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Independent and Absorbent Subsets of BI-Algebras

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Abstract. In this paper, we define the notion of the *right independent* (resp., *left independent*) subsets of BI-algebras. Some of the properties are investigated and get more results in BI-algebras. Moreover, we consider the notion of the *right absorbent* (resp., *left absorbent*) subset. It is proved that in a right distributive BI-algebra X, every right(left) independent subset of X absorbs X from the right. We show that these new concepts are different by presenting several examples. The goal and benefits of our proposed extension of this study are to extend the theory of BI-algebras, and so we enlarge the field of research.

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1 Introduction

J.C. Abbott introduced a class of abstract algebras: *implication algebra* in the sake to formalize the logical connective implication in the classical propositional logic ([1]). W.Y. Chen et al. ([5]) proved that in any implication algebra $(X;*)$ the identity $x*x=y*y$ holds for all $x,y\in X$. We denote the identity $x*x=y*y$ by the constant 0. The notion of BCK-*algebras* was introduced by Y. Imai and K. Iséki ([9]). S. Tanaka introduced an essential class of BCK-algebras, called *commutative* BCK-algebras, which forms a class of lower semilattices [22] (see also [23], [24]). J. Meng showed that implication algebras are dual to *implicative* BCK-algebras ([15]). A. Iorgulescu introduced many interesting generalizations of BCI/BCK-algebras, and the basic properties of such algebras are studied in [10] (see also [11]). A. Walendziak investigated the property of commutativity and implicativity for various generalizations of BCK-algebras ([25], [26]). L.C. Ciungu in [6] defined and investigated some classes of L-algebras and proved equivalent conditions for commutative KL-algebras and CL-algebras. H.S. Kim et al. ([13]) introduced BE-algebras as an extension of commutative BCK-algebras. The notion of d/B-algebras was introduced by J. Neggers and H.S. Kim ([16], [17]). N. Galatos et al. discussed on the generalized bunched implication algebras as residuated lattices with a Heyting implication, and they investigated the relation between Boolean algebras with operators and lattices with operators ([7]). In 2017, A. Borumand Saeid et al. ([4]) introduced BI-*algebras* as an extension of both a (dual) implication algebra and an implicative BCK-algebra, and they investigated some ideals and congruence relations. They showed that every implicative BCK-algebra is a BI-algebra, but the converse is not valid in general. Then R.K. Bandaru introduce the concept of a QI-*algebra*, which is a generalization of a BI-algebra, and gave the relation between ideals and congruence kernels whenever a QI-algebra is distributive ([3]). S.S. Ahn et al. discussed normal subalgebras in BI-algebras and obtained several conditions for obtaining BI-algebra on the non-negative real numbers by using an analytic method ([2]). In 2022, A. Rezaei and S. Soleymani ([21]) defined state ideals on a BI-algebra and gave a characterization of the least state ideal of a BI-algebra. In [18], the authors defined and studied the concept of a (*branch-wise*) *commutative*

BI-algebra and showed that commutative BI-algebras form a class of *lower semilattices.* Recently, A. Rezaei et al. ([19], [20]) defined the notion of a (strongly) right(left) independent subset of a groupoid, and obtain a groupoid having a strongly right(left) independent doubleton's. Moreover, they discussed the notion of dynamic elements with independence. The motivation of this study came from the idea of the converse of *"injective function"*, and then we define the notion of the *right*(*left*) *independent* subsets of BI-algebras. Additionally, new some the properties of BI-algebras are investigated. Moreover, we introduce the notion of the *right*(*left*) *absorbent* subsets of BI-algebras. It is proved that a right distributive BI-algebra X, every right(left) independent subset of X absorbs X from the right. We show that these new concepts are different by presenting several examples.

2 Preliminaries

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

Notice that there are several axiom systems for BCI-algebras. In this paper, we will adopt the following axiom system, introduced by H.S. Li in 1985 (see [14]). An algebra $(X, *, 0)$ of type $(2, 0)$ (i.e. a non-empty set with a binary operation $*$ and a constant 0) is called a BCI-algebra if it satisfies the following axioms (for all $x, y, z \in X$):
(BCI$_1$) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI$_2$) $x * 0 = x$,
(BCI$_3$) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCI-algebra $(X, *, 0)$ is called a BCK-algebra, if it satisfies the following axiom:
(BCK) $0 * x = 0$, for all $x \in X$.

Y.B. Jun et al. ([12]) introduced the notion of a BH-algebra which is a generalization of a $BCK/BCI/BCH$-algebra. An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BH-algebra if it satisfies (BCI$_2$) and the following axioms (for all $x, y \in X$):
(B) $x * x = 0$,
(BH) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Recall that a *BI-algebra* ([4]) is an algebra $(X; *, 0)$ of type $(2, 0)$

satisfies (B) and the following axiom (for all $x, y \in X$):
(BI) $x * (y * x) = x$.

Let $(X; *, 0)$ be a BI-algebra. We introduce a relation $\leq$ on X by $x \leq y$ if and only if $x * y = 0$.

Notice that $\leq$ is not a partially ordered set (poset), but it is only *reflexive*. A BI-algebra X is said to be *right distributive* if it satisfies $(x*y)*z = (x*z)*(y*z)$ for all $x, y, z \in X$ ([4]). A BI-algebra X is said to be *commutative* if it satisfies $x * (x * y) = y * (y * x)$ for all $x, y \in X$ ([18]). As usual, a map $f : X \longrightarrow Y$, where $(X, *, 0)$ and $(Y, \circ, 0)$ are BI-algebras, is called a homomorphism if $f(x * y) = f(x) \circ f(y)$ for any $x, y \in X$. If f is onto (resp., one to one), then f is called an epimorphism (resp., monomorphism). Moreover, if f is a bijection, then f is called an isomorphism.

In what follows, let X denote a BI-algebra unless otherwise specified.

From ([4]) we have (for all $x, y, z, u \in X$):
(p_1) $x * 0 = x$,
(p_2) $0 * x = 0$,
(p_3) $x * y = (x * y) * y$,
(p_4) if $y * x = x$, then $X = \{0\}$,
(p_5) if $x * (y * z) = y * (x * z)$, then $X = \{0\}$,
(p_6) if $x * y = z$, then $z * y = z$ and $y * z = y$,
(p_7) if $(x * y) * (z * u) = (x * z) * (y * u)$, then $X = \{0\}$.

The subsequent list of basic properties of right distributive BI-algebra is borrowed from [4, 18].
(p_8) $x * y \leq x$,
(p_9) $y * (y * x) \leq x$,
(p_{10}) $x * (x * y) \leq x$,
(p_{11}) $(x * z) * (y * z) \leq x * y$,
(p_{12}) if $x \leq y$, then $x * z \leq y * z$,
(p_{13}) $(x * y) * z \leq x * (y * z)$, (i.e., X is a quasi-associative algebra),
(p_{14}) if $x * y = z * y$, then $(x * z) * y = 0$,
(p_{15}) if $x * y = x$, then $y * x = y$,
(p_{16}) if $x * y \neq x$, then $y * x \neq y$,
(p_{17}) if $x * y \neq x$, then $z * x \neq y$,
(p_{18}) if $x * z = 0$, then $y * z \neq x$.

Table 1: Cayley table for the binary operation "$*$".

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

3 Right(Left) Independent Subsets

Given BI-algebra X, we see that a property of an element 0 is if $0 \neq x \neq y$, then $x * 0 = x \neq y = y * 0$. A question arises: is there a subset U of X such that if $x \neq y \in U$, then $x * u \neq y * u$ for all $u \in X$?
Similarly, is there a subset U of X such that if $x \neq y \in U$, then $u * x \neq u * y$ for all $u \in X$?

This motivated us to define the following definition and investigate their properties.

Definition 3.1. A non-empty subset U of X is said to be *right independent* if $x \neq y \in U$, then $x * u \neq y * u$, for all $u \in X$. Also, U is said to be *left independent* if $x \neq y \in U$, then $u * x \neq u * y$, for all $u \in X \setminus \{0\}$. U is said to be *independent* subset of X if it both right and left independent subset of X.

Notice that if U is a right(left) independent subset of X, then $0 \notin U$, since if $x \neq 0 \in U$, by using (B) and (P_2), we get $x * x = 0 = 0 * x$ (resp., $0 * x = 0 = 0 * 0$), for all $x \in X$.

Example 3.2. (i) Let $X := \{0, a, b, c\}$ be a set with a binary operation "$*$" shown in Table 1. Then $(X, *, 0)$ is a BI-algebra (see [4]), but not a right(left) independent of itself, since $b \neq c$, but $b * a = b = c * a$ (resp., since $a \neq c$, but $b * a = b = b * c$). The set $A = \{a, b\}$ is a left independent subset of X ($a * a = 0 \neq a = a * b$, $b * a = b \neq 0 = b * b$, and $c * a = b \neq c = c * b$), but not a right independent subset of X, since $a \neq b$, but $a * c = b = b * c$.

(ii) Let $X := \{0, a, b, c\}$ be a set with a binary operation "$*$" shown in Table 2. Then $(X, *, 0)$ is a BI-algebra (see [18]). The set $A = \{a, b\}$

Table 2: Cayley table for the binary operation “$*$”.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Table 3: Cayley table for the binary operation “$*$”.

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

is a right independent subset of X, but not a left independent subset of X, since $a \neq b$, but $c * a = c = c * b$.

(iii) Let X be a set with $0 \in X$. Define a binary operation “ $*$ ” on X by

$$x * y = \begin{cases} 0 & \text{if } x = y, \\ x & \text{if } x \neq y. \end{cases}$$

Then $(X, *, 0)$ is an implicative BCK-algebra (see [8]), and hence a BI-algebra (see [4]). It is easy to check that X is not an independent subset of X.

(iv) Let $X := \{0, a, b\}$ be a set with a binary operation “$*$” shown in Table 3. Then $(X, *, 0)$ is a BI-algebra (see [18]). The set $A = \{a, b\}$ is an independent subset of X.

(v) Let $X := \{0, a, b\}$ be a set with a binary operation “$*$” shown in Table 4. Then $(X, *, 0)$ is a BI-algebra (see [18]). The set $A = \{a, b\}$ is not a right independent subset of X, since $a \neq b$, but $a * a = 0 = b * a$, nor a left independent subset of X, since $b * a = 0 = b * b$.

(vi) Let $P(X)$ be the power set of X. Define a binary operation $*$ on $P(X)$ by $A * B = A \setminus B$, for all $A, B \in P(X)$. Then $(P(X); *, \emptyset)$ is a commutative BI-algebra (see [18]), but not an independent subset of

Table 4: Cayley table for the binary operation “$*$”.

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	0	0

Table 5: Cayley table for the binary operation “$*$”.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	c	a	b
b	b	c	0	b	b
c	c	c	c	0	b
d	d	c	d	b	0

$P(X)$. Let $X := \{0, 1, 2, 3, 4, 5\}$. Take $A = \{1, 2, 3\}$ and $B = \{1, 2, 4\}$, and so $A \neq B \in P(X)$. Put $C = \{3, 4\}$, we have $A*C = A\setminus C = \{1, 2\} = B \setminus C = B * C$, and so $P(X)$ is not a right independent subset of $P(X)$. Also, if $D = \{1, 2, 5\}$, then $D * A = D \setminus A = \{5\} = D \setminus B = D * B$, and so $P(X)$ is not a left independent subset of $P(X)$.

The following example shows that every right(left) independent subset may not be closed, and so every right(left) independent subset may not be a subalgebra. Also, we see that for every $x \in A$, $x * x = 0 \notin A$.

Example 3.3. Let $X := \{0, a, b, c, d\}$ be a set with a binary operation “$*$” shown in Table 5. Then $(X, *, 0)$ is a BI-algebra (see [18]). The set $A = \{a, d\}$ is an independent subset of X, but not closed, since $a * d = b \notin A$ and $d * a = c \notin A$. Also, $X \setminus (A \cup \{0\}) = \{b, c\}$ is a left independent subset of X, not a right independent subset of X, since $b \neq c$, but $b * a = c = c * a$.

Notice that, for $x \neq 0$ the singleton set $\{x\}$ has no element y in X such that $x \neq y$. It follows that the independent criteria are fulfilled vacuously, and so $\{x\}$ is a right(left) independent subset of X.

Table 6: Cayley table for the binary operation "$*$".

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	d	c
b	b	b	0	b	b
c	c	b	c	0	c
d	d	0	d	d	0

A. Rezaei et al. in [19, Th. 3.5] proved that for every groupoid $(X, *)$ there exists a maximal right(left) independent subsets $\{M_\lambda\}_{\lambda\in\Lambda}$ of X such that $X = \bigcup\limits_{\lambda\in\Lambda} M_\lambda$. Consequently, for every BI-algebra $(X, *, 0)$ there exists a maximal right(left) independent subsets $\{M_\lambda\}_{\lambda\in\Lambda}$ of X such that $X = \bigcup\limits_{\lambda\in\Lambda} M_\lambda \bigcup \{0\}$.

Remark 3.4. By routine calculation we can see that if $A_i \subseteq X$ for $i \in \Lambda$ are right(left) independent subsets of X, then $\bigcap\limits_{i\in\Lambda} A_i$ and $\bigcup\limits_{i\in\Lambda} A_i$ are right(left) independent subsets of X.

Proposition 3.5. *Let $A, B \subseteq X$ and A be a right(left) independent subset of X. Then $A \cap B$ is a right(left) independent subset of X.*

Proof. Assume that A is a right(left) independent subset of X and B is an arbitrary subset of X. Let $x \neq y \in A \cap B$. Since $A \cap B \subseteq A$, we get $x \neq y$ in A. Since A is a right(left) independent subset of X, for all $u \in X$, we have $x * u \neq y * u$ (resp., $u * x \neq u * y$), and so $A \cap B$ is a right(left) independent subset of X. $\square$

The following example shows that the converse of Proposition 3.5, may not be true in general.

Example 3.6. Let $X := \{0, a, b, c, d\}$ be a set with a binary operation "$*$" shown in Table 6. Then $(X, *, 0)$ is a BI-algebra. The set $A = \{a, b, c\}$ is not a right(left) independent subset of X, since $a*d = c = c*d$ (resp., since $b*a = b = b*c$). Also, the set $B = \{a, b, d\}$ is not a right(left) independent subset of X, since $a \neq d$, but $a * a = 0 = d * a$ (resp., since $b * a = b = b * d$). It is easily seen that $A \cap B = \{a, b\}$ is a right(left) independent subset of X.

As a result of the Proposition 3.5, the next corollary is deduced.

Corollary 3.7. *Let $A, B \subseteq X$ and A be a right(left) independent subset of X. Then the following hold:*

(a) $A \setminus (B \cup \{0\})$ is a right(left) independent subset of X,

(b) if $B \subseteq A$, then B is a right(left) independent subset of X.

The following example show that for every right(left) independent subset A, may not be $A \cup B$ a right(left) independent subset of X, where $B \subseteq X$.

Example 3.8. Consider the Example 3.3, $A = \{a, d\}$ and take $B = \{b, c\}$. Hence B is not a right independent subset of X. Then $(A \cup B) \cup \{0\} = X$, which is not a right(left) independent subset of X, since $b \neq c \in X$, but $b * a = c = c * a$, but not a left independent subset of X, since $c \neq d$, but $b * c = b = b * d$. Thus, X is not an independent subset of X. Also, $A \triangle B = (A \cup B) \setminus (A \cap B) = \{a, b, c, d\} \setminus \emptyset = \{a, b, c, d\}$, which is not a right(left) independent subset of X.

Notice that the extension property is not valid for a right(left) independent subset of X. Consider the Examples 3.3 and 3.8. If we take $B = \{a, b, c, d\}$. Hence $A = \{a, d\} \subseteq B = \{a, b, c, d\}$, but B is not a right(left) independent subset of X.

Proposition 3.9. *Let $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ be two BI-algebras, $A \subseteq X$ and $B \subseteq Y$ be right(left) independent subsets of X and Y, respectively. Then $A \times B$ is a right(left) independent subset of $X \times Y$, where $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$ and $\bullet$ is defined by $(x, u) \bullet (y, v) = (x * y, u \circ v)$.*

Proof. Assume that $A \subseteq X$ and $B \subseteq Y$ are right independent subsets of X and Y, respectively, and $(x, y) \neq (u, v) \in A \times B$. Thus, $x \neq u$ or $y \neq v$. Since A and B are right independent subsets of X and Y, respectively, we get for all $z \in X$ and for all $w \in Y$, $x * z \neq u * z$ or $y \circ w \neq v \circ w$, respectively. Hence $(x * z, y \circ w) = (x, y) \bullet (z, w) \neq$

$(u * z, v \circ w) = (u, v) \bullet (z, w)$, for all $(z, w) \in X \times Y$. Therefore, $A \times B$ is a right independent subset of $X \times Y$.
Similarly, for the case left independent subset the proof holds. $\square$

Proposition 3.10. *Let* $(X, *, 0_X)$ *and* $(Y, \circ, 0_Y)$ *be two* BI*-algebras,* $f : X \longrightarrow Y$ *be a homomorphism,* $A \subseteq X$ *and* $B \subseteq Y$ *be right(left) independent subset of* X *and* Y*, respectively. The following statements hold:*

- *(a) If* f *is an isomorphism, then* $f(A)$ *is a right(left) independent subset of* Y*,*
- *(b) If* f *is a monomorphism, then* $f^{-1}(B)$ *is a right(left) independent subset of* X*.*

Proof. (a) Assume that $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ are two BI-algebras, $A \subseteq X$ is a right(left) independent subset of X and f is an isomorphism. Let $y_1 \neq y_2$ in $f(A)$ and $v \in Y$. If $v = 0$, then $u = 0$, and so the proof is obvious. Let $v \neq 0$. Then there are $a_1, a_2 \in A$ such that $f(a_1) = y_1$ and $f(a_2) = y_2$. Since f is a map, we get $a_1 \neq a_2$ (for detail, if $a_1 = a_2$, then $y_1 = f(a_1) = f(a_2) = y_2$, which is a contradiction). Also, since f is an epimorphism, there is $0 \neq u \in X$ such that $f(u) = v$. Since A is a right independent subset of X, we have $a_1 * u \neq a_2 * u$ (resp., $u * a_1 \neq u * a_2$), and since f is monomorphism, we get $f(a_1 * u) \neq f(a_2 * u)$ (resp., $f(u * a_1) \neq f(u * a_2)$). Thus,

$$y_1 \circ v = f(a_1) \circ f(u) = f(a_1 * u) \neq f(a_2 * u) = f(a_2) \circ f(u) = y_2 \circ v,$$

and respectively,

$$v \circ y_1 = f(u) \circ f(a_1) = f(u * a_1) \neq f(u * a_2) = f(u) \circ f(a_2) = v \circ y_1.$$

Therefore $f(A)$ is a right(left) independent subset of Y.

(b) Assume that $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ are two BI-algebras, $B \subseteq Y$ is a right(left) independent subset of Y and f is a monomorphism. Let $a \neq b$ in $f^{-1}(B)$ and $u \in X$. Hence $f(a) \neq f(b)$ in B, since f is a monomorphism. So, $f(a * u) = f(a) \circ f(u) \neq f(b) \circ f(u) = f(b * u)$ (resp., for $u \in X \setminus \{0_X\}$, $f(u * a) = f(u) \circ f(a) \neq f(u) \circ f(b) = f(u * b)$). Since f is a map, we have $a * u \neq b * u$ (resp., $u * a \neq u * b$). Thus, $f^{-1}(B)$ is a right(left) independent subset of X. $\square$

Table 7: Cayley table for the binary operation “$*$”.

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Table 8: Cayley table for the binary operation “$\circ$”.

$\circ$	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	0	0	y
z	z	z	z	0

The following example shows that the condition isomorphism in the Proposition 3.10(a), is necessary.

Example 3.11. Let $X := \{0, a, b\}$ and $Y := \{0, x, y, z\}$ be two sets with the binary operations “$*$” and “$\circ$” shown in Tables 7 and 8. Then $(X, *, 0)$ and $(Y, \circ, 0)$ are BI-algebras. The sets $A = \{a, b\}$ and $B = \{x, z\}$ are independent subsets of X and Y, respectively. Define a map $f : X \longrightarrow Y$ by $0 \mapsto 0$, $a \mapsto x$, $b \mapsto 0$. Then f is not an epimorphism and nor a monomorphism. Also, $f(A) = \{0, x\}$ is not a right independent subset of Y, since $x \neq 0$, but $x \circ y = 0 = 0 \circ y$.

The following example shows that the condition monomorphism in the Proposition 3.10(b), is necessary.

Example 3.12. Let $X := \{0, a, b, c\}$ and $Y := \{0, x, y, z\}$ be two sets with with the binary operations “$*$” and “$\circ$” shown in Tables 9 and 8, respectively. Then $(X, *, 0)$ and $(Y, \circ, 0)$ are two BI-algebras. The sets $A = \{a, b\}$ and $B = \{x, z\}$ are independent subsets of X and Y respectively. Define a map $f : X \longrightarrow Y$ by $0 \mapsto 0$, $a \mapsto x$, $b \mapsto z$ and $c \mapsto z$. Then f is not a monomorphism, since $f(b) = z = f(c)$, but $b \neq c$. Further, $f^{-1}(B) = \{a, b, c\}$ is not a left independent subset of X, since $b \neq c$, but $a * b = a = a * c$.

Table 9: Cayley table for the binary operation "$*$".

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

Proposition 3.13. *Let $X \setminus \{0\}$ be a right(left) independent subset of X. Then $x * y = x$, for all $x \neq y$ in $X \setminus \{0\}$.*

Proof. Assume that $X \setminus \{0\}$ is a right independent subset of X, by (P_3) we have $x * y = (x * y) * y$. Let $x \neq y$ in $X \setminus \{0\}$ such that $x * y \neq x$. If we take $u := y$, then $x * u = x * y = (x * y) * y = (x * y) * u$. This shows that $X \setminus \{0\}$ is not a right independent subset of X, which is a contradiction.

Now, let $X \setminus \{0\}$ be a left independent subset of X and $x \neq y$ in $X \setminus \{0\}$ such that $x * y \neq x$. Using (P_{16}), we get $y * x \neq y$, and so $y * (y * x) \neq y * y = 0$. By (BI) and ($P_1$) we have

$$y * x = (y * x) * (y * (y * x)) \neq (y * x) * 0 = y * x,$$

which is a contradiction. $\square$

Proposition 3.14. *Let $\emptyset \neq A \subseteq X$ and $x \in X$.*

(i) If $a * x = a$, for all $a \in A$, then A is a right independent subset of X.

(ii) If $x * a = a$, for all $a \in A \setminus \{0\}$, then A is a left independent subset of X.

Proof.(i) Assume that $\emptyset \neq A \subseteq X$, $x \neq y \in A$ and $u \in X$. Then $x * u = x \neq y = y * u$. Thus, A is a right independent subset of X.

(ii) Assume that $\emptyset \neq A \subseteq X$, $x \neq y \in A$ and $0 \neq u \in X$. Then $u * x = x \neq y = u * y$. Thus, A is a left independent subset of X. $\square$

Theorem 3.15. *Let A be a right(left) independent subset of X and $x \leq y$ (resp., $y \leq x$), for some $x, y \in A$. Then $x = y$.*

Proof. Assume that A is a right(left) independent subset of X and $x \leq y \in A$ (resp., $y \leq x \in A$). Hence $x * y = 0$ (resp., $y * x = 0$). Let $x \neq y$ in X. Since A is a right(left) independent subset of X and $y \in X$ (resp., $x \in X$), we get $x*y = 0 \neq y*y = 0$ (resp., $y*x = 0 \neq x*x = 0$), which is a contradiction. $\square$

Let A and B be two subsets of X. Define $A * B$ as follows:

$$\begin{aligned} A * B &= \{a * b : a \in A \text{ and } b \in A\} \\ &= \bigcup_{a \in A} (a * B) \\ &= \bigcup_{b \in B} (A * b). \end{aligned}$$

We use the notion $a*B$ (resp., $A*b$) instead of $\{a\}*B$ (resp., $A*\{b\}$). Now, let A, B and C be subsets of X. Then one can see that:

- $\emptyset * \emptyset = \emptyset$, $\emptyset * A = A * \emptyset = \emptyset$,
- $\{0\} * A = \{0 * a : a \in A\} = \{0\}$ and $A * \{0\} = \{a * 0 : a \in A\} = A$,
- $X * X \subseteq X$,
- $A * A \neq A$, $X * A \neq A \neq A * X$, $A * B \neq A$, $A * B \neq B$ and $A * B \neq B * A$, in general.
- if $A \subseteq B$, then $A * C \subseteq B * C$ and $C * A \subseteq C * B$,
- $(A \cap B) * C \subseteq (A * C) \cap (B * C)$,
- $C * (A \cap B) \subseteq (C * A) \cap (C * B)$,
- $(A \cup B) * C = (A * C) \cup (B * C)$,
- $C * (A \cup B) = (C * A) \cup (C * B)$.

Also, if $a \in A$, since $a * a = 0$, then $0 \in a * A$ (resp., $0 \in A * a$), and so $a * A$ (resp., $A * a$) is not an independent subset of X. Also, if $A = \{a\}$ for some $0 \neq a$ i.e., $|A| = 1$, then $x * \{a\} = \{x * a\}$ and $\{a\} * x = \{a * x\}$ are also singleton sets, and so are independent subsets of X, when $x * a \neq 0$ and $a * x \neq 0$. Further, take $A = \{0\}$, using (P_2)

we get $\{0\} * x = \{0 * x\} = \{0\}$, whence it is not an independent subset of X and by (P_1) we have $x * \{0\} = \{x * 0\} = \{x\}$, whence it is an independent subset of X.

Theorem 3.16. *Let $u * v = u$, for all $u, v \in X$, $\emptyset \neq A \subseteq X$ be an independent subset of X and $x \in X$. Then $A * x$ (resp., $x * A$) is too.*

Proof. Since $A \neq \emptyset$, there is at least one element $0 \neq a \in A$. If $A*x = \{a\}$, and so is a singleton set, then the proof is obvious. Assume that $s \neq r \in A*x$ and A is an independent subset of X. Hence there exist a_1, $a_2 \in A$ such that $s = a_1 * x = a_1$ and $r = a_2 * x = a_2$. Hence $a_1 \neq a_2$, and so $s * t = a_1 * t \neq a_2 * t = r * t$ (resp., $t * s = t * a_1 \neq t * a_2 = t * r$) for all $t \in X$. Thus, $A * x$ is an independent subset of X.

Now, suppose $s \neq r \in x * A$ and A is an independent subset of X. Then there exist a_1, $a_2 \in A$ such that $s = x * a_1 = x$ and $r = x * a_2 = x$, which is a contradiction. Thus, $x * A$ is a singleton set, and so is an independent subset of X. $\square$

Corollary 3.17. *Let $X \setminus \{0\}$ be a right(left) independent subset of X and $x \in X$. Then $(X \setminus \{0\}) * x$ (resp., $x * (X \setminus \{0\})$) is too.*

Corollary 3.18. *Let $(X, *, 0)$ be a BI-algebra. Then*

(a) If $x*y = x$ (resp. $x*y = y$) and A or B is a right(left) independent subset of X, then $A * B$ is too.

(b) If $|A| = 1$ or $|B| = 1$, then $A*B$ is a right(left) independent subset of X.

4 Right(Left) Absorbent Subsets

In this section, we define absorbent concept on BI-algebras and investigate several properties in detail. We show that this notions are different with some examples.

Definition 4.1. Let A and B be two subsets of X. We say A absorbs B from the right (resp., from the left) briefly *right absorbent* (resp., *left absorbent*) subset of X if it satisfies:

$$A * B = A \text{ (resp., } B * A = A).$$

A subset A of X absorbs a subset B of X briefly *absorbent* if it is a right and left absorbent of X (i.e., $A * B = B * A = A$).

Example 4.2. (i) Consider the Example 3.3, $A = \{a, d\}$ is an independent subset of X, but does not absorb X since

$$A * X = \{0, a, b, c, d\} \neq A \text{ and } X * A = \{0, b, c\} \neq A.$$

Also, if take $B = \{0, b, c\}$ and $C = \{a, b\}$. Then $C*B = \{a, b\}*\{0, b, c\} = \{0, b, c\} = B$, and so B absorbs C from the left, but does not from the right, since $B * C = \{0, b, c\} * \{a, b\} = \{0, c\} \neq B$.

(ii) Consider the Example 3.2(iv), $\{0, a\} * X = \{0, a\}$, and hence $\{0, a\}$ absorbs X from the right, but does not absorbs X from the left, since $X * \{0, a\} = \{0, a, b\} \neq A$.

Theorem 4.3. *Let $\emptyset \neq A \subseteq X$. Then X absorbs $A * X$ from the right.*

Proof. Assume that $A \subseteq X$. Let $x \in X$ and $a \in A$. Using (BI), we get $x = x * (a * x) \in X * (A * X)$, and so $X \subseteq X * (A * X)$. On the other hand, since $*$ is a binary operation, we have $X * (A * X) \subseteq X$. Thus, $X * (A * X) = X$. □

Theorem 4.4. *Let X be a right distributive BI-algebra, $A \subseteq X$ and B be an independent subset of X. Then $A * B$ absorbs B from the right.*

Proof. Assume that X be a right distributive BI-algebra and $\emptyset \neq A$, $\emptyset \neq B \subseteq X$. Let $x \in A * B$. Then there are $a \in A$ and $b \in B$ such that $x = a * b$. By (P_3), we get $x = a * b = (a * b) * b \in (A * B) * B$, and so $x \in (A * B) * B$. Thus, $A * B \subseteq (A * B) * B$. On the other hand, let $x \in (A * B) * B$. Then there are $a \in A$ and b_1, $b_2 \in B$ such that $x = (a * b_1) * b_2$. If $b_1 = b_2 = b$, then by using (P_3), we get $x = (a * b) * b = a * b \in A * B$, and so $(A * B) * B \subseteq A * B$. If $b_1 \neq b_2$, then $x * b_2 = ((a * b_1) * b_2) * b_2 = (a * b_1) * b_2 = x$. By ($P_{15}$), we get $b_2 * x = b_2$. Since B is a left independent subset of X, we get $a * b_1 \neq a * b_2$. By using right independent property and (P_3), we obtain $(a * b_1) * b_2 \neq (a * b_2) * b_2 = a * b_2$. Hence $x \neq a * b_2$. By the left independent property, we get $b_2 = b_2 * x \neq b_2 * (a * b_2) = b_2$, which is a contradiction. Thus, the proof is complete. □

As an immediate consequence of Definition 4.1, we give some of properties right(left) absorbent subsets in the following:

- if $A = \{0\}$ and $\emptyset \neq B \subseteq X$, then $\{0\} * B = \{0\}$ and $B * \{0\} = B$, (i.e., $\{0\}$ absorbs every non-empty subset B of X from the right, and every non-empty subset B of X absorbs $\{0\}$ from the right, and so $\{0\}$ does not absorbs every non-empty subset B of X from the left),
- if $A = \{x\}$, where $x \neq 0$, then A does not absorbs X from the right and left, since $0 = x * x \in \{x\} * X \cap X * \{x\}$, but $0 \notin A$, and so $A * X \neq A$ (resp., $X * A \neq A$).
- if $0 \in A$, then $X * A = X$,
- if A absorbs B from the right(left) and $A \cap B \neq \emptyset$, then $0 \in A$, and so $X * A = X$,
- if A absorbs B from the right and B absorbs A from the left, then $A = B$,
- if A is a closed subset of X, then $A * A = A$, (i.e., every closed subset absorbs itself from the right and left),
- if A is a closed subset of X and absorbs B from the left, then $A * (B * A) = A$ (i.e., A absorbs $B * A$ from the right),
- if $A \subseteq B$ and B is a closed subset of X, then $A * B \subseteq B$,
- if $A \subseteq B$ and B is a closed subset of X and $0 \in A$, then $B * A = B$,
- if A_i is closed subset of X, for $i \in \{1, 2, \cdots, n\}$, $0 \in A_1$ and $A_1 \subseteq A_2 \cdots \subseteq A_n = X$, then $X * (A_{n-1} * A_{n-2} * \cdots * A_1) = X$,
- if A absorbs B from the right(left) and $A \subseteq B$, then A is a closed subset of X,
- if $A_1 \subseteq A_2 \cdots \subseteq A_n = X$, A_{i-1} absorbs A_i, for $2 \leq i \leq n$, from the right(left), then A_j is a closed subset of X, for $j \in \{1, 2, \cdots, n\}$,
- if A absorbs B from the left and absorbs C from the right, then $(B * A) * C = B * (A * C) = A = B * A = A * C$, and so $B * A$ absorbs C from the right and $A * C$ absorbs B from the left,

- if A absorbs B from the right and C absorbs B from the left, then $A * B * C = A$ (i.e., $A * B$ absorbs C from the right and A absorbs $B * C$ from the right),
- if A absorbs B from the right and left, then $(B * A)^m = A^m$, for all $m \in \mathbb{N}$, where $A^1 = A$, $A^2 = A * A$, $A^3 = A * A * A$ etc.,
- if A absorbs B from the right, then $A * B * A = A^2$,
- if A is a closed subset and absorbs B from the right and left, then $(B * A)^m = A$, for all $m \in \mathbb{N}$,
- if A absorbs B from the left, then for all m, $n \in \mathbb{N}$, $B^m * A = A$, $B * A^n = A^n$ and $B^m * A^n = A^n$,
- if A absorbs C from the right, then for all m, $n \in \mathbb{N}$, $A * C^m = A$, $A^n * C = A^n$ and $A^n * C^m = A$,
- if A absorbs B from the left and absorbs C from the right, then $B * A^s * C = B^m * A^s * C = A^s$, and $B * A^s * C^n = B^m * A^s * C^n = A$, for all m, n, $s \in \mathbb{N}$,

Proposition 4.5. *Let A and B absorbs C from the right(left). Then*

(a) $(A \cap B) * C \subseteq A \cap B$ (resp., $C * (A \cap B) \subseteq A \cap B$),

(b) $(A \cup B) * C = A \cup B$ (resp., $C * (A \cup B) = A \cup B$).

Corollary 4.6. *Let $A_i \neq \emptyset$, for $i \in \Lambda$ and $\{A_i\}_{i \in \Lambda}$ be a family where absorbs C from the right(left). Then $\bigcup_{i \in \Lambda} A_i$ is too.*

Proposition 4.7. *Let A absorbs B_1 and B_2 from the right(left). Then A absorbs $B_1 \cup B_2$ from the right(left).*

(a) $A * (B_1 \cap B_2) \subseteq A$ (resp., $(B_1 \cap B_2) * A \subseteq A$),

(b) $A * (B_1 \cup B_2) = A$ (resp., $(B_1 \cup B_2) * A = A$).

Corollary 4.8. *Let A absorbs the family of non-empty subsets $\{A_i\}_{i \in \Lambda}$ of X, from the right(left). Then A absorbs $\bigcup_{i \in \Lambda} A_i$ from the right(left).*

Proposition 4.9. *Let $A \neq \emptyset$ absorbs X from the right(left). Then A is closed under $*$.*

Proof. Assume that $A \neq \emptyset$ absorbs X from the right and a, $b \in A$. Hence $a * b \in A * A \subseteq A * X = A$ (resp., $a * b \in A * A \subseteq X * A = A$). Thus, $a * b \in A$.

Assume that $A \neq \emptyset$ absorbs X from the left and a, $b \in A$. If $a = 0$ or $b = 0$, then using (P_1), we get $a * 0 = a \in A$ or $b * 0 = b \in A$. Let $a \neq 0$ and $b \neq 0$. Then $a * b \in A * A \subseteq X * A = A$. Thus, $a * b \in A$. □

Theorem 4.10. *If X is a right distributive BI-algebra, then every right(left) independent subset $\emptyset \neq A \subseteq X$ absorbs X from the right.*

Proof. Assume that X is a right distributive BI-algebra and $A \neq \emptyset$ absorbs X from the right. Let $t \in A * X$. Then there exist $0 \neq a \in A$ and $x \in X$ such that $t = a * x$. Using right distributivity, (B) and (P_2), we get $t * a = (a * x) * a = (a * a) * (x * a) = 0 * (x * a) = 0$, and so $t \leq a$. If $t \neq a$, since A is a right independent subset of X, we get $t * a \neq a * a = 0$, which is a contradiction. Thus, $t = a \in A$. Therefore, $A * X \subseteq A$. On the other hand, let $a \in A$. By (P_1), we have $a = a * 0 \in A * X$, and so $A \subseteq A * X$. Thus, $A * X = A$. □

The following example shows that in the Theorem 4.10, the condition right distributivity is necessary.

Example 4.11. Consider the Example 3.2(i), $A = \{a, c\}$ is a right independent subset of X, but does not absorb X from the right, since $A * X = \{0, a, b, c\} \neq A$. Notice that X is not a right distributive BI-algebra, since

$$(b * c) * a = b * a = b \neq 0 = b * b = (b * a) * (c * a).$$

Also, $X * A = \{0, a, b, c\} \neq A$.

The converse of the Theorem 4.10, may not be true in general. For this, consider the Example 3.2(ii) and take $A = \{0\}$, we get A absorbs X from the right, since $\{0\} * X = \{0\}$, but not a right independent subset of X.

Proposition 4.12. *Let $\emptyset \neq A \subseteq X$ absorbs X from the left. Then A is not an independent subset of X.*

Proof. Assume that $\emptyset \neq A \subseteq X$ absorbs X from the left, and so $X * A = A$. Let $a \in A \subseteq X$. Then $0 = a * a \in X * A = A$. Hence $0 \in A$. Thus, A is not an independent subset of X. $\square$

Proposition 4.13. *Let $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ be two BI-algebras, A absorbs C from the right(left) in X and B absorbs D from the right(left) in Y. Then $A \times B$ absorbs $C \times D$ from the right(left) in $X \times Y$.*

Proof. Assume that A absorbs C from the right(left) in X and B absorbs D from the right(left) in Y. Then $A * C = A$ (resp., $C * A = A$) and $B \circ D = B$ (resp., $D \circ B = B$), and so

$$\begin{aligned}(A \times B) \bullet (C \times D) &= \{r \bullet s : r \in A \times B, s \in C \times D\}\\ &= \{r \bullet s : r = (a, b) \in A \times B, s = (c, d) \in C \times D\}\\ &= \{(a, b) \bullet (c, d) : \exists a \in A, \exists b \in B, \exists c \in C, \exists d \in D\}\\ &= \{(a * c, b \circ d) : \exists a \in A, \exists b \in B, \exists c \in C, \exists d \in D\}\\ &= \{(x, y) : x \in A, y \in B\}\\ &= A \times B.\end{aligned}$$

(resp., by a similar argument we have $(C \times D) \bullet (A \times B) = A \times B$). $\square$

Proposition 4.14. *Let $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ be two BI-algebras, $f : X \longrightarrow Y$ be a homomorphism, $A \subseteq X$ and $B \subseteq Y$. The following statements hold:*

(a) if A absorbs C from the right(left), then $f(A)$ absorbs $f(C)$ from the right(left),

(b) if B absorbs D from the right(left), then $f^{-1}(B)$ absorbs $f^{-1}(D)$ from the right(left).

Proof. (a) Assume that $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ are two BI-algebras and $A \subseteq X$ absorbs C from the right(left). Then $A * C = A$ (resp., $C * A = A$). Since f is a homomorphism, we get $f(A) * f(C) = f(A * C) = f(A)$ (resp., $f(C) * f(A) = f(C * A) = f(A)$). Thus, $f(A)$ absorbs $f(C)$ from the right(left).

(b) Assume that $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ are two BI-algebras and $B \subseteq Y$ absorbs D from the right(left). Hence $B \circ D = B$ (resp., $D \circ B =$

B). Let $x \in f^{-1}(B \circ D)$ (resp., $x \in f^{-1}(D \circ B)$). Then $f(x) \in B \circ D = B$, and so $x \in f^{-1}(B)$ (resp., $f(x) \in D \circ B = B$, and so $x \in f^{-1}(B)$). This shows that $f^{-1}(B \circ D) \subseteq f^{-1}(B)$ (resp., $f^{-1}(D \circ B) \subseteq f^{-1}(B)$). On the other hand, let $x \in f^{-1}(B)$. Then $f(x) \in B = B \circ D$ (resp., $f(x) \in B = D \circ B$). Hence $x \in f^{-1}(B \circ D)$ (resp., $x \in f^{-1}(D \circ B)$). It follows that $f^{-1}(B) \subseteq f^{-1}(B \circ D)$ (resp., $f^{-1}(B) \subseteq f^{-1}(D \circ B)$). Therefore $f^{-1}(B) = f^{-1}(B \circ D)$ (resp., $f^{-1}(B) = f^{-1}(D \circ B)$). □
Let $A \subseteq X$ and $t \in X$. Define A^t (resp., A_t) as follows:

$$A^t = \{x \in X : x * t \in A\}, \text{ (resp., } A_t = \{x \in X : t * x \in A\}).$$

Also, we can define:

$$A^t_t = A^t \cap A_t = \{x \in X : \{x * t, t * x\} \subseteq A\}.$$

In what follows, we are going to characterize concepts of independent and absorbent with these subsets:

- $\emptyset^t = \emptyset_t = \emptyset$, $X^t = X_t = X$,
- if $0 \in A^t$, then $0 \in A$,
- if $0 \in A$, then $t \in A^t \cap A_t = A^t_t$,
- $A \subseteq A^{t*a}$, for all $a \in A$,
- $A^0 = A$,
- for all $t \in X$, $t \in \{0\}^t_t$,
- if $0 \in A$, then $A_0 = X$,
- if $0 \notin A$, then $A_0 = \emptyset$,
- if $\emptyset \neq \{a\}^t$, then $a \in \{a\}^t$,
- if $x \in A^t$, then $x * t \in A^t$,
- if $A \subseteq B$, then $A^t \subseteq B^t$ and $A_t \subseteq B_t$, and so $A^t_t \subseteq B^t_t$
- $(A^t)^t = A^t$,

- $(A \cap B)^t = A^t \cap B^t$, $(A \cup B)^t \subseteq A^t \cup B^t$, $(A \setminus B)^t = A^t \setminus B^t$,
- $(A \cap B)_t = A_t \cap B_t$, $(A \cup B)_t \subseteq A_t \cup B_t$, $(A \setminus B)_t = A_t \setminus B_t$,
- $(A \cap B)_t^t = A_t^t \cap B_t^t$, $(A \cup B)_t^t \subseteq A_t^t \cup B_t^t$, $(A \setminus B)_t^t = A_t^t \setminus B_t^t$,
- $(A \times B)^t = A^t \times B^t$ and $(A \times B)_t = A_t \times B_t$,
- $(A \times B)_t^t = A_t^t \times B_t^t$,
- $x \leq t$ if and only if $x \in \{0\}^t$,
- $t \leq x$ if and only if $x \in \{0\}_t$,
- $x \leq t$ and $t \leq x$ if and only if $x \in \{0\}_t^t$,
- if $x \in \{0\}_t^t$ and X is a commutative BI-algebra (or BH-algebra), then $x = t$,
- if $\emptyset \neq A$ is a closed subset of right distributive BI-algebra X, then A^t is too,
- if $\emptyset \neq \{a\}_t$ and X is a right distributive BI-algebra, then $a \leq t$,
- if $\emptyset \neq \{a\}^t$ and $|\{a\}^t| \geq 2$, then it is not a right independent subset of X,
- if $\emptyset \neq \{a\}_t$ and $|\{a\}_t| \geq 2$, then it is not a left independent subset of X,
- if $\emptyset \neq \{a\}_t^t$ and $|\{a\}_t^t| \geq 2$, then it is not an independent subset of X.

Theorem 4.15. *Let $\emptyset \neq A \subseteq X$ absorbs X from the right(left). Then $X = A_t$, for $t \in A$ (resp., $X = A^t$, for $0 \neq t \in A$).*

Proof. Assume that $\emptyset \neq A \subseteq X$ absorbs X from the right(left) and $t \in A$. Then $t * x \in A * X = A$ (resp., $x * t \in X * A = A$, for $0 \neq t \in A$), for all $x \in X$. Hence $x \in A_t$ (resp., $x \in A^t$), and so $X \subseteq A_t$ (resp., $X \subseteq A^t$). Thus, $X = A_t$ (resp., $X = A^t$). $\square$

As a result of the Theorem 4.15, the next corollary is deduced.

Table 10: Cayley table for the binary operation “$*$”.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	0	a	0

Corollary 4.16. *Let $\emptyset \neq A \subseteq X$ absorbs X. Then $X = A_t^t$, for $0 \neq t \in A$.*

Example 4.17. (i) Consider the Example 3.2(i), and take $A = \{0, b, c\}$. Then A absorbs X from the right, but does not absorb X from the left. Sine $X * A = X \neq A$. Also, we have $A_0 = A_b = A_c = X$ (resp., $A^b = A$ and $A^c = X$).

(ii) Let $X := \{0, a, b, c\}$ be a set with a binary operation “$*$” shown in Table 10. Then $(X, *, 0)$ is a BI-algebra. If we take $A = \{0, a, b\}$, then A absorbs X from the right and left. So, we can see that $A_0 = A_a = A_b = X$ (resp., $A^a = A^b = X$). Also, $A_a^a = A_b^b = X$.

Open problem.
There is a partition $\{A_i\}_{i \in \Lambda}$, where A_i absorbs X from the right(left), for $i \in \Lambda$?

5 Conclusions and Future Works

In this paper, we have considered the notion of the *right(left) independent* subsets of BI-algebras as a new concept. Moreover, we have defined the notion of the *right(left) absorbent* subsets. We have shown that these new concepts are different by presenting several examples. Some interrelationships between some subsets of a BI-algebra with these definitions are visualized, and some of the properties are investigated and we have got more results in BI-algebras. As we mentioned in Theorem 4.10, if X is a right distributive BI-algebra, then every right(left) independent subset $\emptyset \neq A \subseteq X$ absorbs X from the right and the converse may not be true in general. As another result of the research, in Proposition 4.12, it is shown that if a subset A absorbs X from the left, then A is not an independent subset of X.

As concerning future works, we will generalize these notions to other algebraic structures and study the relation between them by characterizing the new concepts of independent and absorbent subsets.

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