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Original Research Paper

The Sequences of the Hyperbolic k-Perrin and k-Leonardo Quaternions

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Abstract. In this article, hyperbolic k-Perrin and k-Leonardo quaternions are defined. In this sense, bulletin, are evaluated as sequences of k-Perrin and k-Leonardo, their respective quaternions and thus the definition of hyperbolic quaternions. Thus, there are some algebraic properties around numbers, generating function, Binet's formula and properties inherent to these numbers.

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1 Introduction

Recursive linear sequences are widely studied in pure mathematics, presenting their generalizations and complexifications. Based on that, in this work we will present algebraic properties around the sequences of k-Perrin and k-Leonardo.

Leonardo's sequence is discussed in Catarino and Borges (2020) [1], where it is presented as a recurrent integer sequence that is related to the Fibonacci and Lucas sequence.

Leonardo's sequence corresponds to the following recurrence relationship:

$$Le_n = Le_{n-1} + Le_{n-2} + 1, n \geq 2. \tag{1}$$

For $n+1$ we can rewrite this recurrence relationship as $Le_{n+1} = Le_n + Le_{n-1} + 1$. And yet, subtracting $Le_n - Le_{n+1}$ we obtain another equivalent recurrence relation for this sequence. Watch:

$$\begin{aligned} Le_n - Le_{n+1} &= Le_{n-1} + Le_{n-2} + 1 - Le_n - Le_{n-1} - 1 \\ Le_{n+1} &= 2Le_n - Le_{n-2} \end{aligned} \tag{2}$$

being $Le_0 = Le_1 = 1$ and $Le_2 = 2$ it is initial conditions.

As for the Perrin sequence, this sequence was implicitly mentioned by Édouard Lucas in 1876, known for creating the Lucas sequence and mathematical games with the Tower of Hanoi, but only in 1899, François Perrin defined this sequence. This sequence is defined by the recurrence $Pe_n = Pe_{n-2} + Pe_{n-3}$, $n \geq 3$ and being $Pe_0 = 3, Pe_1 = 0$ and $Pe_2 = 2$ your initial conditions [5].

On the other hand, there are the quaternions, these numbers were developed in 1843 by Willian Rowan Hamilton (1805-1865). The quaternions arise from the attempt to generalize complex numbers in the form $z = a + bi$ in three dimensions [6], the quaternions are presented as formal sums of scalars with usual vectors of three-dimensional space, existing four dimensions. Thus, a quaternion is described by:

$$q = a + bi + cj + dk$$

where a, b, c are real numbers and i, j, k the orthogonal part at the base $\mathbb{R}^3$.

And yet, Horadam (1993) [4] presents the quaternionic product being $i^2 = j^2 = k^2 = -1, ij = k = -ij, jk = i = -kj$ and $ki = j = -ik$.

As for hyperbolic numbers, the set of these numbers $\mathbb{H}$ can be described as:

$$\mathbb{H} = \left\{ z = x + hy | h \notin \mathbb{R}, h^2 = 1, x, y \in \mathbb{R} \right\}.$$

Work on hyperbolic numbers can be found in [2, 3, 7].

Finally, in this article, we will define the hyperbolic quaternions of k-Leonardo and k-Perrin and provide some properties around them.

2 The Hyperbolic Quaternions of k-Perrin and k-Leonardo

The sequence of k-Perrin and k-Leonardo are defined by

$$Pe_{k,n} = Pe_{k,n-2} + kPe_{k,n-3}, n \geq 3,$$

$$Le_{k,n+1} = 2kLe_{k,n} - Le_{k,n-2}, n \geq 2,$$

being $Pe_{k,0} = 3, Pe_{k,1} = 0, Pe_{k,2} = 2$, $Le_{k,0} = Le_{k,1} = 1$ and $Le_{k,2} = 3$ its initial terms.

In turn, we have the characteristic polynomial of the Perrin sequence being $x^3 - x - k = 0$ and Leonardo's being $z^3 - 2kz^2 + 1 = 0$.

Definition 2.1. *The hyperbolic quaternions of k-Perrin and k-Leonardo are given by:*

$$\mathbb{H}Pe_{k,n} = Pe_{k,n} + iPe_{k,n+1} + jPe_{k,n+2} + kPe_{k,n+3},$$
$$\mathbb{H}Le_{k,n} = Le_{k,n} + iLe_{k,n+1} + jLe_{k,n+2} + kLe_{k,n+3},$$

on what $i^2 = j^2 = k^2 = -1$,$ij = k = -ji$,$jk = i = -kj$,$ki = j = -ik$.

According to these definitions, we will carry out a study around the addition, subtraction and multiplication operations of the hyperbolic k-Perrin quaternions.

$$\begin{aligned}\mathbb{H}Pe_{k,n} \pm \mathbb{H}Pe_{k,m} = (Pe_{k,n} \pm Pe_{k,m}) + i(Pe_{k,n+1} \pm Pe_{k,m+1}) + \\ j(Pe_{k,n+2} \pm Pe_{k,m+2}) + k(Pe_{k,n+3} \pm Pe_{k,m+3}),\end{aligned}$$

$$
\begin{aligned}
\mathbb{H}Pe_{k,n}\mathbb{H}Pe_{k,m} = {} & (Pe_{k,n}Pe_{k,m} - Pe_{k,n+1}Pe_{k,m+1} - Pe_{k,n+2}Pe_{k,m+2} - \\
& Pe_{k,n+3}Pe_{k,m+3}) + \\
& i(Pe_{k,n}Pe_{k,m+1} + Pe_{k,n+1}Pe_{k,m} + Pe_{k,n+2}Pe_{k,m+3} - \\
& Pe_{k,n+3}Pe_{k,m+2}) + j(Pe_{k,n}Pe_{k,m+2} + Pe_{k,n+2}Pe_{k,m} - \\
& Pe_{k,n+1}Pe_{k,m+3} + Pe_{k,n+3}Pe_{k,m+1}) + k(Pe_{k,n}Pe_{k,m+3} + \\
& Pe_{k,n+3}Pe_{k,m} + Pe_{k,n+1}Pe_{k,m+2} - Pe_{k,n+2}Pe_{k,m+1}) \\
& \neq \mathbb{H}Pe_{k,m}\mathbb{H}Pe_{k,n}
\end{aligned}
$$

The addition, subtraction and multiplication operations of the hyperbolic k-Leonardo quaternions are performed in similar ways.

The conjugates of the hyperbolic quaternary numbers of k-Perrin and k-Leonardo are represented by:

$$
\begin{aligned}
\overline{\mathbb{H}Pe}_{k,n} &= Pe_{k,n} - iPe_{k,n+1} - jPe_{k,n+2} - kPe_{k,n+3}, \\
\overline{\mathbb{H}Le}_{k,n} &= Le_{k,n} - iLe_{k,n+1} - jLe_{k,n+2} - kLe_{k,n+3}.
\end{aligned}
$$

The norms of the hyperbolic quaternary numbers of k-Perrin and k-Leonardo are represented by:

$$
\begin{aligned}
\|\mathbb{H}Pe_{k,n}\|^2 &= \mathbb{H}Pe_{k,n}\overline{\mathbb{H}Pe}_{k,n} \\
&= Pe^2_{k,n} + Pe^2_{k,n+1} + Pe^2_{k,n+2} + Pe^2_{k,n+3} \\
\|\mathbb{H}Le_{k,n}\|^2 &= \mathbb{H}Le_{k,n}\overline{\mathbb{H}Le}_{k,n} \\
&= Le^2_{k,n} + Le^2_{k,n+1} + Le^2_{k,n+2} + Le^2_{k,n+3}
\end{aligned}
$$

Theorem 2.2. *Let $Pe_{k,n}$ be the n-th term of the sequence of k-Perrin and $Le_{k,n}$ the n-th term of the sequence of k-Leonardo, $\mathbb{H}Pe_{k,n}$ the n-th term of the hyperbolic quaternionic sequence of k-Perrin and $\mathbb{H}Le_{k,n}$ the n-th term of the hyperbolic quaternionic sequence of k-Lerrin, we have* *that for $n \geqslant 1$ the following relations are given:*

$$
\begin{aligned}
&(i)\mathbb{H}Pe_{k,n+3} = \mathbb{H}Pe_{k,n+1} + k\mathbb{H}Pe_{k,n};\\
&(ii)\mathbb{H}Pe_{k,n} - i\mathbb{H}Pe_{k,n+1} - j\mathbb{H}Pe_{k,n+2} -\\
&\quad k\mathbb{H}Pe_{k,n+3} = Pe_{k,n} + Pe_{k,n+2} + Pe_{k,n+4} + Pe_{k,n+6};\\
&(iii)\mathbb{H}Le_{k,n+1} = 2k\mathbb{H}Le_{k,n} - \mathbb{H}Le_{k,n-2};\\
&(iv)\mathbb{H}Le_{k,n} - i\mathbb{H}Le_{k,n+1} + j\mathbb{H}Le_{k,n+2} -\\
&\quad k\mathbb{H}Le_{k,n+3} = Le_{k,n} + Le_{k,n+2} + Le_{k,n+4} + Le_{k,n+6}.
\end{aligned}
$$

Proof. (i) Based on Definition 2.1, we have to:

$$
\begin{aligned}
\mathbb{H}Pe_{k,n+1} + k\mathbb{H}Pe_{k,n} &= Pe_{k,n+1} + iPe_{k,n+2} + jPe_{k,n+3} + kPe_{k,n+4}\\
&+ k(Pe_{k,n} + iPe_{k,n+1} + jPe_{k,n+2} + kPe_{k,n+3})\\
&= (Pe_{k,n+1} + kPe_{k,n}) + i(Pe_{k,n+2} + kPe_{k,n+1})\\
&+ j(Pe_{k,n+3} + kPe_{k,n+2}) + k(Pe_{k,n+4} + kPe_{k,n+3})\\
&= Pe_{k,n+3} + iPe_{k,n+4} + jPe_{k,n+5} + kPe_{k,n+6}\\
&= \mathbb{H}Pe_{k,n+3}
\end{aligned}
$$

For (ii), we have to:

$$
\begin{aligned}
\mathbb{H}Pe_{k,n} - i\mathbb{H}Pe_{k,n+1} + j\mathbb{H}Pe_{k,n+2} - k\mathbb{H}Pe_{k,n+3} &= Pe_{k,n} + iPe_{k,n+1} +\\
&jPe_{k,n+2} + kPe_{k,n+3} -\\
&i(Pe_{k,n+1} + iPe_{k,n+2} +\\
&jPe_{k,n+3} + kPe_{k,n+4}) +\\
&j(Pe_{k,n+2} + iPe_{k,n+3} +\\
&jPe_{k,n+4} + kPe_{k,n+5}) -\\
&k(Pe_{k,n+3} + iPe_{k,n+4} +\\
&jPe_{k,n+5} + kPe_{k,n+6})\\
&= Pe_{k,n} + Pe_{k,n+2} -\\
&kPe_{k,n+3} + jPe_{k,n+4} +\\
&kPe_{k,n+3} + Pe_{k,n+4} -\\
&iPe_{k,n+5} - jPe_{k,n+4} +\\
&iPe_{k,n+5} + Pe_{k,n+6}\\
&= Pe_{k,n} + Pe_{k,n+2} +\\
&Pe_{k,n+4} + Pe_{k,n+6}
\end{aligned}
$$

The statements of items (iii) and (iv) are carried out in a similar way. □

Theorem 2.3. *The generating functions of the hyperbolic quaternary numbers of k-Perrin and k-Leonardo, denoted by $G_{\mathbb{H}Pe_{k,n}}(x)$ and $G_{\mathbb{H}Le_n}(x)$, are:*

$$G_{\mathbb{H}Pe_{k,n}}(x) = \frac{\mathbb{H}Pe_{k,0} + \mathbb{H}Pe_{k,1}x + (\mathbb{H}Pe_{k,2} - \mathbb{H}Pe_{k,0})x^2}{1 - x^2 - kx^3}, \tag{3}$$

$$G_{\mathbb{H}Le_n}(x) = \frac{(\mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x)(1 - 2kx) + \mathbb{H}Le_{k,2}x^2}{1 - 2kx + x^3}. \tag{4}$$

Proof. To define the generating function of the hyperbolic quaternary number k-Perrin (3) let's assume the function:

$$G_{\mathbb{H}Pe_{k,n}}(x) = \sum_{n=0}^{\infty} \mathbb{H}Pe_{k,n}x^n = \mathbb{H}Pe_{k,0} + \mathbb{H}Pe_{k,1}x + \mathbb{H}Pe_{k,2}x^2 + \cdots + \mathbb{H}Pe_{k,n}x^n + \cdots$$

Multiplying both members of equality by $-x^2$ and $-kx^3$:

$$\begin{aligned} -x^2G_{\mathbb{H}Pe_{k,n}}(x) &= -x^2\mathbb{H}Pe_{k,0} - x^3\mathbb{H}Pe_{k,1} - x^4\mathbb{H}Pe_{k,2} - \cdots - \\ &\quad x^{n+2}\mathbb{H}Pe_{k,n} - \cdots \\ -kx^3G_{\mathbb{H}Pe_{k,n}}(x) &= -kx^3\mathbb{H}Pe_{k,0} - kx^4\mathbb{H}Pe_{k,1} - kx^5\mathbb{H}Pe_{k,2} - \cdots - \\ &\quad kx^{n+3}\mathbb{H}Pe_{k,n} - \cdots \end{aligned}$$

So, writing $(1 - x^2 - kx^3)G_{\mathbb{H}Pe_{k,n}}(x)$:

$$\begin{aligned} (1 - x^2 - kx^3)G_{\mathbb{H}Pe_{k,n}}(x) &= \mathbb{H}Pe_{k,0} + \mathbb{H}Pe_{k,1}x + (\mathbb{H}Pe_{k,2} - \mathbb{H}Pe_{k,0})x^2 \\ G_{\mathbb{H}Pe_{k,n}}(x) &= \frac{\mathbb{H}Pe_{k,0} + \mathbb{H}Pe_{k,1}x + (\mathbb{H}Pe_{k,2} - \mathbb{H}Pe_{k,0})x^2}{1 - x^2 - kx^3} \end{aligned}$$

To define the generating function of the quaternary number k-Leonardo hyperbolic (4), denoted by $G_{\mathbb{H}Le_{k,n}}(x)$, let's write a sequence where each term in the sequence corresponds to the coefficients.

$$G_{\mathbb{H}Le_{k,n}}(x) = \sum_{n=0}^{\infty} \mathbb{H}Le_{k,n}x^n$$

Making algebraic manipulations due to the recurrence relation we can write this sequence as:

$$\begin{aligned}
G_{\mathbb{H}Le_{k,n}}(x) &= \mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x + \mathbb{H}Le_{k,2}x^2 + \sum_{n=3}^{\infty} \mathbb{H}Le_{k,n}x^n \\
&= \mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x + \mathbb{H}Le_{k,2}x^2 + \\
&\quad \sum_{n=3}^{\infty}(2k\mathbb{H}Le_{k,n-1} - \mathbb{H}Le_{k,n-3})x^n \\
&= \mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x + \mathbb{H}Le_{k,2}x^2 + 2kx\sum_{n=3}^{\infty} \mathbb{H}Le_{k,n-1}x^{n-1} \\
&\quad -x^3\sum_{n=3}^{\infty} \mathbb{H}Le_{k,n-3}x^{n-3} \\
&= \mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x + \mathbb{H}Le_{k,2}x^2 + \\
&\quad 2kx\left(\sum_{n=0}^{\infty}[\mathbb{H}Le_{k,n}x^n] - \mathbb{H}Le_{k,0} - \mathbb{H}Le_{k,1}x\right) - \\
&\quad x^3\sum_{n=0}^{\infty} \mathbb{H}Le_{k,n}x^n \\
&= \mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x + \mathbb{H}Le_{k,2}x^2 - 2kx\mathbb{H}Le_{k,0} - \\
&\quad 2kx^2\mathbb{H}Le_{k,1} + 2kx\sum_{n=0}^{\infty} \mathbb{H}Le_{k,n}x^n - x^3\sum_{n=0}^{\infty} \mathbb{H}Le_{k,n}x^n
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x + \mathbb{H}Le_{k,2}x^2 - 2kx\mathbb{H}Le_{k,0} - 2kx^2\mathbb{H}Le_{k,1} + \\
&\quad 2kxG_{\mathbb{H}Le_{k,n}} - t^3G_{\mathbb{H}Le_{k,n}}
\end{aligned}$$

So, we have:

$$\begin{aligned}
G_{\mathbb{H}Le_{k,n}}(x) - 2kxG_{\mathbb{H}Le_{k,n}} + x^3G_{\mathbb{H}Le_{k,n}} &= \mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x + \mathbb{H}Le_{k,2}x^2 \\
&\quad -2kx\mathbb{H}Le_{k,0} - 2kx^2\mathbb{H}Le_{k,1}
\end{aligned}$$

$$\begin{aligned}
G_{\mathbb{H}Le_{k,n}}(x)(1-2kx+x^3) &= \mathbb{H}Le_{k,0}(1-2kx) + \mathbb{H}Le_{k,1}x(1-2kx) + \\
&\quad \mathbb{H}Le_{k,2}x^2 \\
G_{\mathbb{H}Le_{k,n}}(x) &= \frac{(\mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}x)(1-2kx) + \mathbb{H}Le_{k,2}x^2}{1-2kx+x^3}.
\end{aligned}$$

□

Theorem 2.4. *For $n \geq 0$, we have that Binet's formula for the hyperbolic quaternary numbers of k-Perrin and k-Leonardo are:*

$$\mathbb{H}Pe_{k,n} = \varphi(l_1)^n + \omega(l_2)^n + \theta(l_3)^n,$$

$$\mathbb{H}Le_{k,n} = \alpha(x_1)^n + \beta(x_2)^n + \gamma(x_3)^n.$$

where l_1, l_2 and l_3 are the roots of the characteristic equation of the quaternionic sequence of k-Perrin hyperbolic and x_1, x_2, x_3 the roots of the characteristic equation of the quaternionic sequence of k-Leonardo hyperbolic and φ, ω, θ, α, β and γ the coefficients equal to:

$$\begin{aligned}
\varphi &= \frac{\mathbb{H}Pe_{k,0}(l_2l_3^2 - l_2^2l_3) + \mathbb{H}Pe_{k,1}(l_2^2 - l_3^2) + \mathbb{H}Pe_{k,2}(l_3 - l_2)}{l_3l_1^2 + l_3^2l_2 + l_1l_2^2 - l_3^2l_1 - l_1^2l_2 - l_3l_2^2};\\
\omega &= \frac{\mathbb{H}Pe_{k,0}(l_3l_1^2 - l_3^2l_1) + \mathbb{H}Pe_{k,1}(l_3^2 - l_1^2) + \mathbb{H}Pe_{k,2}(l_1 - l_3)}{l_3l_1^2 + l_3^2l_2 + l_1l_2^2 - l_3^2l_1 - l_1^2l_2 - l_3l_2^2};\\
\theta &= \frac{\mathbb{H}Pe_{k,0}(l_1l_2^2 - l_1^2l_2) + \mathbb{H}Pe_{k,1}(l_1^2 - l_2^2) + \mathbb{H}Pe_{k,2}(l_2 - l_1)}{l_3l_1^2 + l_3^2l_2 + l_1l_2^2 - l_3^2l_1 - l_1^2l_2 - l_3l_2^2};\\
\alpha &= \frac{\mathbb{H}Le_{k,2} + (-x_2 - x_3)\mathbb{H}Le_{k,1} + (x_2x_3)\mathbb{H}Le_{k,0}}{x_1^2 - x_1x_2 - x_1x_3 + x_2x_3};\\
\beta &= \frac{\mathbb{H}Le_{k,2} + (-x_1 - x_3)\mathbb{H}Le_{k,1} + (x_1x_3)\mathbb{H}Le_{k,0}}{x_2^2 - x_2x_3 - x_1x_2 + x_1x_3};\\
\gamma &= \frac{\mathbb{H}Le_{k,2} + (-x_1 - x_2)\mathbb{H}Le_{k,1} + (x_1x_2)\mathbb{H}Le_{k,0}}{x_3^2 + x_1x_2 - x_1x_3 - x_2x_3}.
\end{aligned}$$

Proof. Binet's formula can be represented as follows [9]:

$$\mathbb{H}Pe_{k,n} = \varphi(l_1)^n + \omega(l_2)^n + \theta(l_3)^n.$$

So, one has to $n = 0$, get $\varphi + \omega + \theta = \mathbb{H}Pe_{k,0}$, for $n = 1$, got up $\varphi l_1 + \omega l_2 + \theta l_3 = \mathbb{H}Pe_{k,1}$ and for $n = 2$, has $\varphi l_1^2 + \omega l_2^2 + \theta l_3^2 = \mathbb{H}Pe_{k,2}$. With that, you can build a system of linear equations as follows:

$$\begin{cases} \varphi + \omega + \theta = \mathbb{H}Pe_{k,0} \\ \varphi l_1 + \omega l_2 + \theta l_3 = \mathbb{H}Pe_{k,1} \\ \varphi l_1^2 + \omega l_2^2 + \theta l_3^2 = \mathbb{H}Pe_{k,2} \end{cases}$$

Solving the linear system, using Cramer's rule, the coefficients found were: $\varphi = \dfrac{\mathbb{H}Pe_{k,0}(l_2l_3^2 - l_2^2l_3) + \mathbb{H}Pe_{k,1}(l_2^2 - l_3^2) + \mathbb{H}Pe_{k,2}(l_3 - l_2)}{l_3l_1^2 + l_3^2l_2 + l_1l_2^2 - l_3^2l_1 - l_1^2l_2 - l_3l_2^2}$,
$\omega = \dfrac{\mathbb{H}Pe_{k,0}(l_3l_1^2 - l_3^2l_1) + \mathbb{H}Pe_{k,1}(l_3^2 - l_1^2) + \mathbb{H}Pe_{k,2}(l_1 - l_3)}{l_3l_1^2 + l_3^2l_2 + l_1l_2^2 - l_3^2l_1 - l_1^2l_2 - l_3l_2^2}$ and
$\theta = \dfrac{\mathbb{H}Pe_{k,0}(l_1l_2^2 - l_1^2l_2) + \mathbb{H}Pe_{k,1}(l_1^2 - l_2^2) + \mathbb{H}Pe_{k,2}(l_2 - l_1)}{l_3l_1^2 + l_3^2l_2 + l_1l_2^2 - l_3^2l_1 - l_1^2l_2 - l_3l_2^2}$

The proof for the quaternary number of k-Leonardo hyperbolic is carried out in an analogous way. $\square$

3 Properties of Hyperbolic k-Perrin and k-Leonardo Quaternions

Next, some properties inherent to the hyperbolic quaternary numbers of k-Perrin and k-Leonardo are studied.

Propriety 3.1. *The sum of the n first quaternary numbers of hyperbolic k-Leonardo is given by:*

$$\begin{aligned}\sum_{m=3}^{n} \mathbb{H}Le_{k,m} &= 2k\mathbb{H}Le_{k,n-2} + 2k\mathbb{H}Le_{k,n-1} - (\mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}) + \\ &\quad (2k-1)\sum_{s=2}^{n-3} \mathbb{H}Le_{k,s}.\end{aligned}$$

Proof. Using the recurrence relation of the hyperbolic k-Leonardo quaternions, with $n \in \mathbb{N}$, we have that:

$$\mathbb{H}Le_{k,n+1} = 2k\mathbb{H}Le_{k,n} - \mathbb{H}Le_{k,n-2} \tag{5}$$

Thus, evaluating the relationship given in Equation (5), in values of $n \geq 2$, we get:

$$\begin{array}{rcl} \mathbb{H}Le_{k,3} & = & 2k\mathbb{H}Le_{k,2} - \mathbb{H}Le_{k,0} \\ \mathbb{H}Le_{k,4} & = & 2k\mathbb{H}Le_{k,3} - \mathbb{H}Le_{k,1} \\ \mathbb{H}Le_{k,5} & = & 2k\mathbb{H}Le_{k,4} - \mathbb{H}Le_{k,2} \\ \mathbb{H}Le_{k,6} & = & 2k\mathbb{H}Le_{k,5} - \mathbb{H}Le_{k,3} \\ \mathbb{H}Le_{k,7} & = & 2k\mathbb{H}Le_{k,6} - \mathbb{H}Le_{k,4} \\ \vdots & & \\ \mathbb{H}Le_{k,n-2} & = & 2k\mathbb{H}Le_{k,n-3} - \mathbb{H}Le_{k,n-5} \\ \mathbb{H}Le_{n-1} & = & 2k\mathbb{H}Le_{k,n-2} - \mathbb{H}Le_{k,n-4} \\ \mathbb{H}Le_{k,n} & = & 2k\mathbb{H}Le_{k,n-1} - \mathbb{H}Le_{k,n-3} \end{array}$$

Through successive cancellations, the following results are obtained:

$$\begin{array}{rcl} \displaystyle\sum_{m=3}^{n} \mathbb{H}Le_{k,m} & = & (2k-1)\mathbb{H}Le_{k,2} - \mathbb{H}Le_{k,0} + (2k-1)\mathbb{H}Le_{k,3} \\ & & -\mathbb{H}Le_{k,1} + (2k-1)\mathbb{H}Le_{k,4} + \cdots + (2k-1)\mathbb{H}Le_{k,n-3} + \\ & & (2k-1)\mathbb{H}Le_{k,n-3} + 2k\mathbb{H}Le_{k,n-2} + 2k\mathbb{H}Le_{k,n-1} \\ & = & 2k\mathbb{H}Le_{k,n-2} + 2k\mathbb{H}Le_{k,n-1} - (\mathbb{H}Le_{k,0} + \mathbb{H}Le_{k,1}) + \\ & & (2k-1)\displaystyle\sum_{s=2}^{n-3} \mathbb{H}Le_{k,s}. \end{array}$$

□

Propriety 3.2. *The sum of the even indices numbers of the hyperbolic k-Leonardo quaternions is given by:*

$$\sum_{m=3}^{n} \mathbb{H}Le_{k,2m} = 2k\mathbb{H}Le_{k,2n-1} - \mathbb{H}Le_{k,1} + (2k-1)\sum_{s=3}^{2n-3} \mathbb{H}Le_{k,s}.$$

Proof. Using the recurrence relation of the hyperbolic k-Leonardo quaternions, with $n \in \mathbb{N}$, we have that:

$$\mathbb{H}Le_{k,n+1} = 2k\mathbb{H}Le_{k,n} - \mathbb{H}Le_{k,n-2}$$

Thus, evaluating the recurrence relation, in values of $n \geq 2$, we obtain:

$$\begin{aligned}
\mathbb{H}Le_{k,4} &= 2k\mathbb{H}Le_{k,3} - \mathbb{H}Le_{k,1}\\
\mathbb{H}Le_{k,6} &= 2k\mathbb{H}Le_{k,5} - \mathbb{H}Le_{k,3}\\
\mathbb{H}Le_{k,8} &= 2k\mathbb{H}Le_{k,7} - \mathbb{H}Le_{k,5}\\
&\vdots\\
\mathbb{H}Le_{k,2n-2} &= 2k\mathbb{H}Le_{k,2n-3} - \mathbb{H}Le_{k,2n-5}\\
\mathbb{H}Le_{k,2n} &= 2k\mathbb{H}Le_{k,2n-1} - \mathbb{H}Le_{k,2n-3}
\end{aligned}$$

Through successive cancellations, the following results are obtained:

$$\begin{aligned}
\sum_{m=2}^{n} \mathbb{H}Le_{k,2m} &= (2k-1)\mathbb{H}Le_{k,3} - \mathbb{H}Le_{k,1} + (2k-1)\mathbb{H}Le_{k,5} + \cdots +\\
&\quad (2k-1)\mathbb{H}Le_{k,2n-3} + 2k\mathbb{H}Le_{k,2n-1}\\
&= 2k\mathbb{H}Le_{k,2n-1} - \mathbb{H}Le_{k,1} + (2k-1)\sum_{s=3}^{2n-3} \mathbb{H}Le_{k,s}.
\end{aligned}$$

□

Propriety 3.3. *The sum of the odd index numbers of the hyperbolic k-Leonardo quaternions is given by:*

$$\sum_{m=2}^{n} \mathbb{H}Le_{k,2m-1} = 2k\mathbb{H}Le_{k,2n-2} - \mathbb{H}Le_{k,0} + (2k-1)\sum_{s=2}^{2n-4} \mathbb{H}Le_{k,s}.$$

Proof. Using the recurrence relation of the hyperbolic k-Leonardo quaternions, with $n \in \mathbb{N}$, we have that:

$$\mathbb{H}Le_{k,n+1} = 2k\mathbb{H}Le_{k,n} - \mathbb{H}Le_{k,n-2}$$

Thus, evaluating the recurrence relationship, in values of $n \geq 2$, we get:

$$\begin{aligned}
\mathbb{H}Le_{k,3} &= 2k\mathbb{H}Le_{k,2} - \mathbb{H}Le_{k,0}\\
\mathbb{H}Le_{k,5} &= 2k\mathbb{H}Le_{k,4} - \mathbb{H}Le_{k,2}\\
\mathbb{H}Le_{k,7} &= 2k\mathbb{H}Le_{k,6} - \mathbb{H}Le_{k,4}\\
&\vdots\\
\mathbb{H}Le_{k,2n-3} &= 2k\mathbb{H}Le_{k,2n-4} - \mathbb{H}Le_{k,2n-6}\\
\mathbb{H}Le_{k,2n-1} &= 2k\mathbb{H}Le_{k,2n-2} - \mathbb{H}Le_{k,2n-4}
\end{aligned}$$

Through successive cancellations, the following results are obtained:

$$\begin{aligned}\sum_{m=2}^{n} \mathbb{H}Le_{k,2m-1} &= (2k-1)\mathbb{H}Le_{k,2} - \mathbb{H}Le_{k,0} + (2k-1)\mathbb{H}Le_{k,4} + \cdots + \\ &\quad (2k-1)\mathbb{H}Le_{k,2n-4} + 2k\mathbb{H}Le_{k,2n-2} \\ &= 2k\mathbb{H}Le_{k,2n-2} - \mathbb{H}Le_{k,0} + (2k-1)\sum_{s=2}^{2n-4} \mathbb{H}Le_{k,s}.\end{aligned}$$

□

For the properties of the hyperbolic k-Perrin quaterns, we will use the recurrence relation of the hyperbolic k-Perrin quaterns, with $n \in \mathbb{N}$, we have that:

$$\mathbb{H}Pe_{k,n} = \mathbb{H}Pe_{k,n-2} + k\mathbb{H}Pe_{k,n-3}$$

We can reorganize it and present it as:

$$\mathbb{H}Pe_{k,n-2} = \mathbb{H}Pe_{k,n+1} - k\mathbb{H}Pe_{k,n-1} \tag{6}$$

Thus, evaluating the relationship given in Equation (6), in values of $n \geq 3$, we obtain:

$$\begin{aligned}\mathbb{H}Pe_{k,1} &= \mathbb{H}Pe_{k,4} - k\mathbb{H}Pe_{k,2} \\ \mathbb{H}Pe_{k,2} &= \mathbb{H}Pe_{k,5} - k\mathbb{H}Pe_{k,3} \\ \mathbb{H}Pe_{k,3} &= \mathbb{H}Pe_{k,6} - k\mathbb{H}Pe_{k,4} \\ \mathbb{H}Pe_{k,4} &= \mathbb{H}Pe_{k,7} - k\mathbb{H}Pe_{k,5} \\ &\vdots \\ \mathbb{H}Pe_{k,2n-1} &= \mathbb{H}Pe_{k,2n+2} - k\mathbb{H}Pe_{k,2n} \\ \mathbb{H}Pe_{k,2n} &= \mathbb{H}Pe_{k,2n+3} - k\mathbb{H}Pe_{k,2n+1} \\ \mathbb{H}Pe_{k,2n+1} &= \mathbb{H}Pe_{k,2n+4} - k\mathbb{H}Pe_{k,2n+2}\end{aligned}$$

Propriety 3.4. *The sum of even-index hyperbolic k-Perrin quaternary numbers can be described as:*

$$\begin{aligned}\sum_{m=1}^{n} \mathbb{H}Pe_{k,2m} &= (1-k)\mathbb{H}Pe_{k,5} - k\mathbb{H}Pe_{k,3} + \cdots + (1-k)\mathbb{H}Pe_{k,2n+1} + \\ &\quad \mathbb{H}Pe_{k,2n+3}.\end{aligned}$$

Proof. Through successive cancellations, the following results are obtained:

$$\begin{aligned}
\sum_{m=1}^{n} \mathbb{H}Pe_{k,2m} &= \mathbb{H}Pe_{k,2} + \mathbb{H}Pe_{k,4} + \cdots + \mathbb{H}Pe_{k,2n} \\
&= (\mathbb{H}Pe_{k,5} - k\mathbb{H}Pe_{k,3}) + (\mathbb{H}Pe_{k,7} - k\mathbb{H}Pe_{k,5}) + \cdots + \\
&\quad (\mathbb{H}Pe_{k,2n+3} - k\mathbb{H}Pe_{k,2n+1}) \\
&= (1-k)\mathbb{H}Pe_{k,5} - k\mathbb{H}Pe_{k,3} + \cdots + (1-k)\mathbb{H}Pe_{k,2n+1} + \\
&\quad \mathbb{H}Pe_{k,2n+3}.
\end{aligned}$$

□

Propriety 3.5. *The sum of odd-index hyperbolic k-Perrin quaternary numbers can be described as:*

$$\begin{aligned}
\sum_{m=1}^{n} \mathbb{H}Pe_{k,2m-1} &= (1-k)\mathbb{H}Pe_{k,4} - k\mathbb{H}Pe_{k,2} + \cdots + (1-k)\mathbb{H}Pe_{k,2n} + \\
&\quad \mathbb{H}Pe_{k,2n+2}.
\end{aligned}$$

Proof. Through successive cancellations, the following results are obtained:

$$\begin{aligned}
\sum_{m=1}^{n} \mathbb{H}Pe_{k,2m} &= \mathbb{H}Pe_{k,1} + \mathbb{H}Pe_{k,3} + \cdots + \mathbb{H}Pe_{k,2n-1} \\
&= (\mathbb{H}Pe_{k,4} - k\mathbb{H}Pe_{k,2}) + (\mathbb{H}Pe_{k,6} - k\mathbb{H}Pe_{k,4}) + \cdots + \\
&\quad (\mathbb{H}Pe_{k,2n+2} - k\mathbb{H}Pe_{k,2n}) \\
&= (1-k)\mathbb{H}Pe_{k,4} - k\mathbb{H}Pe_{k,2} + \cdots + (1-k)\mathbb{H}Pe_{k,2n} + \\
&\quad \mathbb{H}Pe_{k,2n+2}.
\end{aligned}$$

□

Propriety 3.6. *The sum of the first n terms of the hyperbolic k-Perrin quaternions is given by:*

$$\begin{aligned}
\sum_{m=1}^{n} \mathbb{H}Pe_{k,m} &= (1-k)(\mathbb{H}Pe_{k,4} + \mathbb{H}Pe_{k,5}) - k(\mathbb{H}Pe_{k,2} + \mathbb{H}Pe_{k,3}) + \cdots + \\
&\quad (1-k)\mathbb{H}Pe_{k,2n+1} + \mathbb{H}Pe_{k,2n+3}.
\end{aligned}$$

Proof. Through successive cancellations, the following results are obtained:

$$\begin{aligned}
\sum_{m=1}^{n} \mathbb{H}Pe_{k,m} &= \mathbb{H}Pe_{k,1} + \mathbb{H}Pe_{k,2} + \mathbb{H}Pe_{k,3} + \cdots + \mathbb{H}Pe_{k,n} \\
&= (\mathbb{H}Pe_{k,4} - k\mathbb{H}Pe_{k,2}) + (\mathbb{H}Pe_{k,5} - k\mathbb{H}Pe_{k,3}) + \\
&\quad (\mathbb{H}Pe_{k,6} - k\mathbb{H}Pe_{k,4}) + \cdots + (\mathbb{H}Pe_{k,2n+3} - \\
&\quad k\mathbb{H}Pe_{k,2n+1}) \\
&= (1-k)\mathbb{H}Pe_{k,4} - k\mathbb{H}Pe_{k,2} + (1-k)\mathbb{H}Pe_{k,5} - \\
&\quad k\mathbb{H}Pe_{k,3} + \cdots + (1-k)\mathbb{H}Pe_{k,2n+1} + \mathbb{H}Pe_{k,2n+3} \\
&= (1-k)(\mathbb{H}Pe_{k,4} + \mathbb{H}Pe_{k,5}) - k(\mathbb{H}Pe_{k,2} + \mathbb{H}Pe_{k,3}) + \\
&\quad \cdots + (1-k)\mathbb{H}Pe_{k,2n+1} + \mathbb{H}Pe_{k,2n+3}.
\end{aligned}$$

□

4 Conclusion

The study discussed allowed the introduction of the hyperbolic quaternary sequences of k-Perrin and k-Leonardo, thus carrying out an evolution in light of the mathematical complexification process of these generalized sequences. As soon as possible, it was possible to discuss some mathematical properties and theorems, showing the mathematical rigor of the primitive sequences, Perrin and Leonardo.

In fact, it can be noted that for the values of $k = 1$, we get the Perrin and Leonardo sequences in their primitive form. Furthermore, to emphasize the generalization of these hyperbolic quaternionic sequences, their shape for the recurrence k and their respective properties were studied.

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