}}dt\right| \\
&\leq \frac{\|g\|_{0}}{\Gamma(\alpha)}\int_{x}^{+\infty} t^{1-\alpha}\frac{t^{1-\alpha}}{t^{2}(t-x)^{1-\alpha}}dt \\
&= \frac{\Gamma(\alpha)}{x^{\alpha}\Gamma(2\alpha)}\|g\|_{0} \\
&\leq \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\|g\|_{0}.
\end{aligned}$$

Which implies that

$$\left\|J_{-}^{\alpha}g\right\|_{0} \leq \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\|g\|_{0}.$$

□

Definition 2.6. *([18]). For $0 < \alpha < 1$, the modified Liouville fractional derivative of order α of a function g is defined by*

$$\begin{aligned}
D_{-}^{\alpha}g(x) &= -x^{2}\frac{d}{dx}I_{-}^{1-\alpha}g(x) \\
&= -\tfrac{x^{2}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{+\infty}(xt)^{\alpha}\frac{g(t)}{t^{2}(t-x)^{\alpha}}dt, \text{ for all } x \in [1,+\infty).
\end{aligned}$$

Property 2 (See [18, Page 71])**.** For all $0 < \alpha < 1$, $\beta < 1$ with $\alpha+\beta < 1$ and $x \geq 1$, we have

$$\begin{aligned}
D_{-}^{\alpha}x^{\beta} &= -x^{2}\frac{d}{dx}I_{-}^{1-\alpha}x^{\beta} \\
&= -x^{2}\frac{d}{dx}\frac{\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)}x^{\beta+\alpha-1} \\
&= -x^{2}\frac{\Gamma(1-\beta)}{\Gamma(2-\beta-\alpha)}\frac{d}{dx}x^{\beta+\alpha-1} \\
&= \frac{\Gamma(1-\beta)}{\Gamma(1-\beta-\alpha)}x^{\beta+\alpha}.
\end{aligned}$$

Notation 3 (See [18, Page 71])**.** We note $\mathcal{AC}[1;+\infty)$ the following space

$$g\in\mathcal{AC}[1;+\infty)\Leftrightarrow g(x)=\widetilde{c}+\int_x^{+\infty}\frac{\psi(t)}{t^2}dt,$$

with $\psi\in\pounds(1;+\infty)$ and $\widetilde{c}\in\mathbb{R}$.

We have the following result.

Theorem 2.7. *([18, Theorem 1]). Let $0<\alpha<1$. If $g\in\pounds(1;+\infty)$ and $J_-^{1-\alpha}g\in\mathcal{AC}[1;+\infty)$, then we have*

$$\left(J_-^{\alpha}D_-^{\alpha}g\right)(x)=g(x)-\frac{\lim\limits_{x\to+\infty}J_-^{1-\alpha}g(x)}{\Gamma(\alpha)}x^{1-\alpha}.$$

Lemma 2.8. *Let $0<\alpha<1$ and assume that $g:[1,+\infty)\to\mathbb{R}$ is continuous.*

(i) If $\lim\limits_{x\to+\infty}x^{\alpha-1}g(x)=b$ with $b\in\mathbb{R}$, then $\lim\limits_{x\to+\infty}J_-^{1-\alpha}g(x)=b\Gamma(\alpha)$.

(ii) If $\lim\limits_{x\to+\infty}J_-^{1-\alpha}g(x)=b$ with $b\in\mathbb{R}$ and if $\lim\limits_{x\to+\infty}x^{\alpha-1}g(x)$ exists, then $\lim\limits_{x\to+\infty}x^{\alpha-1}g(x)=\dfrac{b}{\Gamma(\alpha)}$.

Proof. The proof is similar to that of Lemma 3.2 in [19], so we omit it. □

Now, we consider the following problem

$$\begin{cases} D_-^{\alpha}u(x)=h(x,u(x)),\ x\in[1,+\infty),\\ \lim\limits_{x\to+\infty}x^{\alpha-1}g(x)=b_2, \end{cases}\tag{3}$$

where $0<\alpha<1$, $h:[1,+\infty)\times G\to\mathbb{R}$ is a function such that $h(x,y)\in\pounds(1;+\infty)$ for any $y\in G$ with G is an open set in $\mathbb{R}$ and $b_2\in\mathbb{R}$.

As a consequence of Lemma 2.8 and Corollary 1 in [18, Page 73], we obtain the following result.

Theorem 2.9. *$u\in C(1;+\infty)$ is a solution for the problem (3) if, and only if, is a solution of the following Volterra equation*

$$u(x)=b_2x^{1-\alpha}+\frac{1}{\Gamma(\alpha)}\int_x^{+\infty}(xt)^{1-\alpha}\frac{h(t,u(t))}{t^2(t-x)^{1-\alpha}}dt.$$

2.2 Compactness criterion in an unbounded interval and fixed point theorems

Let

$$C_l = \left\{ u \in C\left(\overline{I}, \mathbb{R}\right), \ \lim_{x \to +\infty} u(x) \text{ exists} \right\}.$$

Note that $(C_l, \|.\|)$ is a Banach space, where

$$\|v\| = \sup_{x \in \overline{I}} |v(x)|.$$

The following proposition gives the compactness criterion of Corduneanu.

Proposition 2.10. *[See [9, Page 62]]Let $F \subset C_l$ be a set satisfying the following conditions:*

(i) F is bounded in C_l;

(ii) the functions belonging to F are equicontinuous on any compact interval of $\overline{I}$;

(iii) the functions from F are equiconvergent, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $\left| u(x) - \lim_{x \to +\infty} u(x) \right| < \varepsilon$, for any $x > T(\varepsilon)$ and $u \in F$.

Then F is compact in C_l.

From the preceding Proposition, we obtain the following result.

Lemma 2.11. *Let $X \subset C(1; +\infty)$ be a nonempty set satisfying the following conditions:*

(i) $\left\{ x^{\alpha-1} u(x) : \ u \in X \right\}$ is uniformly bounded;

(ii) $\left\{ x^{\alpha-1} u(x) : \ u \in X \right\}$ is equicontinuous on any compact interval of $[1, +\infty)$,

(iii) $\left\{ x^{\alpha-1} u(x) : \ u \in X \right\}$ is equiconvergent at $+\infty$.

Then X is compact in $C(1; +\infty)$.

For the proof of our main results, we use the following fixed point theorems.

Theorem 2.12. *([4]) (Schauder's fixed point theorem). Let C be a closed, convex subset of a normed linear space E. Then every compact, continuous map $F : C \to C$ has at least one fixed point.*

Definition 2.13. *([4]). Let (X, d) be a metric space. A map $S : X \to X$ is a contraction if for all u_1, $u_2 \in X$, we have*

$$d\left(S(u_1), S(u_2)\right) \leq \theta d\left(u_1, u_2\right) \text{ with } \theta < 1. \tag{4}$$

Theorem 2.14. *([4]). (Banach's fixed point theorem). Let (X, d) be a nonempty complete metric space and let $S : X \to X$ be a contraction. Then S has a unique fixed point $u \in X$.*

3 Main Results

Definition 3.1. *A function $u \in C\left(1; +\infty\right)$ is a solution for the problem (1) if $D_{-}^{\alpha} u \in C\left(\left[1, +\infty\right), \mathbb{R}\right)$ and satisfies (1).*

The first result of this paper is the following.

Theorem 3.2. *Assume that the following hypothesis is satisfied*

(H1) There exists $L > 0$ such that

$$\left|f\left(x, u\right) - f\left(x, v\right)\right| \leq L\left|u - v\right|, \text{ for all } x \in \left[1, +\infty\right) \text{ and } u,\ v \in \mathbb{R}.$$

Then the problem (1) admits a unique solution.

Proof. We consider the operator T defined by

$$\begin{aligned} T: \quad & C\left(1; +\infty\right) \to C\left(1; +\infty\right) \\ & u \mapsto Tu\left(x\right) = a x^{1-\alpha} + \frac{1}{\Gamma\left(\alpha\right)} \int_{x}^{+\infty} \left(xt\right)^{1-\alpha} \frac{f\left(t, u\left(t\right)\right)}{t^2\left(t - x\right)^{1-\alpha}} dt, \end{aligned}$$

and we use the following norm

$$\left\|v\right\|_* = \sup_{x \in \overline{I}} e^{-\dfrac{\lambda}{x}} \left|x^{\alpha-1} v\left(x\right)\right|,$$

where $\lambda > 0$ and $v \in C(1;+\infty)$.

Since the norms $\|.\|_*$ and $\|.\|_0$ are equivalent, then $(C(1;+\infty), \|.\|_*)$ is a Banach space.

Now we are going to show that T is a contraction on $(C(1;+\infty), \|.\|_*)$.

Let u_1, $u_2 \in C(1;+\infty)$, then for all $x \in \overline{I}$, one has

$$
\begin{aligned}
& e^{-\frac{\lambda}{x}} \left| x^{\alpha-1}((Tu_1)(x) - (Tu_2)(x)) \right| \\
= & \frac{e^{-\frac{\lambda}{x}}}{\Gamma(\alpha)} \left| \int_x^{+\infty} t^{1-\alpha} \frac{(f(t,u_1(t)) - f(t,u_2(t)))}{t^2 (t-x)^{1-\alpha}} dt \right| \\
\leq & \frac{Le^{-\frac{\lambda}{x}}}{\Gamma(\alpha)} \left| \int_x^{+\infty} t^{1-\alpha} \frac{|u_1(t) - u_2(t)|}{t^2 (t-x)^{1-\alpha}} dt \right| \\
\leq & \frac{L\|u_1 - u_2\|_*}{\Gamma(\alpha)} \int_x^{+\infty} e^{-\lambda\left(\frac{1}{x} - \frac{1}{t}\right)} \frac{t^{2-2\alpha}}{t^2 (t-x)^{1-\alpha}} dt \\
= & \frac{L\|u_1 - u_2\|_*}{x^\alpha \Gamma(\alpha)} \int_1^{+\infty} e^{-\frac{\lambda}{x}\left(1 - \frac{1}{v}\right)} \frac{v^{2-2\alpha}}{v^2 (v-1)^{1-\alpha}} dv \\
= & \frac{L\|u_1 - u_2\|_*}{x^\alpha \Gamma(\alpha)} \int_1^{+\infty} e^{-\frac{\lambda}{x}\left(1 - \frac{1}{v}\right)} \left(1 - \frac{1}{v}\right)^{\alpha-1} v^{1-\alpha} \frac{dv}{v^2}.
\end{aligned}
$$

If we put the change of variables $w = 1 - \frac{1}{v}$, we obtain

$$
e^{-\frac{\lambda}{x}} \left| x^{\alpha-1}((Tu_1)(x) - (Tu_2)(x)) \right| \leq \frac{L\|u_1 - u_2\|_*}{x^\alpha \Gamma(\alpha)} J, \tag{5}
$$

where

$$
J = \int_0^1 e^{-\frac{\lambda w}{x}} w^{\alpha-1} (1-w)^{\alpha-1} dw.
$$

Now, we have

$$J = J_1 + J_2,$$

where

$$J_1 = \int_0^{\frac{1}{2}} e^{-\dfrac{\lambda w}{x}} \left(w\left(1-w\right)\right)^{\alpha-1} dw,$$

and

$$J_2 = \int_{\frac{1}{2}}^{1} e^{-\dfrac{\lambda w}{x}} \left(w\left(1-w\right)\right)^{\alpha-1} dw.$$

We have

$$\begin{aligned}
J_1 &\leq \frac{1}{2^{\alpha-1}} \int_0^{\frac{1}{2}} e^{-\dfrac{\lambda w}{x}} w^{\alpha-1} dw \\
&= 2\left(\frac{x}{2\lambda}\right)^{\alpha} \int_0^{\frac{\lambda}{2x}} e^{-y} y^{\alpha-1} dy \\
&\leq 2\left(\frac{x}{2\lambda}\right)^{\alpha} \int_0^{+\infty} e^{-y} y^{\alpha-1} dy \\
&= 2\left(\frac{x}{2\lambda}\right)^{\alpha} \Gamma\left(\alpha\right).
\end{aligned}$$

That is

$$J_1 \leq 2\left(\frac{x}{2\lambda}\right)^{\alpha} \Gamma\left(\alpha\right). \tag{6}$$

On the other hand, we have

$$
\begin{aligned}
J_2 &\leq \frac{1}{2^{\alpha-1}} \int_{\frac{1}{2}}^{1} e^{-\dfrac{\lambda w}{x}} (1-w)^{\alpha-1}\, dw \\
&\leq \frac{e^{-\dfrac{\lambda}{2x}}}{2^{\alpha-1}} \int_{\frac{1}{2}}^{1} (1-w)^{\alpha-1}\, dw \\
&= \frac{e^{-\dfrac{\lambda}{2x}}}{\alpha 2^{2\alpha-1}}.
\end{aligned}
$$

That is

$$
J_2 \leq \frac{e^{-\dfrac{\lambda}{2x}}}{\alpha 2^{2\alpha-1}}. \tag{7}
$$

Then from (5), (6) and (7), we obtain

$$\begin{aligned}
& e^{-\frac{\lambda}{x}}\left|x^{\alpha-1}((Tu_1)(x)-(Tu_2)(x))\right| \\
\leq\ & \frac{L\|u_1-u_2\|_*}{x^\alpha\Gamma(\alpha)}\left(2\left(\frac{x}{2\lambda}\right)^\alpha\Gamma(\alpha)+\frac{e^{-\frac{\lambda}{2x}}}{\alpha 2^{2\alpha-1}}\right) \\
=\ & \frac{L\|u_1-u_2\|_*}{2^{\alpha-1}x^\alpha\Gamma(\alpha)}\left(\left(\frac{x}{\lambda}\right)^\alpha\Gamma(\alpha)+\frac{e^{-\frac{\lambda}{2x}}}{\alpha 2^{\alpha}}\right) \\
=\ & \frac{L\|u_1-u_2\|_*}{2^{\alpha-1}\lambda^\alpha\Gamma(\alpha)}\left(\Gamma(\alpha)+\left(\frac{\lambda}{2x}\right)^\alpha\frac{e^{-\frac{\lambda}{2x}}}{\alpha}\right) \\
\leq\ & \frac{L\|u_1-u_2\|_*}{2^{\alpha-1}\lambda^\alpha\Gamma(\alpha)}\left(\Gamma(\alpha)+\frac{1}{\alpha}\right) \\
=\ & \frac{L\|u_1-u_2\|_*}{2^{\alpha-1}\lambda^\alpha}\left(1+\frac{1}{\Gamma(\alpha+1)}\right).
\end{aligned}$$

Which implies that

$$\|Tu_1-Tu_2\|\leq\frac{L}{2^{\alpha-1}\lambda^\alpha}\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\|u_1-u_2\|_*.$$

Now if we choose $\lambda>\left(\frac{L}{2^{\alpha-1}}\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right)^{\frac{1}{\alpha}}$, we obtain

$$\frac{L}{2^{\alpha-1}\lambda^\alpha}\left(1+\frac{1}{\Gamma(\alpha+1)}\right)<1.$$

Then by Banach's fixed point theorem the operator T admits a unique fixed point and consequently from Theorem 2.9, it follows that the problem (1) admits a unique solution.

The proof of our first result is complete. □

Remark 3.3. *The idea of the proof of Theorem 3.2 is similar to that of Theorem 3.1 in [14].*

Now our next result is based on Schauder's fixed point theorem.

We put by definition

$$E = \left(C\left(1;+\infty\right), \|.\|_0\right),$$

and

$$X = \left\{u \in C\left(1;+\infty\right):\ \|u\|_0 \leq r\right\},$$

where $r > |a|$.

We assume the following hypotheses are satisfied.

(H2) $|f(x,u)| \leq \varphi(x) F\left(x^{\alpha-1}|u|\right)$, for all $x \in [1,+\infty)$ and $u \in \mathbb{R}$ with $\varphi \in C(1;+\infty)$ and $F : [0,+\infty) \to [0,+\infty)$ is continuous nondecreasing.

(H3) $\dfrac{\Gamma(\alpha) F(r)}{\Gamma(2\alpha)} \|\varphi\|_0 \leq r - |a|.$

We have the following result.

Theorem 3.4. *If the hypotheses (H2) and (H3) are satisfied, then the problem (1) admits at least one solution.*

Proof. We consider the operator T defined by

$$\begin{aligned} T:\quad X &\longrightarrow C\left(1;+\infty\right) \\ u &\longmapsto (Tu)(x) = ax^{1-\alpha} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} (xt)^{1-\alpha} \frac{f(t,u(t))}{t^2 (t-x)^{1-\alpha}} dt. \end{aligned}$$

Step 1: $TX \subseteq X$.

For $u \in X$, we have

$$\begin{aligned}
\left|x^{\alpha-1}(Tu)(x)\right| &= \left|a + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} t^{1-\alpha} \frac{f(t,u(t))}{t^2 (t-x)^{1-\alpha}} dt\right| \\
&\leq |a| + \frac{1}{\Gamma(\alpha)} \left|\int_x^{+\infty} \frac{\varphi(t) F\left(t^{\alpha-1}|u(t)|\right)}{t^{\alpha+1}(t-x)^{1-\alpha}} dt\right| \\
&\leq |a| + \frac{F(r)}{\Gamma(\alpha)} \left|\int_x^{+\infty} \frac{\varphi(t)}{t^{\alpha+1}(t-x)^{1-\alpha}} dt\right| \\
&\leq |a| + \frac{\Gamma(\alpha) F(r)}{x^{\alpha}\Gamma(2\alpha)} \|\varphi\|_0 \\
&\leq |a| + \frac{\Gamma(\alpha) F(r)}{\Gamma(2\alpha)} \|\varphi\|_0 \\
&\leq r.
\end{aligned}$$

Which implies that

$$\|Tu\|_0 \leq r,$$

and consequently, we have $TX \subseteq X$.

Step 2: TX is uniformly bounded in E.

The proof follows from **Step 1**.

Step 3: The set TX is equiconvergent at $+\infty$.

Let $u \in X$, we have

$$\begin{aligned}
\left|x^{\alpha-1}(Tu)(x) - a\right| &= \left|\frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(1 - \frac{x}{t}\right)^{\alpha-1} \frac{f(t,u(t))}{t^2} dt\right| \\
&\leq \frac{\Gamma(\alpha) F(r)}{x^{\alpha}\Gamma(2\alpha)} \|\varphi\|_0 .
\end{aligned}$$

Which implies that

$$\lim_{x \to +\infty} \left|x^{\alpha-1}(Tu)(x) - a\right| = 0.$$

Which means that

$$\lim_{x \to +\infty} x^{\alpha-1}(Tu)(x) = a.$$

Step 4: The set TX is equicontinuous on any compact interval of $[1, +\infty)$.

For $u \in TX$ and $1 \leq x_1 \leq x_2 < +\infty$, one has

$$\begin{aligned}
& \left| x_2^{\alpha-1} (Tu)(x_2) - x_1^{\alpha-1} (Tu)(x_1) \right| \\
= & \frac{1}{\Gamma(\alpha)} \left| \int_{x_2}^{+\infty} t^{1-\alpha} \frac{f(t,u(t))}{t^2 (t-x_2)^{1-\alpha}} dt - \int_{x_1}^{+\infty} t^{1-\alpha} \frac{f(t,u(t))}{t^2 (t-x_1)^{1-\alpha}} dt \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \left| \int_{x_2}^{+\infty} \left((t-x_2)^{\alpha-1} - (t-x_1)^{\alpha-1} \right) \frac{f(t,u(t))}{t^{\alpha+1}} dt \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \int_{x_1}^{x_2} t^{1-\alpha} \frac{f(t,u(t))}{t^2 (t-x_1)^{1-\alpha}} dt \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_{x_2}^{+\infty} \left((t-x_2)^{\alpha-1} - (t-x_1)^{\alpha-1} \right) \frac{|f(t,u(t))|}{t^{\alpha+1}} dt \\
& + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} t^{1-\alpha} \frac{|f(t,u(t))|}{t^2 (t-x_1)^{1-\alpha}} dt.
\end{aligned}$$

That is

$$\left| x_2^{\alpha-1} (Tu)(x_2) - x_1^{\alpha-1} (Tu)(x_1) \right| \leq F_1(x_1,x_2) + F_2(x_1,x_2),$$

where

$$F_1(x_1,x_2) = \frac{1}{\Gamma(\alpha)} \int_{x_2}^{+\infty} \left((t-x_2)^{\alpha-1} - (t-x_1)^{\alpha-1} \right) \frac{|f(t,u(t))|}{t^{\alpha+1}} dt,$$

and

$$F_2(x_1,x_2) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} t^{1-\alpha} \frac{|f(t,u(t))|}{t^2 (t-x_1)^{1-\alpha}} dt.$$

On the other hand, we have

$$\begin{aligned}
&F_1(x_1,x_2)\\
&\leq \frac{1}{\Gamma(\alpha)}\int_{x_2}^{+\infty}\left((t-x_2)^{\alpha-1}-(t-x_1)^{\alpha-1}\right)\frac{|f(t,u(t))|}{t^{\alpha+1}}dt\\
&=\frac{1}{\Gamma(\alpha)}\int_{1}^{+\infty}\chi_{[x_2,+\infty[}(t)\left((t-x_2)^{\alpha-1}-(t-x_1)^{\alpha-1}\right)\frac{|f(t,u(t))|}{t^{\alpha+1}}dt.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{1}^{+\infty}\chi_{[x_2,+\infty[}(t)(t-x_1)^{\alpha-1}\frac{|f(t,u(t))|}{t^{\alpha+1}}dt\\
\leq\ &\int_{1}^{+\infty}\chi_{[x_2,+\infty[}(t)(t-x_2)^{\alpha-1}\frac{|f(t,u(t))|}{t^{\alpha+1}}dt<+\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{x_1\to x_2}\chi_{[x_2,+\infty[}(t)(t-x_1)^{\alpha-1}\frac{|f(t,u(t))|}{t^{\alpha+1}}\\
=\ &\chi_{[x_2,+\infty[}(t)(t-x_2)^{\alpha-1}\frac{|f(t,u(t))|}{t^{\alpha+1}},
\end{aligned}$$

we get

$$\begin{aligned}
&\lim_{x_1\to x_2}\int_{1}^{+\infty}\chi_{[x_2,+\infty[}(t)(t-x_1)^{\alpha-1}\frac{|f(t,u(t))|}{t^{\alpha+1}}dt\\
=\ &\int_{1}^{+\infty}\chi_{[x_2,+\infty[}(t)(t-x_2)^{\alpha-1}\frac{|f(t,u(t))|}{t^{\alpha+1}}dt.
\end{aligned}$$

Then if we put by definition $\delta_3(\varepsilon)=\varepsilon\Gamma(\alpha)$, we obtain

$$\forall\varepsilon>0,\ \exists\ \delta_3(\varepsilon)>0,\ \left(|x_1-x_2|<\delta_3(\varepsilon)\Rightarrow F_1(x_1,x_2)\right)<\varepsilon). \tag{8}$$

Also, we have

$$
\begin{aligned}
F_2(x_1,x_2) &\leq \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2} t^{1-\alpha}\frac{|f(t,u(t))|}{t^2(t-x_1)^{1-\alpha}}dt \\
&\leq \frac{F(r)\|\varphi\|_0}{\Gamma(\alpha)}\int_{x_1}^{x_2}(t-x_1)^{\alpha-1}dt \\
&= \frac{F(r)\|\varphi\|_0}{\Gamma(\alpha+1)}(x_2-x_1)^{\alpha}.
\end{aligned}
$$

Then if we put by definition $\delta_4(\varepsilon) = \dfrac{\varepsilon\Gamma(\alpha+1)}{F(r)\|\varphi\|_0+1}$, we obtain

$$
\forall\varepsilon>0,\ \exists\ \delta_4(\varepsilon)>0,\ (|x_1-x_2|<\delta_4(\varepsilon)\Rightarrow F_2(x_1,x_2))<\varepsilon). \tag{9}
$$

Then by (8) and (9) and if we put $\delta_5(\varepsilon)=\min(\delta_3(\varepsilon),\delta_4(\varepsilon))$, we obtain $\forall\varepsilon>0$, $\exists\ \delta_5(\varepsilon)>0$ such that

$$
|x_1-x_2|<\delta_5(\varepsilon)\Rightarrow\left|x_2^{\alpha-1}(Tu)(x_2)-x_1^{\alpha-1}(Tu)(x_1)\right|<\varepsilon.
$$

Step 5: The operator T is continuous.

The proof follows from Lemma 2.5.

In conclusion the operator T satisfies the assumptions of Schauder's fixed point theorem and consequently from Theorem 2.9 it follows that problem (1) admits at least one solution. □

4 Application

To illustrating the application of our results, we give two examples.

4.1 Example 1

We consider the problem

$$
\begin{cases}
D_{-}^{\alpha}u(x) = h_1(u(x))+\widetilde{h}(x),\ x\in[1,+\infty), \\
\lim\limits_{x\to+\infty} x^{\alpha-1}u(x) = a,
\end{cases} \tag{10}
$$

where $0 < \alpha < 1$, $h_1 : \mathbb{R} \to \mathbb{R}$ is of class C^1 with bounded derivative, $\widetilde{h} : [1, +\infty) \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$. In this example, we have

$$f(x, u) = h_1(u(x)) + \widetilde{h}(x).$$

For u_1, $u_2 \in \mathbb{R}$ and $x \in [1, +\infty)$, one has

$$\begin{aligned} |f(x, u_1) - f(x, u_2)| &= |h_1(u_1(x)) - h_1(u_2(x))| \\ &\leq \sup_{v \in \mathbb{R}} |h_1'(v)| . |u_1 - u_2| \\ &\leq k. |u_1 - u_2|, \end{aligned}$$

where k is a real number.

Then the assumption of Theorem 3.2 is satisfied and therefore we obtain the existence of a unique solution for the problem (10).

4.2 Example 2

We consider the following problem

$$\begin{cases} D_{-}^{\frac{1}{2}} u(x) = \varphi(x) . F\left(\dfrac{|u(x)|}{\sqrt{x}}\right), \, x \in [1, +\infty), \\ \lim\limits_{x \to +\infty} \dfrac{u(x)}{\sqrt{x}} = a, \end{cases} \tag{11}$$

where $\varphi(x) = \sqrt{\dfrac{x}{\pi}}$ and $F\left(\dfrac{|u(x)|}{\sqrt{x}}\right) = \ln\left(1 + \dfrac{|u(x)|}{\sqrt{x}}\right) - a$ with $a \in \mathbb{R}$. We have

$$\begin{aligned} \frac{\Gamma\left(\frac{1}{2}\right) F(r)}{\Gamma(1)} \|\varphi\|_0 &= \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}} (\ln(1 + r) - a) \\ &\leq r - a. \end{aligned}$$

Then from Theorem 3.4, we obtain the existence of solutions for the problem (11).

5 Conclusion

In this paper based on Banach's and Schauder's fixed point theorems, we gave some results concerning the existence and uniqueness of solutions for a class of terminal value problems involving modified Liouville fractional derivatives. We note that our result can be applied to terminal value problems for fractional differential equations involving modified Liouville fractional derivatives with advanced arguments and we can generalize the results obtained in [27] . Also, we note that terminal value problems for differential equations with advanced arguments arise, for example in biological cellular growth models and theoretical physics (see [8] and [26]). On the other hand, it could be interesting to generalize the results obtained in ([7], [10], [17] and [30]) to study the qualitative analysis of fractional integro-differential equations involving modified Liouville fractional derivatives.

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