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Original Research Paper

A Method of Weighted Residuals for Solving Fractional Boundary Value Problems

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Abstract. In this article, we propose an approximation scheme for solving fractional boundary value problems with a finite element called the method of weighted residuals. The fractional derivatives are taken in the Caputo and Riemann–Liouville sense. Numerical examples are provided to show that the numerical method is easy to apply and computationally efficient.

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Keywords and Phrases: Riemann-Liouville fractional derivative, Caputo fractional derivative, Fractional differential equations, Weighted residuals method, Galerkin method

1 Introduction

Recently, fractional-order models are palyed a crucial role more than integer-order forms for many engineering sciences subjects. Fractional derivatives supply an exceptional instrument for the explanation of memory and inherited properties of variant techniques and materials. This is the principal benefit of fractional differential problems in analogy with classical integer-order equations([1, 3, 9, 18]. Fractional differential

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equations appear in many science and engineering fields for modelling of various processes in the areas of polymer rheology, electrodynamics, chemistry, physics, aerodynamics and etc. Consequently, the subject of fractional differential equations is achieving much attention and importance. For instance see([1]-[8], [13]-[19], [22]-[24]) and the references therein.

In this paper we utilize weighted residuals method with simple base, for solving linear and nonlinear boundary value problems of fractional order. Different kind of examples of linear and nonlinear fractional boundary value problems are given to demonstrate the ability of the proposed method.

This article has been arranged as follows: Section 2 gives preliminary definitions. Section 3 discusses the principal results of this study, in which weighted residuals method has been executed on the boundary value problems of fractional order. Finally, in section 4 two practical examples are provided.

2 Preliminaries

To being, we need to define fractional integrals and derivatives. First, we introduce the Riemann-Liouville fractional derivative operator J_a^α.

Definition 2.1. *Let $\alpha \in \mathbb{R}^+$. The operator J_a^α, defined on the usual Lebesgue space $L_1[a,b]$ by*

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \tag{1}$$

$$J_a^0 f(t) = f(t),$$

for $a \leq t \leq b$, is called the Riemann-Liouville fractional integral operator of order α.

Properties of this operator can be found in [13]. For $f \in L_1[a,b], \alpha, \beta \geq 0$ and $\gamma > -1$, we mention only the following:

$$J_a^\alpha f(t) \; exists \; for \; almost \; every \; t \in [a,b],$$

$$J_a^{\alpha} J_a^{\beta} f(t) = J_a^{\alpha+\beta} f(t),$$

$$J_a^{\alpha} J_a^{\beta} f(t) = J_a^{\beta} J_a^{\alpha} f(t),$$

$$J_a^{\alpha} (t-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (t-a)^{\alpha+\gamma}.$$

Definition 2.2. *The fractional derivative of $f(t)$ in the Riemann-Liouville sense is defined by*

$$D_a^{\alpha} f(t) = D^m J_a^{m-\alpha} f(t) = \frac{d^m}{dt^m} \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f(s) ds, \quad (2)$$

where $m \in N$ and satisfies the relation $m-1 < \alpha \leq m$, and $f \in L_1[a, b]$. For $m-1 < \alpha \leq m$, $t > a$ and $\gamma > -1$, we have:

$$D_a^{\alpha} k = \frac{k(t-a)^{-\alpha}}{\Gamma(1-\alpha)},$$

$$D_a^{\alpha} (t-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha},$$

$$D_a^{\alpha} J_a^{\alpha} f(t) = f(t).$$

For more properties of this operator see [20, 21].

Certainly, we declare that the definition of Riemann-Liouville fractional derivative play a main role in the progress of fractional calculus. However, the initial conditions required for the physical interpretation of fractional initial value problems can hardly provide. Likewise, The same is true for fractional boundary value problems. The Caputo fractional derivative $D_*^{\alpha} f(t)$ solves this problem. Importantly, the Caputo fractional derivative converts the conventional nth derivative of the function $f(t)$ as $\alpha \to n$ and the initial conditions of fractional differential equations are maintained like ordinary differential equations with integer derivatives. Another difference between the two derivatives is that the Caputo fractional derivative is zero for a constant, while the Riemann-Liouville fractional derivative is not zero. For more details, see [7, 13, 15].

Definition 2.3. *The fractional derivative of $f(t)$ in the Caputo sense is defined by*

$$D_*^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad (3)$$

$$D_*^\alpha J^\alpha f(t) = f(t),$$

for $m-1 < \alpha \leq m, m \in N, t > 0$. Also, if $m-1 < \alpha \leq m$, $t > a$ then

$$D_*^\alpha k = 0,$$

$$D_*^\alpha (J_a^\alpha f(t)) = f(t),$$

$$J_a^\alpha (D_*^\alpha f(t)) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(t-a)^k}{k!}.$$

3 Weighted Residuals Method

Suppose that we have the boundary value problem of fractional order

$$D^\alpha [y(t)] + L[y(t)] + N[y(t)] = f(t), \quad m-1 < \alpha \leq m, \quad (4)$$

$$y(a) = \alpha, \quad y(b) = \beta, \quad a \leq t \leq b,$$

where α and β are constants. The term $D^\alpha [y(t)]$ denotes a linear fractional differential operator, $L[y(t)]$ is a linear differential operator, $N[y(t)]$ is a nonlinear operator and $f(t)$ is a given function.
We will approximate the solution $y(t)$ as

$$\widehat{y}(t) = \sum_{i=0}^{n} c_i \phi_i(t), \quad (5)$$

where n is the number of unknown parameters, and each ϕ_i is an independent basis function. Hence, we denote $\widehat{y}(t)$ as the trial functions. The goal in method of weighted residuals is the determination of the $(n+1)$ scalars $\{c_i\}_{i=0}^n$.
Therefore an error or residual will exist

$$E(t) = R(t) = D^\alpha [\widehat{y}(t)] + L[\widehat{y}(t)] + N[\widehat{y}(t)] - f(t) \neq 0. \quad (6)$$

The notion in the method of weighted residuals is to force the residual to zero over the domain $T = [a, b]$ in some average sense. That is

$$\int_T R(t)W_i(t)dt = 0, \qquad i = 0, 1, \cdots, n, \tag{7}$$

where $\{W_i\}_{i=0}^n$ are the test functions or weights. A good choice of basis functions for boundary value problems of fractional order are the fractional power polynomials

$$\varphi_i(t) = \left\{ t^{\frac{c}{d}+\frac{e}{f}i} \right\}_{i=0}^n$$

where c, d, e and f are constants. The result is a set of $(n + 1)$ algebraic equations for the unknown constants c_i. There are (at least) three method of weighted residuals sub-methods, according to the choices for the W_i's. These three methods are:

1. *Least Squares method*
2. *Sub − domain method*,
3. *Galerkin method.*

Each of these will be explained below [10, 12].

3.1 Least squares method

If the continuous summation of all the squared residuals is minimized, the reasonability behind the name can be observed. To put it another way, a minimum of

$$S = \int_T R(t)R(t)dt = \int_T R^2(t)dt. \tag{8}$$

In order to reach a minimum of this scalar function, the derivatives of S must be zero with respect to all the unknown parameters. That is,

$$\begin{aligned} 0 &= \frac{\partial S}{\partial c_i} \\ &= 2\int_T R(t)\frac{\partial R}{\partial c_i}dt. \end{aligned} \tag{9}$$

The weight functions are apeared to be

$$W_i = 2\frac{\partial R}{\partial c_i},$$

Nonetheless, 2 can be omitted because it cancels in the equation. Hence, the weight functions for the method of least squares are only the residual derivatives with regard to the unknown constants:

$$W_i = \frac{\partial R}{\partial c_i}.$$

3.2 Sub-domain method

This method is not exactly a member of the weighted residual family beacause doesn't use weighting factors explicity. However, it can be brought up a modification of the collocation method. The method is to reduce the residual weight to zero at fixed points in the domain as well as over different subsections of the domain. To achieve this, the weight functions are set to unity, and for evaluateing all unknown parameters, the integral is broken over the entire domain into a sufficient number of subdomains.
That is,

$$\int_T R(t)W_i(t)dt = \sum_i \left(\int_{T_i} R(t)dt\right) = 0, \quad i = 0, 1, \cdots, n. \tag{10}$$

3.3 Galerkin method

In this method, fractional power polynomials are selected as weight functions. That is,

$$W_i = t^{\frac{c}{d}+\frac{e}{f}i}, \quad i = 0, 1, \cdots, n.$$

In which case, for approximating (the $\varphi_i's$) the basis functions were designated as fractional power polynomials.

4 Numerical Examples

In the present section, two examples are presented in order to show the ability and efficiency of the proposed method. The algorithms are performed by Maple 12 with 10 digits precision. For two examples, we take

$t_0 = 0, t_i = t_0 + 0.05\ i$ for $i = 1, \cdots, 20$ and results for $n = 1, 2$ and 3 are reported.

RMS errors

The L_2 norm or Euclidean norm is a suitable scalar index for showing the closeness of two functions. Often in engineering, this measure is called the root-mean squared (RMS) error and can be defined as

$$E_{RMS} = \frac{\sqrt{\int (y(t) - \widehat{y}(t))^2 dt}}{\int dt},$$

which in discrete terms can be evaluated as

$$E_{RMS} = \sqrt{\frac{\sum\limits_{i=1}^{n} (y_i - \widehat{y}_i)^2}{n}}.$$

Example 4.1. Consider the linear boundary value problem of Riemann-Liouville fractional order

$$y''(t) + \sin t D^{0.5} y(t) + t y(t) = f(t), \quad 0 < t < 1, \tag{11}$$

with the boundary condition:

$$y(0) = y(1) = 0,$$

where

$$f(t) = t^9 - t^8 + 56t^6 - 42t^5 + \sin t\left(\frac{32768}{6435\sqrt{\pi}} t^{7.5} - \frac{2048}{429\sqrt{\pi}} t^{6.5}\right), \tag{12}$$

and the exact solution is $y(t) = t^8 - t^7$ [6].
Let's solve the above example by the method of weighted residuals using a fractional power polynomial functions as a basis. That is, let the approximating function $\widehat{y}(t)$ be

$$\widehat{y}(t) = \sum_{i=0}^{n} c_i t^{\frac{c}{d} + \frac{e}{f} i}.$$

Table 1: Numerical results for Example 1 using $\varphi_i(t) = \{t^{6+\frac{95}{100}i}\}_{i=0}^{n}$

n	$E_{RMS} - Galerkin$	$E_{RMS} - LeastSquares$	$E_{RMS} - Subdomain$
1	0.06337614509	0.06201871147	0.05913693352
2	0.01942054388	0.02013470606	0.01609329029
3	0.02147222606	0.02395010463	0.01667516339

Table 2: Numerical results for Example 1 using $\varphi_i(t) = \{t^{8-\frac{95}{100}i}\}_{i=0}^{n}$

n	$E_{RMS} - Galerkin$	$E_{RMS} - LeastSquares$	$E_{RMS} - Subdomain$
1	0.02403628953	0.02342287470	0.01948231563
2	0.01895034797	0.01994715241	0.01654689139
3	0.02192845645	0.02441210961	0.01653205299

By applying the boundary condition and calculating the second derivative and 0.5 Riemann-Liouville derivative of $\widehat{y}(t)$ the residual $R(t)$ could be found:

$$R(t) = \widehat{y}''(t) + \sin t D^{0.5}\, \widehat{y}(t) + t\widehat{y}(t) - f(t), \tag{13}$$

The numerical results are summarized in Tables 1 and 2.

Example 4.2. Consider the nonlinear boundary value problem of Caputo fractional order

$$D_*^{0.25} y(t) + t y^2(t) = f(t), \qquad 0 < t < 1, \tag{14}$$

with the boundary condition

$$y(0) = 0, \qquad y(1) = 1,$$

where

$$f(t) = \frac{32}{21\Gamma(0.75)} t^{1.75} + t^5,$$

and the exact solution is $y(t) = t^2$[11].
We solve the present example by the method of weighted residuals using a fractional power polynomial function as a basis. That is, let the approximating function $\widehat{y}(t)$ be

$$\widehat{y}(t) = \sum_{i=0}^{n} c_i t^{\frac{c}{d} + \frac{e}{f} i}.$$

Table 3: Numerical results for Example 2 using $\varphi_i(t) = \{t^{\frac{175}{100}+\frac{25}{100}i}\}_{i=0}^{n}$

n	$E_{RMS} - Galerkin$	$E_{RMS} - LeastSquares$	$E_{RMS} - Subdomain$
1	0.0	0.0	0.000003244042158
2	0.0	0.0	0.000006365831408
3	0.0	0.0	0.000007910210400

Table 4: Numerical results for Example 2 using $\varphi_i(t) = \{t^{2-\frac{95}{100}i}\}_{i=0}^{n}$

n	$E_{RMS} - Galerkin$	$E_{RMS} - LeastSquares$	$E_{RMS} - Subdomain$
1	0.0	0.00001625833120	0.000001214985793
2	0.00001292837411	0.00001901002242	0.000005589105048
3	-	-	-

By applying the boundary condition and calculating 0.25 Caputo derivative of $\widehat{y}(t)$ the residual $R(t)$ could be found:

$$R(t) = D_*^{0.25}\widehat{y}(t) + t\widehat{y}^2(t) - \frac{32}{21\Gamma(0.75)}t^{1.75} - t^5. \tag{15}$$

The computational results are summarized in Tables 3 and 4.

In the weighted residuals solutions, the basis of $\{t^{2-\frac{95}{100}i}\}_{i=0}^{n}$ form for $n = 3, \cdots$ are not used because the Caputo derivative of order 0.25 of the approximated function $\widehat{y}(t)$

$$\widehat{y}(t) = \sum_{i=0}^{n} c_i t^{2-\frac{95}{100}i}, \qquad n = 3, 4, \cdots$$

does not exist.

5 Conclusion

In this paper, the method of weighted residuals for approximate solution of linear and nonlinear boundary value problems of fractional order is introduced and proposed. Moreover, a comparison between the exact solution and three sub-method of weighted residuals method using the fractional power polynomials basis, shows that the error of the approximation is small and usually only a few iterations leading to very accurate solutions. Two examples of boundary value problems of fractional order were solved by weighted residuals to illustrate the efficiency and accuracy of the method. By this method in Example 2, we found the exact solution.

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