

On the Local Cohomology Modules Defined by a Pair of Ideals and Serre Subcategory

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Abstract. This paper is concerned with the relation between local cohomology modules defined by a pair of ideals and Serre classes of R -modules. Let R be a commutative Noetherian ring, I, J be two ideals of R and M be an R -module. Let $\mathfrak{a} \in \tilde{W}(I, J)$ and $t \in \mathbb{N}_0$ be such that $Ext_R^t(R/\mathfrak{a}, M) \in S$ and $Ext_R^j(R/\mathfrak{a}, H_{I,J}^i(M)) \in S$ for all $i < t$ and all $j \geq 0$. Then for any submodule N of $H_{I,J}^t(M)$ such that $Ext_R^1(R/\mathfrak{a}, N) \in S$, we obtain $Hom_R(R/\mathfrak{a}, H_{I,J}^t(M)/N) \in S$.

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1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring, I, J denote two ideals of R , and M denotes an arbitrary R -module. By \mathbb{N}_0 , we shall mean the set of non-negative integers. For basic results, notations and terminologies not given in this paper, the reader is referred to [7, 21], if necessary.

As a generalization of the usual local cohomology modules, Takahashi, Yoshino and Yoshizawa [21], introduce the local cohomology modules with respect to a pair of ideals (I, J) . To be more precise, let $W(I, J) = \{p \in Spec(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some positive integer } n\}$ and $\tilde{W}(I, J)$ denotes the set of ideals \mathfrak{a} of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n . For an R -module M , we consider the (I, J) -torsion submodule $\Gamma_{I,J}(M)$ of M which consists of all elements x

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of M with $Supp(Rx)$ in $W(I, J)$. Furthermore, for an integer i , the local cohomology functor $H_{I,J}^i$ with respect to (I, J) is defined to be the i -th right derived functor of $\Gamma_{I,J}$. Also $H_{I,J}^i(M)$ is called the i -th local cohomology module of M with respect to (I, J) . If $J = 0$, then $H_{I,J}^i$ coincides with the ordinary local cohomology functor H_I^i with the support in the closed subset $V(I)$.

Recently, some authors approached the study of properties of these extended modules, see for Example [8, 9, 18, 22].

It is well known that an important problem in commutative algebra is to determine when the R -module $Hom_R(R/I, H_I^i(M))$ is finite. Grothendieck in [13] conjectured the following:

If R is a Noetherian ring, then for any ideal I of R and any finite R -module M , the modules $Hom_R(R/I, H_I^i(M))$ are finite for all $i \geq 0$.

In [14], Hartshorne gave a counterexample to Grothendieck's conjecture and he defined the concept of I -cofinite modules to generalize the conjecture. In [6], Brodmann and Lashgari showed that if, for a finite R -module M and an integer t , the local cohomology modules $H_I^0(M)$, $H_I^1(M)$, \dots , $H_I^{t-1}(M)$ are finite, then R -module $Hom_R(R/I, H_I^t(M))$ is finite and so $Ass(H_I^t(M)/N)$ is a finite set for any finite submodule N of $H_I^t(M)$. A refinement of this result for I -minimax R -modules which has been proved in [4] is as follows. Recall that an R -module M is said to be an I -minimax if the I -relative Goldie dimension of any quotient module of M is finite, see [4].

Theorem 1.1. *Let M be an I -minimax R -module and t be a non-negative integer such that $H_I^i(M)$ is I -minimax for all $i < t$. Then for any I -minimax submodule N of $H_I^t(M)$, the R -module*

$$Hom_R(R/I, H_I^t(M)/N)$$

is I -minimax.

Also authors in [1, 3] studied local cohomology modules by means of Serre subcategories. As a consequence, for an arbitrary Serre subcategory \mathcal{S} , authors in [3] showed the following result.

Theorem 1.2. *Let $s \in \mathbb{N}_0$ be such that*

$$Ext_R^s(R/I, M) \in \mathcal{S}$$

and

$$Ext_R^j(R/I, H_I^i(M)) \in \mathcal{S}$$

for all $i < s$ and all $j \geq 0$. Let N be a submodule of $H_I^s(M)$ such that $Ext_R^1(R/I, N) \in \mathcal{S}$. Then $Hom_R(R/I, H_I^s(M)/N) \in \mathcal{S}$.

The aim of the present paper is to generalize the concept of an I -cominimax R -module, introduced in [4], to an arbitrary Serre subcategory \mathcal{S} , to verify situations in which the R -module $\text{Hom}_R(R/I, H_{I,J}^i(M))$ belongs to \mathcal{S} . To approach it, we use the methods of [3, 4]. Our paper consists of four sections as follows.

In Section 2, by using the concept of (I, J) -relative Goldie dimension, we introduce the (I, J) -minimax R -modules and we study some properties of them, (see Proposition 2.7).

In Section 3, for an arbitrary Serre subcategory \mathcal{S} , we define (\mathcal{S}, I, J) -cominimax R -modules. This concept of R -modules can be considered as a generalization of I -cofinite R -modules [13], I -cominimax R -modules [4], and (I, J) -cofinite R -modules [22]. Also, as a main result of our paper, we prove the following. (See Theorem 3.13).

Theorem 1.3. *Let $\mathfrak{a} \in \tilde{W}(I, J)$. Let $t \in \mathbb{N}_0$ be such that $\text{Ext}_R^t(R/\mathfrak{a}, M) \in \mathcal{S}$ and $\text{Ext}_R^j(R/\mathfrak{a}, H_{I,J}^i(M)) \in \mathcal{S}$ for all $i < t$ and all $j \geq 0$. Then for any submodule N of $H_{I,J}^t(M)$ such that $\text{Ext}_R^1(R/\mathfrak{a}, N) \in \mathcal{S}$, we have $\text{Hom}_R(R/\mathfrak{a}, H_{I,J}^t(M)/N) \in \mathcal{S}$.*

One can see, by replacing various Serre classes with \mathcal{S} and using Theorem 1.3, the main results of [2, Theorem 1.2], [3, Theorem 2.2], [4, Theorem 4.2], [5, Lemma 2.2], [6], [11, Corollary 2.7], [15], [16, Corollary 2.3], and [22, Theorems 2.3, 2.5] are obtained (see Theorem 3.13, Corollary 3.14).

At last, in Section 4, as an application of the results of the previous sections, we give the following consequence about finiteness of associated primes of local cohomology modules. (See Proposition 4.1, Corollary 4.2, Notations 2.6, and 3.3)

Proposition 1.4. *Let $t \in \mathbb{N}_0$ be such that $\text{Ext}_R^t(R/I, M) \in \mathcal{S}_{I,J}$ and $H_{I,J}^i(M) \in \mathcal{C}(\mathcal{S}_{I,J}, I, J)$ for all $i < t$. Let N be a submodule of $H_{I,J}^t(M)$ such that $\text{Ext}_R^1(R/I, N)$ belongs to $\mathcal{S}_{I,J}$. If $\text{Supp}(H_{I,J}^t(M)/N) \subseteq V(I)$ then $\text{Gdim} H_{I,J}^t(M)/N < \infty$ and so $H_{I,J}^t(M)/N$ has finitely many associated primes; in particular, for $N = JH_{I,J}^t(M)$.*

2. Serre Classes and (I, J) -Minimax Modules

Recall that for an R -module H , the Goldie dimension of H is defined as the cardinality of the set of indecomposable submodules of $E(H)$, the injective hull of H , which appear in a decomposition of $E(H)$ in to direct sum of indecomposable submodules. Therefore, H is said to have finite Goldie dimension if H does not contain an infinite direct sum of non-zero submodules, or equivalently $E(H)$

decomposes as a finite direct sum of indecomposable (injective) submodules. We shall use $Gdim H$ to denote the Goldie dimension of H . For a prime ideal \mathfrak{p} , let $\mu^0(\mathfrak{p}, H)$ denotes the 0-th Bass number of H with respect to the prime ideal \mathfrak{p} . Also, an R -module M is said to minimax if there is a finite submodule N of M such that the quotient module M/N is Artinian. It is known that $\mu^0(\mathfrak{p}, H) > 0$ iff $\mathfrak{p} \in Ass(H)$. It is clear by the definition of the Goldie dimension that $Gdim H = \sum_{\mathfrak{p} \in Ass(H)} \mu^0(\mathfrak{p}, H) = \sum_{\mathfrak{p} \in Spec(R)} \mu^0(\mathfrak{p}, H)$. Also, the (I, J) -relative Goldie dimension of H is defined as $Gdim_{I,J} H := \sum_{\mathfrak{p} \in W(I,J)} \mu^0(\mathfrak{p}, H)$. (See [18, Definition 3.1]). If $J = 0$, then $Gdim_{I,J} H = Gdim_I H$ (see [10, Definition 2.5]); moreover if $I = 0$, we obtain $Gdim_{I,J} H = Gdim H$. It is known that when R is a Noetherian ring, an R -module H is minimax if and only if any homomorphic image of H has finite Goldie dimension (see [12, 23, 24]). This motivates the definition of (I, J) -minimax modules.

Definition 2.1. *An R -module M is said to be minimax with respect to (I, J) or (I, J) -minimax if the (I, J) -relative Goldie dimension of any quotient module of M is finite.*

Remarks 2.2. *By Definition 2.1, it is clear that*

- (i) $Gdim_I M \leq Gdim_{I,J} M \leq Gdim M$.

These inequalities maybe strict (see [18, Definition 3.1]).

- (ii) *For a Noetherian ring R , an R -module M is minimax iff for any R -submodule N of M , $Gdim M/N < \infty$. Therefore, by (i), in the Noetherian case, the class of I -minimax R -modules contains the class of (I, J) -minimax R -modules and it contains the class of minimax R -modules.*

Example and Remarks 2.3. It is easy to see that

- (i) Every quotient of finite modules, Artinian modules, Matlis reflexive modules and linearly compact modules have finite (I, J) -relative Goldie dimension, and so all of them are (I, J) -minimax modules.
- (ii) If $I = 0$, then $W(I, J) = Spec(R) = V(I)$ and so an R -module M is minimax iff it is (I, J) -minimax iff it is I -minimax.
- (iii) If $J = 0$, then $W(I, J) = V(I)$, so that an R -module M is (I, J) -minimax iff it is I -minimax.
- (iv) let M be an I -torsion module. Then, by [10, Lemma 2.6] and [18, Lemma 3.3], M is minimax iff it is (I, J) -minimax iff it is I -minimax.
- (v) If M is (I, J) -torsion module, then M is minimax iff it is (I, J) -minimax. (By the definition and [18, Lemma 3.3]). Specially, when $(0) \in \tilde{W}(I, J)$.

- (vi) By [21, Corollary (1.8) (2)], the class of (I, J) -torsion is a Serre subcategory of R -modules. Therefore, by part(v), in this category, the concept of minimax modules coincides with the concept of (I, J) -minimax modules; specially, for the (I, J) -torsion module of $H_{I,J}^i(M)$ ($i \geq 0$).
- (vii) If $Min(M) \subseteq W(I, J)$ and $Gdim_{I,J} M < \infty$, then by [18, Lemma 3.3] and definition, $Gdim M < \infty$, and so $|Ass(M)| < \infty$.

The following proposition shows that the class of (I, J) -minimax R -modules is a Serre subcategory.

Proposition 2.4. *Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of R -modules. Then M is (I, J) -minimax if and only if M' and M'' are both (I, J) -minimax.*

Proof. One can obtain the result, by replacing $W(I, J)$ with $V(I)$ and a modification of the proof of Proposition 2.3 of [4]. \square

Remark 2.5. *Recall that a class \mathcal{S} of R -modules is a “Serre subcategory” or “Serre class” of the category of R -modules, when it is closed under taking submodules, quotients and extensions. For example, the following class of R -modules are Serre subcategory.*

- (a) *The class of Zero modules.*
- (b) *The class of Noetherian modules.*
- (c) *The class of Artinian modules.*
- (d) *The class of R -modules with finite support.*
- (e) *The class of all R -modules M with $\dim_R M \leq n$, where n is a non-negative integer.*
- (f) *The class of minimax modules and the class of I -cofinite minimax R -modules. (see [17, Corollary 4.4]).*
- (g) *The class of I -minimax R -modules. (see [4, Proposition 2.3]).*
- (h) *The class of I -torsion R -modules and the class of (I, J) -torsion R -modules. (see [21, Corollary 1.8]).*
- (i) *The class of (I, J) -minimax R -modules. (Proposition 2.4).*

Notations 2.6. In this paper, the following notations are used for the following Serre subcategories:

“ \mathcal{S} ” for an arbitrary Serre class of R -modules;

“ \mathcal{S}_0 ” for the class of minimax R -modules;

“ \mathcal{S}_I ” for the class of I -minimax R -modules;

“ $\mathcal{S}_{I,J}$ ” for the class of (I, J) -minimax R -modules.

Using the above notations and Remark 2.2, we have $\mathcal{S}_0 \subseteq \mathcal{S}_{I,J} \subseteq \mathcal{S}_I$.
Now, we exhibit some of the properties of $\mathcal{S}_{I,J}$.

Proposition 2.7. *Let I, J, I', J' be ideals of R and M be an R -module. Then*

- (i) $\mathcal{S}_{I,J} = \mathcal{S}_{\sqrt{I},J} = \mathcal{S}_{\sqrt{I},\sqrt{J}} = \mathcal{S}_{I,\sqrt{J}}$.
- (ii) *If $I^n \subseteq \sqrt{J}$, for some $n \in \mathbb{N}$ (or equally, if R/J is an I -torsion R -module), then $\mathcal{S}_0 = \mathcal{S}_{I,J}$.*
- (iii) *If $I^n \subseteq \sqrt{I'}$, for some $n \in \mathbb{N}$, then $\mathcal{S}_{I,J} \subseteq \mathcal{S}_{I',J}$.*
- (iv) *If $J^n \subseteq \sqrt{J'}$, for some $n \in \mathbb{N}$, then $\mathcal{S}_{I,J'} \subseteq \mathcal{S}_{I,J}$.*
- (v) *If $I^n \subseteq \sqrt{I'}$, for some $n \in \mathbb{N}$ and M is (I', J) -torsion, then $M \in \mathcal{S}_{I,J}$ iff $M \in \mathcal{S}_0$ iff $M \in \mathcal{S}_{I',J}$.*
- (vi) *If $J^n \subseteq \sqrt{J'}$, for some $n \in \mathbb{N}$ and M is (I, J) -torsion, then $M \in \mathcal{S}_{I,J}$ iff $M \in \mathcal{S}_0$ iff $M \in \mathcal{S}_{I,J'}$.*

Proof. All these statements follow easily from [21, Propositions 1.4, 1.6] and Remark 2.3. As an illustration, we just prove statement (iii).

Let $H \in \mathcal{S}_{I,J}$. Since $I^n \subseteq \sqrt{I'}$, we have $W(\sqrt{I'}, J) \subseteq W(I, J)$, by [21, Proposition 1.6]. Now, since H is (I, J) -minimax, the assertion follows from the definition. \square

Lemma 2.8. (i) *If $N \in \mathcal{S}$ and M is a finite R -module, then for any submodule H of $Ext_R^i(M, N)$ and T of $Tor_R^i(M, N)$, we have*

$$Ext_R^i(M, N)/H \in \mathcal{S}$$

and $Tor_R^i(M, N)/T \in \mathcal{S}$, for all $i \geq 0$.

(ii) *For all $i \geq 0$, we have $Ext_R^i(R/I, M) \in \mathcal{S}_{I,J}$ iff $Ext_R^i(R/I, M) \in \mathcal{S}_0$ iff $Ext_R^i(R/I, M) \in \mathcal{S}_I$.*

Proof. (i) The result follows from [3, Lemma 2.1].

(ii) Since, for all i , $Ext_R^i(R/I, M)$ and $Tor_R^i(R/I, M)$ are (I, J) -torsion R -modules, the assertion holds by Remark 2.3 (iv). \square

The following proposition can be thought of as a generalization of Proposition 2.6 of [4], in case of $J = 0$ and $\mathcal{S} = \mathcal{S}_I$

Proposition 2.9. *Let $\text{Min}(M) \subseteq W(I, J)$. If $M \in \mathcal{S}$, then $H_{I,J}^i(M) \in \mathcal{S}$ for all $i \geq 0$.*

Proof. By hypothesis and [21, Corollary 1.7], M is (I, J) -torsion R -module and so $H_{I,J}^0(M) = \Gamma_{I,J}(M) = M$. Therefore, $H_{I,J}^i(M) = 0$ for all $i \geq 1$, by [21, Corollary 1.13]. Thus the assertion holds. \square

By applying the same method of the proof of [4, Theorem 2.7] and the properties of Serre subcategory, for $\mathcal{S} = \mathcal{S}_I$, we can obtain the following theorem, which is a generalization of [4, Theorem 2.7]. Some applications of this result will appear in Section 3.

Theorem 2.10. *Let M be a finite R -module and N an arbitrary R -module. Let $t \in \mathbb{N}_0$. Then the following conditions are equivalent:*

- (i) $\text{Ext}_R^i(M, N) \in \mathcal{S}$ for all $i \leq t$.
- (ii) For any finite R -module H with $\text{Supp}(H) \subseteq \text{Supp}(M)$, $\text{Ext}_R^i(H, N) \in \mathcal{S}$ for all $i \leq t$.

Corollary 2.11. *Let $r \in \mathbb{N}_0$. Then, for any R -module M , the following conditions are equivalent:*

- (i) $\text{Ext}_R^i(R/I, M) \in \mathcal{S}$ for all $i \leq r$.
- (ii) For any ideal \mathfrak{a} of R with $\mathfrak{a} \supseteq I$, $\text{Ext}_R^i(R/\mathfrak{a}, M) \in \mathcal{S}$ for all $i \leq r$.
- (iii) For any finite R -module N with $\text{Supp}(N) \subseteq V(I)$, $\text{Ext}_R^i(N, M) \in \mathcal{S}$ for all $i \leq r$.
- (iv) For any $\mathfrak{p} \in \text{Min}(I)$, $\text{Ext}_R^i(R/\mathfrak{p}, M) \in \mathcal{S}$ for all $i \leq r$.

Proof. Apply the same method of the proof of [4, Corollary 2.8]. \square

3. (\mathcal{S}, I, J) -Cominimax Modules and $H_{I,J}^i(M)$

Recall that M is said to be (I, J) -cofinite if M has support in $W(I, J)$ and $\text{Ext}_R^i(R/I, M)$ is a finite R -module for each $i \geq 0$ (see [22, Definition 2.1]). In fact this definition is a generalization of I -cofinite modules, which is introduced by Hartshorne in [14]. Considering an arbitrary Serre subcategory of R -modules instead of finitely generated one, we can give a generalization of (I, J) -cofinite modules as follows.

Definition 3.1. *Let R be a Noetherian ring and I, J be two ideals of R . For the Serre subcategory \mathcal{S} of the category of R -modules, an R -module M*

is called an (\mathcal{S}, I, J) -cominimax module precisely when $\text{Supp}(M) \subseteq W(I, J)$ and $\text{Ext}_R^i(R/I, M) \in \mathcal{S}$ for all $i \geq 0$.

Remark 3.2. By applying various Serre classes of R -modules in 3.1, we may obtain different concepts. But in view of [21, Proposition 1.7], the class of (\mathcal{S}, I, J) -cominimax R -modules is contained in the class of (I, J) -torsion R -modules. Moreover, for every R -module M and all $i \geq 0$, $\text{Ext}_R^i(R/I, M)$ is I -torsion, so by Lemma 2.8 (ii), we have $\text{Ext}_R^i(R/I, M) \in \mathcal{S}_{I, J}$ iff $\text{Ext}_R^i(R/I, M) \in \mathcal{S}_0$. In other words, the class of (\mathcal{S}_0, I, J) -cominimax R -modules and the class of $(\mathcal{S}_{I, J}, I, J)$ -cominimax R -modules are the same. Also, since $\text{Supp}(M) \subseteq V(I)$ implies that $\text{Supp}(M) \subseteq W(I, J)$, hence the class of $(\mathcal{S}_{I, J}, I, J)$ -cominimax R -modules contains the class of (\mathcal{S}_I, I, J) -cominimax R -modules.

Notation 3.3. For a Serre class \mathcal{S} of R -modules and two ideals I, J of R , we use $\mathcal{C}(\mathcal{S}, I, J)$ to denote the class of all (\mathcal{S}, I, J) -cominimax R -modules.

Example and Remark 3.4. (i) Let $N \in \mathcal{S}$ be such that $\text{Supp}(N) \subseteq W(I, J)$. Then it follows from Lemma 2.8 (i) that $N \in \mathcal{C}(\mathcal{S}, I, J)$.

(ii) Let N be a pure submodule of R -module M . By using the following exact sequence $0 \rightarrow \text{Ext}_R^i(R/I, N) \rightarrow \text{Ext}_R^i(R/I, M) \rightarrow \text{Ext}_R^i(R/I, M/N) \rightarrow 0$, for all $i \geq 0$, [19, Theorem 3.65], $M \in \mathcal{C}(\mathcal{S}, I, J)$ iff $N, M/N \in \mathcal{C}(\mathcal{S}, I, J)$; in particular, when $\mathcal{S} = \mathcal{S}_{I, J}$.

Proposition 3.5. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of R -modules such that two of the modules belong to $\mathcal{C}(\mathcal{S}, I, J)$. Then the third one belongs to $\mathcal{C}(\mathcal{S}, I, J)$.

Proof. The assertion follows from the induced long exact sequence

$$\cdots \rightarrow \text{Ext}_R^i(R/I, M) \rightarrow \text{Ext}_R^i(R/I, M'') \rightarrow \text{Ext}_R^{i+1}(R/I, M') \rightarrow \cdots$$

and Lemma 2.8 (i). \square

An immediate consequence of Proposition 3.5 and Lemma 2.8 is as follows.

Proposition 3.6. Let I, J, I', J' are ideals of R . Then

- (i) $M \in \mathcal{C}(\mathcal{S}, I, J)$ iff $M \in \mathcal{C}(\mathcal{S}, \sqrt{I}, J)$ iff $M \in \mathcal{C}(\mathcal{S}, I, \sqrt{J})$ iff $M \in \mathcal{C}(\mathcal{S}, \sqrt{I}, \sqrt{J})$.
- (ii) If M is I -cominimax, then $M \in \mathcal{C}(\mathcal{S}_{I, J}, I, J)$.
- (iii) If $\text{Min}(M) \subseteq W(I', J)$, $\text{Ext}_R^i(R/I, M) \in \mathcal{S}_{I, J}$, and $I^n \subseteq \sqrt{I'}$ for some $n \in \mathbb{N}$ and all $i \geq 0$, then $M \in \mathcal{C}(\mathcal{S}_{I, J}, I, J)$ and $M \in \mathcal{C}(\mathcal{S}_{I', J}, I', J)$. In particular, if $M \in \mathcal{C}(\mathcal{S}_{I, J}, I, J)$, then we have $M \in \mathcal{C}(\mathcal{S}_{I', J}, I', J)$.

- (iv) If $\text{Min}(M) \subseteq W(I, J)$ and $J^n \subseteq \sqrt{J'}$ for some $n \in \mathbb{N}$, then $M \in \mathcal{C}(\mathcal{S}_{I,J}, I, J)$ iff $M \in \mathcal{C}(\mathcal{S}_{I,J'}, I, J')$.

Proof. (i) Since $V(I) = V(\sqrt{I})$, the assertions follow from [21, Proposition 1.6], Corollary 2.11, and Definition 3.1.

(ii) By assumption and Lemma 2.8 (ii), $\text{Supp}(M) \subseteq V(I) \subseteq W(I, J)$ and $\text{Ext}_R^i(R/I, M) \in \mathcal{S}_{I,J}$ for all $i \geq 0$.

(iii), (iv) Apply [21, Propositions 1.6, 1.7], Corollary 2.11, and Proposition 2.7 (iii),(iv). \square

The following Remark plays an important role in the proof of our main theorems in this section.

Remark 3.7. In view of proof [21, Theorem 3.2], $\Gamma_{\mathfrak{a}}(M) \subseteq \Gamma_{I,J}(M)$, for any $\mathfrak{a} \in \tilde{W}(I, J)$. Thus $\Gamma_{I,J}(M) = 0$ implies that $\Gamma_{\mathfrak{a}}(M) = 0$, for all $\mathfrak{a} \in \tilde{W}(I, J)$. Now, let $\bar{M} = M/\Gamma_{I,J}(M)$ and $E = E_R(\bar{M})$ be the injective hull of R -module \bar{M} . Put $L = E/\bar{M}$. Since $\Gamma_{I,J}(\bar{M}) = 0$, then $\Gamma_{I,J}(E) = 0$ and also for any $\mathfrak{a} \in \tilde{W}(I, J)$, we have $\Gamma_{\mathfrak{a}}(\bar{M}) = 0 = \Gamma_{\mathfrak{a}}(E)$. In particular, the R -module $\text{Hom}_R(R/\mathfrak{a}, E)$ is zero. Now, from the exact sequence $0 \rightarrow \bar{M} \rightarrow E \rightarrow L \rightarrow 0$, and applying $\text{Hom}_R(R/\mathfrak{a}, -)$ and $\Gamma_{I,J}(-)$, we have the following isomorphisms

$$\text{Ext}_R^i(R/\mathfrak{a}, L) \cong \text{Ext}_R^{i+1}(R/\mathfrak{a}, \bar{M}) \text{ and } H_{I,J}^i(L) \cong H_{I,J}^{i+1}(M),$$

for any $\mathfrak{a} \in \tilde{W}(I, J)$ and all $i \geq 0$. In particular, $\text{Ext}_R^i(R/I, L) \cong \text{Ext}_R^{i+1}(R/I, \bar{M})$.

Proposition 3.8. Let $t \in \mathbb{N}_0$ be such that $H_{I,J}^i(M) \in \mathcal{C}(\mathcal{S}, I, J)$ for all $i < t$. Then $\text{Ext}_R^i(R/I, M) \in \mathcal{S}$ for all $i < t$.

Proof. We use induction on t . When $t = 0$, there is nothing to prove. For $t = 1$, since $\text{Hom}_R(R/I, \Gamma_{I,J}(M)) = \text{Hom}_R(R/I, M)$, and $\Gamma_{I,J}(M)$ is (\mathcal{S}, I, J) -cominimax, the result is true. Now, suppose that $t \geq 2$ and the case $t - 1$ is settled. The exact sequence $0 \rightarrow \Gamma_{I,J}(M) \rightarrow M \rightarrow \bar{M} \rightarrow 0$ induced the long exact sequence

$$\cdots \rightarrow \text{Ext}_R^i(R/I, \Gamma_{I,J}(M)) \rightarrow \text{Ext}_R^i(R/I, M) \rightarrow \text{Ext}_R^i(R/I, \bar{M}) \rightarrow \cdots$$

Since $\Gamma_{I,J}(M) \in \mathcal{C}(\mathcal{S}, I, J)$, we have $\text{Ext}_R^i(R/I, \Gamma_{I,J}(M)) \in \mathcal{S}$ for all $i \geq 0$. Therefore, it is enough to show that $\text{Ext}_R^i(R/I, \bar{M}) \in \mathcal{S}$ for all $i < t$. For this purpose, let $E = E_R(\bar{M})$ and $L = E/\bar{M}$. Now, by Remark 3.8, for all $i \geq 0$, we get the isomorphisms $H_{I,J}^i(L) \cong H_{I,J}^{i+1}(M)$ and $\text{Ext}_R^i(R/I, L) \cong \text{Ext}_R^{i+1}(R/I, \bar{M})$. Now, by assumption, $H_{I,J}^{i+1}(M) \in \mathcal{C}(\mathcal{S}, I, J)$ for all $i < t - 1$, and so $H_{I,J}^i(L) \in \mathcal{C}(\mathcal{S}, I, J)$ for all $i < t - 1$. Thus, by the inductive hypothesis, $\text{Ext}_R^i(R/I, L) \in \mathcal{S}$ and so $\text{Ext}_R^{i+1}(R/I, \bar{M}) \in \mathcal{S}$ for all $i < t - 1$. \square

The next corollary generalizes Proposition 3.7 of [4].

Corollary 3.9. *Let $H_{I,J}^i(M) \in \mathcal{C}(\mathcal{S}, I, J)$ for all $i \geq 0$. Then $\text{Ext}_R^i(R/I, M) \in \mathcal{S}$ for all $i \geq 0$; particularly, when \mathcal{S} is the class of I -minimax modules or the class of (I, J) -minimax modules.*

The Proposition 3.8 of [4] can be obtained from the following theorem when $J = 0$ and $\mathcal{S} = \mathcal{S}_I$.

Theorem 3.10. *Let $\text{Ext}_R^i(R/I, M) \in \mathcal{S}$ for all $i \geq 0$. Let $t \in \mathbb{N}_0$ be such that $H_{I,J}^i(M) \in \mathcal{C}(\mathcal{S}, I, J)$, for all $i \neq t$, then $H_{I,J}^t(M) \in \mathcal{C}(\mathcal{S}, I, J)$.*

Proof. We use induction on t . If $t = 0$, we must prove that $\text{Ext}_R^i(R/I, \Gamma_{I,J}(M)) \in \mathcal{S}$ for all $i \geq 0$. By the exact sequence

$$\dots \rightarrow \text{Ext}_R^{i-1}(R/I, \bar{M}) \rightarrow \text{Ext}_R^i(R/I, \Gamma_{I,J}(M)) \rightarrow \text{Ext}_R^i(R/I, M) \rightarrow \dots$$

and the hypothesis, it is enough to show that $\text{Ext}_R^i(R/I, \bar{M}) \in \mathcal{S}$ for all $i \geq 0$. Now, by Remark 3.7 and our assumption, we obtain $H_{I,J}^i(L) \in \mathcal{C}(\mathcal{S}, I, J)$. Therefore Corollary 3.9 implies that $\text{Ext}_R^i(R/I, \bar{M}) \in \mathcal{S}$ for all $i \geq 0$ (note that $\text{Ext}_R^0(R/I, \bar{M}) = 0$). Now suppose, inductively, that $t > 0$ and the result has been proved for $t - 1$. By Remark 3.7, it is easy to show that L satisfies in our inductive hypothesis. Therefore, the assertion follows from $H_{I,J}^t(M) \cong H_{I,J}^{t-1}(L)$. \square

Corollary 3.11. *Let $M \in \mathcal{S}$ and $t \in \mathbb{N}_0$ be such that $H_{I,J}^i(M)$ is (\mathcal{S}, I, J) -cominimax for all $i \neq t$. Then $H_{I,J}^t(M)$ is (\mathcal{S}, I, J) -cominimax.*

Proof. This is an immediate consequence of Lemma 2.8 (i) and Theorem 3.11. \square

Corollary 3.12. *Let I be a principal ideal and J be an arbitrary ideal of R . Let $M \in \mathcal{S}$. Then $H_{I,J}^i(M)$ is (\mathcal{S}, I, J) -cominimax for all $i \geq 0$.*

Proof. For $i = 0$, since $H_{I,J}^0(M)$ is a submodule of M and $M \in \mathcal{S}$, it turns out that $H_{I,J}^0(M)$ is (\mathcal{S}, I, J) -cominimax, by Remark 3.4 (i). Now, let $I = aR$. By [21, Definition 2.2 and Theorem 2.4], we have $H_{I,J}^i(M) \cong H^i(C_{I,J}^\bullet \otimes_R M) = 0$ for all $i > 1$. Therefore the result follows from Theorem 3.10. \square

Now we are prepared to prove the main theorem of this section, which is a generalization of one of the main results of [3, Theorem 2.2] and also [22, Theorem 2.3].

Theorem 3.13. *Let $\mathfrak{a} \in \tilde{W}(I, J)$. Let $t \in \mathbb{N}_0$ be such that $\text{Ext}_R^t(R/\mathfrak{a}, M) \in \mathcal{S}$*

and $Ext_R^j(R/\mathfrak{a}, H_{I,J}^i(M)) \in \mathcal{S}$ for all $i < t$ and all $j \geq 0$. Then for any submodule N of $H_{I,J}^t(M)$ such that $Ext_R^1(R/\mathfrak{a}, N) \in \mathcal{S}$, we have $Hom_R(R/\mathfrak{a}, H_{I,J}^t(M)/N) \in \mathcal{S}$; in particular, for $\mathfrak{a} = I$.

Proof. Considering the following long exact sequence

$$\cdots \rightarrow Hom_R(R/\mathfrak{a}, H_{I,J}^t(M)) \rightarrow Hom_R(R/\mathfrak{a}, H_{I,J}^t(M)/N) \rightarrow Ext_R^1(R/\mathfrak{a}, N) \rightarrow \cdots,$$

since $Ext_R^1(R/\mathfrak{a}, N) \in \mathcal{S}$, it is enough to show that $Hom_R(R/\mathfrak{a}, H_{I,J}^t(M)) \in \mathcal{S}$. To do this, we use induction on t . When $t = 0$, since $Hom_R(R/\mathfrak{a}, \Gamma_{I,J}(M)) = Hom_R(R/\mathfrak{a}, M) \in \mathcal{S}$, the result is obtained. Next, we assume that $t > 0$ and that the claim is true for $t - 1$. Let $\bar{M} = M/\Gamma_{I,J}(M)$. Then, by the long exact sequence

$$\cdots \rightarrow Ext_R^t(R/\mathfrak{a}, M) \rightarrow Ext_R^t(R/\mathfrak{a}, \bar{M}) \rightarrow Ext_R^{t+1}(R/\mathfrak{a}, \Gamma_{I,J}(M)) \rightarrow \cdots,$$

and assumption, we conclude that $Ext_R^j(R/\mathfrak{a}, \bar{M}) \in \mathcal{S}$. Now, by using notation of Remark 3.7, it is easy to see that L satisfies the inductive hypothesis. So that we get $Hom_R(R/\mathfrak{a}, H_{I,J}^{t-1}(L)) \in \mathcal{S}$ and therefore, $Hom_R(R/\mathfrak{a}, H_{I,J}^t(M)) \in \mathcal{S}$, as required. \square

The main results of [4, Theorem 4.2], [5, Theorem 2.2], [2, Theorem 2.1], [15], [11, Corollary 2.7], and [16, Corollary 2.3] are all special cases of the next corollary, by replacing various Serre classes with \mathcal{S} and $J = 0$.

Corollary 3.14. *Let $t \in \mathbb{N}_0$ be such that $Ext_R^t(R/I, M) \in \mathcal{S}$ and $H_{I,J}^i(M) \in \mathcal{C}(\mathcal{S}, I, J)$ for all $i < t$. Then for any submodule N of $H_{I,J}^t(M)$ and any finite R -module M' with $Supp(M') \subseteq V(I)$ and $Ext_R^1(M', N) \in \mathcal{S}$, we have $Hom_R(M', H_{I,J}^t(M)/N) \in \mathcal{S}$.*

Proof. Apply Theorem 3.13 and Corollary 2.11. \square

Proposition 3.15. *Let $t \in \mathbb{N}_0$ be such that $H_{I,J}^i(M) \in \mathcal{C}(\mathcal{S}, I, J)$ for all $i < t$. Then the following statements hold:*

- (i) *If $Ext_R^t(R/I, M) \in \mathcal{S}$, then $Hom_R(R/I, H_{I,J}^t(M)) \in \mathcal{S}$.*
- (ii) *If $Ext_R^{t+1}(R/I, M) \in \mathcal{S}$, then $Ext_R^1(R/I, H_{I,J}^t(M)) \in \mathcal{S}$.*
- (iii) *If $Ext_R^i(R/I, M) \in \mathcal{S}$ for all $i \geq 0$, then $Hom_R(R/I, H_{I,J}^{t+1}(M)) \in \mathcal{S}$ iff $Ext_R^2(R/I, H_{I,J}^t(M)) \in \mathcal{S}$.*

Proof. (i) Apply Corollary 3.14 or Theorem 3.13.

(ii) We proceed by induction on t . If $t = 0$, then by the long exact sequence

$$\begin{aligned}
(*) \quad 0 &\rightarrow Ext_R^1(R/I, \Gamma_{I,J}(M)) \rightarrow Ext_R^1(R/I, M) \rightarrow Ext_R^1(R/I, \bar{M}) \\
&\rightarrow Ext_R^2(R/I, \Gamma_{I,J}(M)) \rightarrow Ext_R^2(R/I, M) \rightarrow Ext_R^2(R/I, \bar{M}) \\
&\vdots \\
&\rightarrow Ext_R^i(R/I, \Gamma_{I,J}(M)) \rightarrow Ext_R^i(R/I, M) \rightarrow Ext_R^i(R/I, \bar{M}) \\
&\rightarrow Ext_R^{i+1}(R/I, \Gamma_{I,J}(M)) \rightarrow \cdots
\end{aligned}$$

and $Ext_R^1(R/I, \bar{M}) \in \mathcal{S}$, the result follows. Suppose that $t > 0$ and the assertion is true for $t-1$. Since $\Gamma_{I,J}(M) \in \mathcal{C}(\mathcal{S}, I, J)$, so $Ext_R^i(R/I, \Gamma_{I,J}(M)) \in \mathcal{S}$ for all $i \geq 0$, and so by (*), $Ext_R^{t+1}(R/I, \bar{M}) \in \mathcal{S}$. Now, by the notations of Remark 3.7, it is easy to see that R -module L satisfies the inductive hypothesis and so $Ext_R^1(R/I, H_{I,J}^{t-1}(M)) \in \mathcal{S}$. Now, the result follows from $H_{I,J}^{t-1}(L) \cong H_{I,J}^t(M)$.

(iii) (\Rightarrow) We use induction on t . Let $t = 0$. Then considering the long exact sequence (*), it is enough to show that $Ext_R^1(R/I, \bar{M}) \in \mathcal{S}$. By Remark 3.7, we have

$$\begin{aligned}
Ext_R^1(R/I, \bar{M}) &\cong Hom_R(R/I, L) \\
&\cong Hom_R(R/I, \Gamma_{I,J}(L)) \\
&\cong Hom_R(R/I, H_{I,J}^1(M)),
\end{aligned}$$

as required. Suppose $t > 0$ and the assertion is true for $t-1$. Since $\Gamma_{I,J}(M) \in \mathcal{C}(\mathcal{S}, I, J)$, we have $Ext_R^i(R/I, \Gamma_{I,J}(M)) \in \mathcal{S}$ for all $i \geq 0$. Therefore the exactness of sequence (*) implies that $Ext_R^i(R/I, \bar{M}) \in \mathcal{S}$ for all $i \geq 0$. Again by using the notations of Remark 3.7, we get $Ext_R^i(R/I, L) \in \mathcal{S}$, for all $i \geq 0$, and also $Hom_R(R/I, H_{I,J}^t(L)) \cong Hom_R(R/I, H_{I,J}^{t+1}(M)) \in \mathcal{S}$. Now, by inductive hypothesis, $Ext_R^2(R/I, H_{I,J}^{t-1}(L)) \in \mathcal{S}$ and hence $Ext_R^2(R/I, H_{I,J}^t(M)) \in \mathcal{S}$, as required.

(\Leftarrow) This part can be proved by the same method of (\Rightarrow), using induction on t , the following exact sequence

$$Ext_R^1(R/I, M) \rightarrow Ext_R^1(R/I, \bar{M}) \rightarrow Ext_R^2(R/I, \Gamma_{I,J}(M)),$$

and Remark 3.7. \square

4. Finiteness Properties of Associated Primes

In this short section, we obtain some results, as some applications of previous sections, about associated prime ideals of local cohomology modules and also finiteness properties of them.

Proposition 4.1. *Let $t \in \mathbb{N}_0$ be such that $\text{Ext}_R^t(R/I, M) \in \mathcal{S}_{I,J}$ and $H_{I,J}^i(M) \in \mathcal{C}(\mathcal{S}_{I,J}, I, J)$ for all $i < t$. Let N be a submodule of $H_{I,J}^t(M)$ such that $\text{Ext}_R^1(R/I, N) \in \mathcal{S}_{I,J}$. If $\text{Supp}(H_{I,J}^t(M)/N) \subseteq V(I)$, then $\text{Gdim}(H_{I,J}^t(M)/N) < \infty$ and so $H_{I,J}^t(M)/N$ has finitely many associated primes.*

Proof. By using Theorem 3.14, for the Serre class $\mathcal{S}_{I,J}$, we have $\text{Hom}_R(R/I, H_{I,J}^t(M)/N) \in \mathcal{S}_{I,J}$. Hence, by Lemma 2.8 (ii), $\text{Hom}_R(R/I, H_{I,J}^t(M)/N) \in \mathcal{S}_0$, as required. \square

Corollary 4.2. *Let $t \in \mathbb{N}_0$ be such that $\text{Ext}_R^t(R/I, M)$ and $H_{I,J}^i(M)$ are (I, J) -minimax R -modules for all $i < t$. Let N be a submodule of $H_{I,J}^t(M)$ such that $\text{Supp}(H_{I,J}^t(M)/N) \subseteq V(I)$ and $\text{Ext}_R^1(R/I, N)$ is (I, J) -minimax. Then $H_{I,J}^t(M)/N$ has finite Goldie dimension and so $\text{Ass}(H_{I,J}^t(M)/N)$ is a finite set; in particular for $N = JH_{I,J}^t(M)$.*

Proof. For the first part, apply Remark 3.4 and Proposition 4.1. Since by [21, Corollary 1.9], $H_{I,J}^i(M)/JH_{I,J}^i(M)$ is I -torsion, the last part immediately follows from the first. \square

Corollary 4.3. (See [20, Theorem 4]) *Let $t \in \mathbb{N}_0$ be such that $\text{Ext}_R^t(R/I, M)$ is a finite R -module. If $H_I^i(M)$ is I -cofinite for all $i < t$ and $H_I^t(M)$ is minimax.*

Proof. In Proposition 3.15 (i), apply \mathcal{S} as the class of finite R -modules, and $J = 0$. Therefore, $\text{Hom}_R(R/I, H_I^t(M))$ is finite R -module. Now, use [17, Proposition 4.1]. \square

Corollary 4.4. *Let the situation be as in Corollary 4.3. Then the following statements hold:*

- (i) *If $\text{Ext}_R^{t+1}(R/I, M)$ is finite, then $\text{Hom}_R(R/I, H_I^{t+1}(M))$ and $\text{Ext}_R^1(R/I, H_I^t(M))$ are finite and so $\text{Ass}(H_I^{t+1}(M))$ is a finite set.*
- (ii) *If $\text{Ext}_R^i(R/I, M)$ is finite for all $i \geq 0$, then $\text{Ext}_R^2(R/I, H_I^t(M))$ is finite.*

Proof. (i) By Corollary 4.3, we conclude that $H_I^t(M)$ is I -cofinite. So $H_I^i(M)$ is I -cofinite for all $i < t + 1$. Now, using Proposition 3.15 (i),(ii).

(ii) The result follows from (i) and Proposition 3.15 (iii). \square

References

- [1] M. Aghapournahr and L. Melkersson, Local cohomology and serre subcategories, *J. Algebra*, 320 (2008), 1275-1287.
- [2] J. Asadollahi, K. Khashyarmanesh, and S. Salarian, On the finiteness properties of the generalized local cohomology modules, *Commun Algebra*, 30 (2002), 859-867.
- [3] M. Asgharzadeh and M. Tousi, A unified approach to local cohomology modules using Serre classes, *Can Math. Bulletin*, 53 (2010), 577-586.
- [4] J. Azami, R. Naghipour, and B. Vakili, Finiteness properties of local cohomology modules for \mathfrak{a} -minimax modules, *Proc. Amer. Math. Soc.*, 137 (2009), 439-448.
- [5] K. Bahmanpour and R. Naghipour, On the cofiniteness of local cohomology modules, *Proc. Amer. Math. Soc.*, 136 (2008), 2359-2363.
- [6] M. P. Brodmann and A. Lashgari Faghani, A finiteness result for associated primes of local cohomology modules, *Proc. Amer. Math. Soc.*, 128 (2000), 2851-2853.
- [7] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction With Geometric Applications*, Cambridge university press, 1998.
- [8] L. Chu, Top local cohomology modules with respect to a pair of ideals, *Proc. Amer. Math. Soc.*, 139 (2011), 777-782.
- [9] L. Chu and Q. Wang, Some results on local cohomology modules defined by a pair of ideals, *J. Math. Kyoto. Uni.*, 49 (2009), 193-200.
- [10] K. Divaani-Aazar and M. A. Esmkhani, Artinianness of local cohomology modules of ZD-modules, *Commun Algebra*, 33 (2005), 2857-2863.
- [11] K. Divaani-Aazar and A. Mafi, Associated primes of local cohomology modules, *Proc. Amer. Math. Soc.*, 133 (2005), 655-660.
- [12] C. Faith and M. D. Herbera, Endomorphism rings and tensor products of linearly compact modules, *Commun Algebra*, 54 (1997), 1215-1255.
- [13] A. Grothendieck, *Cohomologie Locale des Faisceaux Cohérents et Théorèmes de Lefschetz Locaux et Globaux*. SGA 2, North-Holland, Amsterdam: Séminaire de géométrie algébrique du Bois-Marie, 1968.
- [14] R. Hartshorne, Affine duality and cofiniteness, *Invent Math.*, 9 (1970), 154-164.

- [15] K. Khashyarmansh and S. Salarian, On the associated primes of local cohomology modules, *Commun Algebra*, 27 (1999), 6191-6198.
- [16] B. Lorestani, P. Sahandi, and S. Yassemi, Artinian local cohomology modules, *Can Math. Bulletin*, 53 (2010), 577-586.
- [17] L. Melkersson, Modules cofinite with respect to an ideal, *J. Algebra*, 285 (2005), 649-668.
- [18] S. Payrovi and M. L. Parsa, Artinianness of local cohomology modules defined by a pair of ideals, *B. Malays. Math. Sci. So.*, 53 (2010), 577-586.
- [19] J. J. Rotman, *An Introduction to Homological Algebra*, Academic press, 1979.
- [20] S. S. Laleh, M. Y. Sadeghi, and M. Hanifi Mostaghim, Some results on the cofiniteness of local cohomology modules, *Czech Math. J.*, 62 (2012), 105-110.
- [21] R. Takahashi, Y. Yoshino, and T. Yoshizawa, Local cohomology based on a nonclosed support defined by a pair of ideals, *J. Pure. Appl. Algebra*, 213 (2009), 582-600.
- [22] T. Tehranian and A. P. Talemi, Cofiniteness of local cohomology based on a nonclosed support defined by a pair of ideals, *Bull Iran Math. Soc.*, 36 (2010), 145-155.
- [23] T. Zink, Endlichkeitsbedingungen für modulen über einem Noetherschen ring, *Math. Nachr.*, 164 (1974), 239-252.
- [24] H. Zöschinger, Minimax-moduln, *J. Algebra*, 102 (1986), 1-32.

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