Journal of Mathematical Extension
Vol. 15, SI-NTFCA, (2021) (29)1-17
URL: https://doi.org/10.30495/JME.SI.2021.2221
ISSN: 1735-8299
Original Research Paper

Spectral Method for Third-Kind Volterra Integral Equation

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Abstract. This present work will focus on the spectral approach to a class of third-kind Volterra integral equation using the Boubaker polynomials as a basis function. In this approach, the operational matrix of fractional integration and the operational matrix of multiplication are utilized. The strategy that adopts here is expanding the unknown function in terms of Boubaker polynomials with unknown coefficients. Then, by using the given operational matrices, the problem under study is reduced to a problem easier to solve. The error bound of the suggested approximation is investigated. Some examples are implemented to display the efficiency and applicability of the recommended scheme.

AMS Subject Classification: 31A10; 26A33; 74G15
Keywords and Phrases: Third-kind Volterra integral equation, Boubaker polynomials, Operational matrix.

1 Introduction

The integral equations are widely applied in various fields such as mathematics, physics, mechanics of structures and engineering. Many problems concerning the mechanics of structures can be formulated equivalently as either a differential or integral equation. The integral equations

Received: November 2021; Published: February 2022
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are as important as differential equations. In mathematics, an integral equation is an equation in which an unknown function appears under the integral sign. Two of the most popular types of integral equations are Fredholm integral equations and Volterra integral equations. The integral equation with constant integration limits is called Fredholm integral equation, while if one limit of integration is a variable, it is a Volterra integral equation. Applications of Fredholm's integral equations arise in boundary issues. Also, Volterra equations are applied in the dynamics and rheology of constructions. The numerical solution and analysis of these equations have been investigated by many authors. In this paper, the following third-kind Volterra integral equation (VIEs) will be considered

$$\tau^{\mu}\nu(\tau) = \rho(\tau) + \int_0^{\tau} (\tau - s)^{-\gamma}\kappa(\tau, s)\nu(s)ds \quad \tau \in I = [0, T], \tag{1}$$

where $\gamma \in [0, 1), \mu > 0, \rho(\tau) = \tau^{\mu}\rho_1(\tau)$ and κ are continuous on I and $D := \{(\tau, s) : \tau \in I,\ 0 \leq s \leq \tau\}$, respectively. κ for $\mu + \gamma \geq 1$ has the form

$$\kappa(\tau, s) = s^{\mu+\mu-1}\kappa_1(\tau, s),$$

where $\kappa_1 \in C(D)$. So far, this class of equations has been considered by many scientists. The first study of the mentioned equation was reported by Evans in 1910 and 1911 [8]. The Existence, uniqueness, and regularity of solutions to Eq. (1) were carried out by Seyed Allaei et al. in [3]. They have obtained necessary conditions for converting Eq. (1) into a cordial VIEs. This model studied in [19, 20]. The collocation method has been investigated for the third-kind VIEs by Seyed Allaei et al. in [2] and the multistep collocation method has been explained by Shayanfard in 2019 [18]. In [13], the authors have introduced a numerical technique based on hat functions to solve the third-kind of VIEs. Recently, polynomials have received a great deal of attention in solving integral equations because such problems can be converted into finding the solution of a system of algebraic equations using these polynomials. In this way, differential equations can be solved much more simply. In the current research work, a numerical scheme based on Boubaker polynomials is proposed for solving the third-kind of VIEs. We first expand the solution using the Boubaker polynomials with unknown coefficients. In this process, the integral operator is approximated by the operational matrix of fractional integral. The advantage of this scheme is that it is transforming the given equations into a set of algebraic equations by using the mentioned matrix. The mentioned polynomials have been carried out to various types of equations. The Boubaker polynomials were first introduced by Boubaker [5] for solving a one-dimensional formulation of the heat transfer equation. In [10], the authors reported temperature 3D profiling in cryogenic cylindrical devices. Love's equation and Boltzmann diffusion equation were studied using the Boubaker polynomials [6, 11]. The numerical study on optimal control problems using the Boubaker polynomials was undertaken by Kafash et al. [9]. The mentioned problems of fractional order were reported by Rabiei [16]. The application of these polynomials to solve the multi-order fractional differential equations was discussed in [7]. It is also worth mentioning that the integral equations (1) and fractional differential equations are very similar. In (1), the integral operator alters to the Riemann-Liouville operator of orde $1-\gamma$, when $\kappa(t,\tau)=1$. This paper is arranged as follows: In Section 2, a brief discussion of definitions and relations of the fractional calculus are presented. After that, some relevant properties of the Boubaker polynomials and function approximation are given. Also, The error bound of the suggested approximations is investigated. The operational matrix of fractional integration and the operational matrix of multiplication are provided in Section 3. The proposed approach is described in Section 4. Section 6 is devoted to some examples to show the validity of the sugested approach. Finally, the conclusion is given in the last section.

2 Preliminaries

This section is devoted to some applicable definitions of the fractional calculus [4, 15]. Then, the shifted Boubaker polynomials and function approximation are presented [5, 6].

Definition 2.1. The fractional integral for a function $\nu(t)$ is given in

Riemann-Liouville (RL) sense by

$$\mathcal{I}_\tau^\gamma \nu(\tau) = \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau - s)^{\gamma-1} \nu(s) ds, \quad \gamma > 0,$$

some properties of this operator are given as:

$$\begin{aligned} \mathcal{I}^\mu \mathcal{I}^\gamma \nu(\tau) &= \mathcal{I}^{\mu+\gamma} \nu(\tau), \\ \mathcal{I}^\mu \mathcal{I}^\gamma \nu(\tau) &= \mathcal{I}^\gamma \mathcal{I}^\mu \nu(\tau), \\ \mathcal{I}^\gamma \tau^\eta &= \frac{\Gamma(\eta+1)}{\Gamma(\gamma+\eta+1)} \tau^{\gamma+\eta}. \end{aligned}$$

Definition 2.2. For a function $\nu(\tau)$, the Caputo definition of fractional derivative is defined by

$$\mathcal{D}_\tau^\gamma \nu(\tau) = \begin{cases} \frac{1}{\Gamma(n-\gamma)} \int_0^\tau \frac{\nu^{(n)}(s)}{(\tau-s)^{1+\gamma-n}} ds, & n-1 < \gamma < n, \quad n \in \mathbb{N}, \\ \dfrac{d^n}{d\tau^n} \nu(\tau), & \gamma = n. \end{cases}$$

The relationship between these two operators ia as follow:

$$\mathcal{I}_\tau^\gamma \mathcal{D}_\tau^\alpha \nu(\tau) = \nu(\tau) - \sum_{k=0}^{n-1} \nu^{(k)}(0^+) \frac{\tau^k}{k!}.$$

Definition 2.3. Boubaker polynomials is given by [5, 6, 7, 14, 17]:

$$B_i(\tau) = \sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^p \left[\frac{(i-4p)}{(i-p)} C_{i-p}^p \right] \tau^{i-2p}, \quad i \geq 0, \tag{2}$$

where

$$C_{i-p}^p = \frac{(i-p)!}{p!(i-2p)!}.$$

The symbol $\lfloor . \rfloor$ denotes the floor function.

The Boubaker polynomials can be determined with the following recurrence relation:

$$\begin{cases} B_0(\tau) = 1, \\ B_1(\tau) = \tau, \\ B_m(\tau) = \tau B_{m-1}(\tau) - B_{m-2}(\tau), \quad \text{for} \quad m \geq 2. \end{cases}$$

Let $\nu(\tau) \in L^2[0,1]$, then it can be approximated in terms of the Boubaker polynomials as follows:

$$\nu(\tau) = \sum_{j=0}^{\infty} \nu_j B_j(\tau),$$

where ν_j can be obtained by

$$\nu_j = Q^{-1} \int_0^1 \nu(\tau) B_j(\tau) d\tau, \quad j = 0, 1, \ldots \tag{3}$$

in which

$$Q = \int_0^1 B_i(\tau) B_j(\tau) d\tau,$$

In particular applications, $\nu(t)$, we have

$$\nu_n = \sum_{j=0}^{m} \nu_j B_j(\tau) = V^T \Phi(\tau),$$

with

$$\begin{aligned} &\Phi(t) = \left[B_0(\tau), B_1(\tau), \ldots, B_m(\tau)\right], \\ &V^T = \left[\nu_0, \nu_1, \ldots, \nu_m\right], \end{aligned}$$

Lemma 2.4. *Suppose $\nu \in C^{m+1}$ and $S_n = Span\{B_0(\tau), B_1(\tau), \ldots, B_m(\tau)\}$. If $\nu_n = V^T \Phi(\tau)$ be the best approximation ν out of S_n then*

$$\parallel \nu(\tau) - V^T \Phi(\tau) \parallel_{L^2[0,1]} \leq \frac{\hat{\varepsilon}}{(m+1)!\sqrt{2m+3}},$$

where $\hat{\varepsilon} = max_{\tau \in [0,1]} \left| \nu^{(n+1)}(\tau) \right|$.

Proof: According to the Taylor expansion, we define

$$\varphi(\tau) = \nu(0) + \tau \nu'(0) + \frac{\tau^2}{2!} \nu''(0) + \ldots + \frac{\tau^m}{m!} \nu^{(m)}(0),$$

Also, we know

$$\mid \nu(\tau) - \varphi(\tau) \mid \leq \nu^{(m+1)}(\zeta) \frac{\tau^{m+1}}{(m+1)!}, \quad \zeta \in (0,1), \tag{4}$$

Since $V^T\Phi(\tau)$ is the best approximation for $\nu(t)$, then using (4), we have

$$\begin{aligned}
\| \nu - V^T\Phi(\tau) \|^2_{L^2[0,1]} &\leq \| \nu - \varphi(\tau) \|^2_{L^2[0,1]} \\
&= \int_0^1 | \nu - \varphi(t) |^2 \, dt \\
&\leq \int_0^1 [\nu^{(m+1)}(\zeta) \frac{\tau^{m+1}}{(m+1)!}]^2 d\tau \\
&\leq \frac{\hat{\varepsilon}^2}{(m+1)!^2} \int_0^1 \tau^{2m+2} d\tau \\
&= \frac{\hat{\varepsilon}^2}{(m+1)!^2(2m+3)},
\end{aligned}$$

by taking the square roots we have the upper bound and this complete proof.

3 The Operational Matrix

In this section, we will consider the procedure of obtaining the operational matrices of fractional integral and multiplication for Boubaker polynomials [7, 16].

3.1 The operational matrix of fractional integration

The RL integral of $\Phi(t)$ can be approximated as

$$\mathcal{I}_\tau^\gamma \Phi(\tau) \simeq P^\gamma \Phi(\tau),$$

where P^γ is an $(m+1) \times (m+1)$ matrix which is called the operational matrix of fractional integration. By utilizing the definition of the Boubaker polynomials (2) for $i \geq 2$ and by appling the RL integral, we yield

$$\begin{aligned}
\mathcal{I}_\tau^\gamma B_i(\tau) &= \sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^p \left[\frac{(i-4p)}{(i-p)} C_{i-p}^p \right] \mathcal{I}_\tau^\gamma \tau^{i-2p} \\
&= \sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^p \frac{(i-p-1)!(i-4p)}{p! \, \Gamma(i-2p+\gamma+1)} \tau^{i-2p+\gamma}. \qquad (5)
\end{aligned}$$

Now we approximate $\tau^{i-2p+\gamma}$ in terms of the Boubaker polynomials as

$$\tau^{i-2p+\gamma} \simeq \sum_{q=0}^{m} c_{p,q} B_q(\tau), \tag{6}$$

by substituting Eq. (6) in Eq. (5), we get

$$\begin{aligned} \mathcal{I}_\tau^\gamma B_i(\tau) &\simeq \sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^p \frac{(i-p-1)!(i-4p)}{p!\, \Gamma(i-2p+\gamma+1)} \sum_{q=0}^{m} c_{p,q} B_q(\tau) \\ &= \sum_{q=0}^{m} \left(\sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} \theta_{i,q,p} \right) B_q(\tau), \end{aligned} \tag{7}$$

where $\theta_{i,q,p}$ is

$$\theta_{i,q,p} = (-1)^p \frac{(i-p-1)!(i-4p)}{p!\, \Gamma(i-2p+\gamma+1)} c_{p,q},$$

and rewrite Eq. (7) for $i = 2, \cdots, m$

$$\mathcal{I}_t^\gamma B_i(t) \simeq \left[\sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} \theta_{i,0,p}, \sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} \theta_{i,1,p}, \cdots, \sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} \theta_{i,m,p} \right] \Phi(\tau), \tag{8}$$

for $i = 0, 1$

$$\mathcal{I}_\tau^\gamma B_i(\tau) = \frac{1}{\Gamma(\gamma+i+1)} \tau^{\gamma+i}, \quad i = 0, 1,$$

according Eq. (3), we can express $t^{\gamma+i}$ in terms of Boubaker polynomials as

$$\tau^{\gamma+i} \simeq \sum_{q=0}^{m} \mathcal{V}_{i,q} B_q(\tau), \tag{9}$$

A combination of Eqs. (8) and (9) leads

$$P^{\gamma}=\begin{bmatrix} \frac{\mathcal{V}_{0,0}}{\Gamma(\gamma+1)} & \frac{\mathcal{V}_{0,1}}{\Gamma(\gamma+1)} & \cdots & \frac{\mathcal{V}_{0,m}}{\Gamma(\gamma+1)} \\ \frac{\mathcal{V}_{1,0}}{\Gamma(\gamma+2)} & \frac{\mathcal{V}_{1,1}}{\Gamma(\gamma+2)} & \cdots & \frac{\mathcal{V}_{mm}}{\Gamma(\gamma+m)} \\ \sum_{p=0}^{1}\theta_{2,0,p} & \sum_{p=0}^{1}\varphi_{2,1,p} & \cdots & \sum_{p=0}^{1}\theta_{2,m,p} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{p=0}^{\lfloor \frac{m}{2}\rfloor}\theta_{m,0,p} & \sum_{p=0}^{\lfloor \frac{m}{2}\rfloor}\theta_{m,1,p} & \cdots & \sum_{p=0}^{\lfloor \frac{m}{2}\rfloor}\theta_{m,m,p} \end{bmatrix}.$$

where P^{γ} is the operational matrix of fractional integration for Boubaker polynomials.

3.2 The operational matrix of multiplication

Suppose V is an arbitrary vector, then we have the following form as:

$$V^T\Phi(\tau)\phi(\tau)^T \simeq \Phi(\tau)^T\tilde{V}, \tag{10}$$

where $\hat{V}$ is the operational matrix of multiplication for the Boubaker polynomials. Now, we approximate $\nu^T\Phi(x)\phi(t)^T$ in terms of the Boubaker polynomials

$$V^T\Phi(\tau)\phi(\tau)^T = [\nu_0(\tau), \dots, \nu_m(\tau)],$$

in which

$$\nu_i(\tau) \simeq \sum_{j=0}^{m} \tilde{\nu}_{i,j}B_j(\tau) = \tilde{V_i}^T\Phi(\tau), \tag{11}$$

where

$$\tilde{V_i} = [\tilde{\nu}_{i,0}, \tilde{\nu}_{i,1}, \dots, \tilde{\nu}_{i,m}]^T.$$

Now, So by considering $V_i = [\nu_i^0, \nu_i^1, \dots, \nu_i^m]^T$ and

$$\nu_i^k = \langle \nu_i(\tau)B_k(\tau)\rangle,$$

by using (11), we yeild

$$\nu_i^k = \left\langle \sum_{j=0}^{m} \tilde{\nu}_{i,j}B_j(\tau), B_k(\tau)\right\rangle = \sum_{j=0}^{m} \tilde{\nu}_{i,j}\langle B_j(\tau), B_k(\tau)\rangle,$$

consequently, we have

$$\tilde{V_i}^T = V_i^T Q^{-1},$$

and therefore, the operational matrix of multiplication was obtained.

4 Numerical Scheme Constructing the Method

This section is devoted to a numerical scheme based on Boubaker polynomials to solve (1). To this end, we exapand the following functions in terms of Boubaker polynomials to obtain numerical solution of Eq. (1) as

$$\tau^{\mu} \simeq \chi^T \Phi(\tau), \tag{12}$$
$$\nu(\tau) \simeq V^T \Phi(\tau), \tag{13}$$
$$\rho(\tau) \simeq \Lambda^T \Phi(\tau), \tag{14}$$
$$\kappa(\tau, s) \simeq \Phi^T(\tau) K \Phi(s), \tag{15}$$

where $V = [\nu_0, \nu_1, \ldots, \nu_m]$ is an unknown vector to be determined. By replacing Eq. (12)-(15) in Eq. (1), we yield

$$V^T \Phi(\tau)\Phi^T(\tau)\chi \simeq \Lambda^T \Phi(\tau) + \Phi^T(\tau) K \int_0^{\tau} (\tau - s)^{-\gamma} \Phi(s)\Phi^T(s) V ds. \tag{16}$$

Using (5) and (10), Eq. (16) can be written as

$$\begin{aligned} V^T \tilde{\chi}\Phi(\tau) &\simeq \Lambda^T + \Phi^T(\tau) K \tilde{V} \int_0^{\tau} (\tau - s)^{-\gamma} \Phi(s) ds \\ &= \Lambda^T + \Gamma(1-\gamma)\Phi^T(\tau) K \tilde{V} \mathcal{I}_t^{1-\gamma} \Phi(\tau) \\ &= \Lambda^T + \Gamma(1-\gamma)\Phi^T(\tau) K \tilde{V} P^{1-\gamma} \Phi(\tau). \end{aligned}$$

Therefore, we get to the following system

$$V^T \tilde{\chi} - \Lambda^T - \Gamma(1-\gamma)\Phi^T(t) K \tilde{V} P^{1-\gamma} = 0. \tag{17}$$

Then, the system obtained from Eq. (17), can be solved significantly simpler to evaluate the unknown vectors V. Consequently, we can determine the approximate solution of $\nu(t)$ from Eqs. (13).

5 Numerical Examples

In this section, we have implemented the suggested approach in the Section 4 for some examples.

Example 5.1. The first example to be considered is a particular case of the third-kind VIEs as the following form [2, 13]

$$\tau^{\frac{2}{3}}\nu(t) = \rho(\tau) + \int_0^\tau \frac{\sqrt{3}}{3\pi} s^{\frac{1}{3}}(\tau - s)^{-\frac{2}{3}}\nu(s)ds \quad \tau \in I = [0, 1], \tag{18}$$

where

$$\rho(\tau) = \tau^{\frac{47}{12}}\left(1 - \frac{\Gamma(\frac{1}{3})\Gamma(\frac{55}{12})}{\pi\sqrt{3}\Gamma(\frac{55}{12})}\right).$$

For this problem, there exist $\nu(\tau) = \tau^{\frac{13}{4}}$. Eq. (18) is a version of an Abel-type nonlinear equation related to Ligthill's model for the temperature distribution on the surface of a projectile moving through a laminar layer [1, 12]. We applied the suggested method for this example. Table 1 demonstrates the absolute error of $\nu(\tau)$ at different values of m. The exact and numerical solutions of $\nu(t)$ for $m = 4, 8$ are compared in Fie. 1. Fig. 2 demonstrates the absolute error of $\nu(\tau)$ for $m = 10$.

Table 1: Absolute errors of $\nu(\tau)$ at $m = 4, 10$ for Example 5.1.

τ	$m = 4$	$m = 10$
0.1	7.22950×10^{-5}	7.39394×10^{-6}
0.2	1.79692×10^{-4}	1.60666×10^{-5}
0.3	5.56510×10^{-4}	2.35453×10^{-5}
0.4	2.25499×10^{-4}	1.99577×10^{-5}
0.5	2.89103×10^{-4}	1.06700×10^{-5}
0.6	7.89823×10^{-5}	3.51947×10^{-5}
0.7	3.19913×10^{-4}	2.73020×10^{-5}
0.8	5.80912×10^{-4}	9.26617×10^{-6}
0.9	9.83001×10^{-5}	3.54259×10^{-5}
1.0	2.04156×10^{-2}	3.14177×10^{-6}

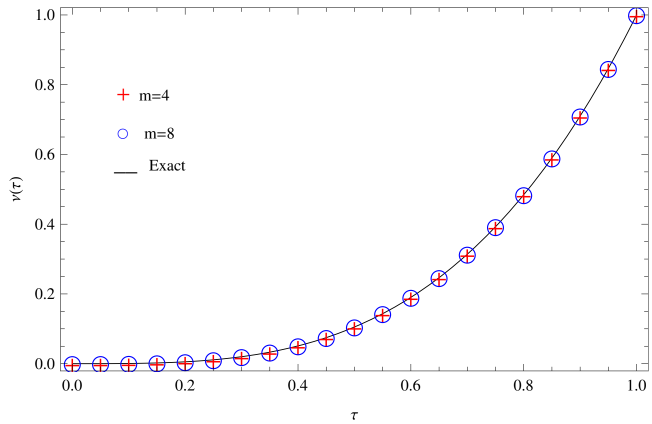

Figure 1: Comparison of the exact solution and numerical solutions for Example 5.1

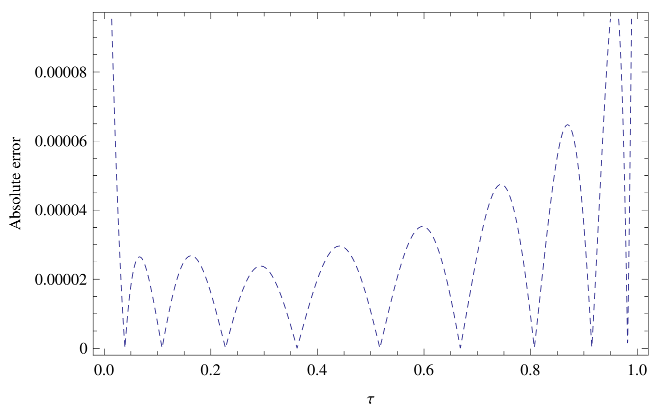

Figure 2: Absolute error of $\nu(\tau)$ at $m = 10$ for Example 5.1

Example 5.2. The another example to be considered is used in the modelling of some heat conduction problems with mixed-type boundary conditions [2, 13]

$$\tau\nu(\tau) = \frac{6}{7}t^3\sqrt{\tau} + \int_0^\tau \frac{1}{2}\nu(s)ds, \qquad \tau \in I = [0, 1],$$

For this problem, there exist $\nu(\tau) = \tau^{\frac{5}{2}}$. We applied the suggested method for this example. Table 2 displays the absolute error of $\nu(\tau)$ at different values of m. The obtained numerical solutions are compared with the exact solution for $m = 4, 8$ in Fie. 3. Fig. 4 demonstrates the absolute error of $\nu(\tau)$ for $m = 12$.

Table 2: Absolute errors of $\nu(\tau)$ at $m = 4, 8, 12$ for Example 5.1.

τ	$m = 4$	$m = 8$	$m = 12$
0.1	6.75453×10^{-5}	1.67833×10^{-5}	2.16633×10^{-6}
0.2	3.52039×10^{-4}	1.35238×10^{-5}	1.43583×10^{-6}
0.3	4.01156×10^{-4}	8.96428×10^{-5}	1.11726×10^{-6}
0.4	4.15311×10^{-4}	9.60197×10^{-6}	9.99087×10^{-7}
0.5	3.40305×10^{-4}	8.26090×10^{-6}	9.04193×10^{-7}
0.6	2.44923×10^{-4}	6.53335×10^{-6}	7.80942×10^{-7}
0.7	2.06809×10^{-4}	7.04011×10^{-6}	7.85543×10^{-7}
0.8	2.45683×10^{-4}	6.50333×10^{-6}	6.70279×10^{-7}
0.9	2.80168×10^{-5}	5.49367×10^{-6}	6.67733×10^{-7}
1.0	9.79413×10^{-5}	3.84593×10^{-6}	4.88508×10^{-7}

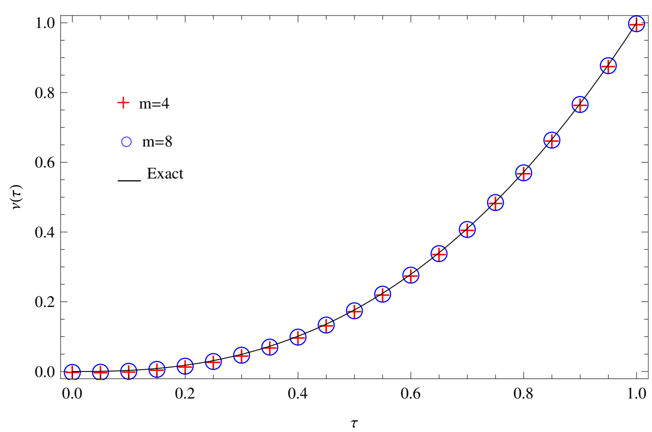

Figure 3: Comparison of the exact solution and numerical solutions for Example 5.2

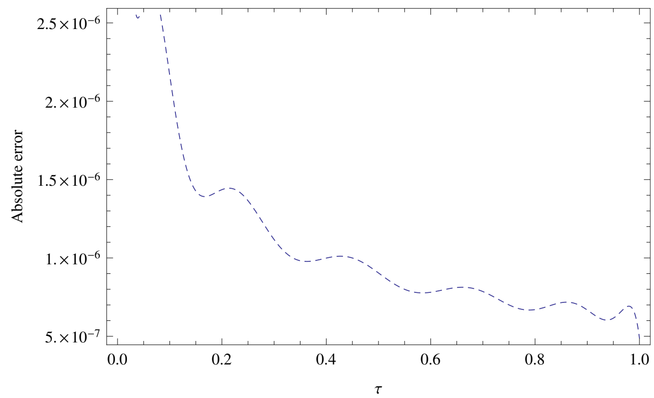

Figure 4: Absolute error of $\nu(\tau)$ at $m = 12$ for Example 5.2

Example 5.3. As final example, we consider the following the third-kind VIEs [13]

$$\tau^{\frac{3}{2}}\nu(\tau) = \rho(\tau) + \int_0^{\tau} \frac{\sqrt{2}}{2\pi} s(\tau - s)^{-\frac{1}{2}}\nu(s)ds, \qquad \tau \in [0, 1],$$

where

$$\rho(\tau) = \tau^{\frac{33}{10}} \left(1 - \frac{\Gamma(\frac{19}{5})}{\sqrt{2\pi}\Gamma(\frac{43}{10})}\right).$$

where the exact solution is $\nu(\tau) = \tau^{\frac{9}{5}}$. So like the previous process, we applied the suggested method for this example. Table 3 displays the absolute error of $\nu(\tau)$ at different values of m. In Fie. 5, the numerical results at $m = 4, 8$ were compared with the exact solution for validating the method.

6 Conclusion

In this paper, the spectral method was investigated for solving the third-kind VIEs. The proposed technique was presented to find the numerical solution of the mentioned equations based on the Boubaker polynomials. First, the operational matrices of fractional integration and multiplication were achieved. Then, the unknown function in terms of Boubaker

Table 3: Absolute errors of $\nu(\tau)$ at $m = 4, 10$ for Example 5.3.

τ	$m = 4$	$m = 10$
0.1	1.75262×10^{-3}	9.92046×10^{-4}
0.2	4.27859×10^{-4}	3.21049×10^{-4}
0.3	5.73541×10^{-4}	2.23037×10^{-5}
0.4	8.65173×10^{-4}	2.15771×10^{-5}
0.5	3.31430×10^{-4}	1.37100×10^{-5}
0.6	3.92188×10^{-4}	3.51305×10^{-5}
0.7	1.07391×10^{-4}	8.16281×10^{-5}
0.8	2.68400×10^{-4}	5.90158×10^{-5}
0.9	2.70037×10^{-4}	4.12784×10^{-5}
1.0	7.58123×10^{-4}	7.91058×10^{-5}

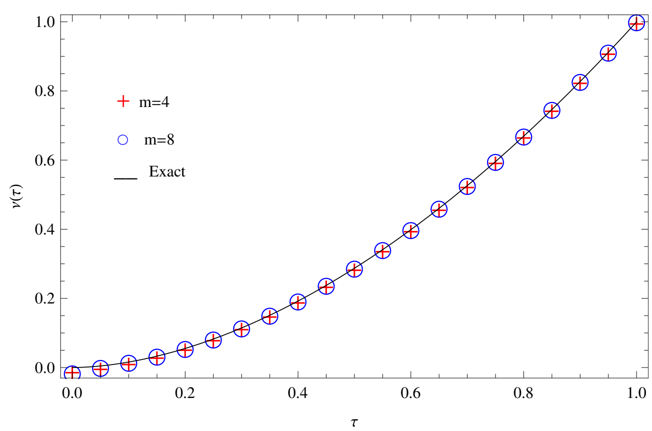

Figure 5: Comparison of the exact solution and numerical solutions for Example 5.3

polynomials was approximated. Finally, the mentioned problem has been transformed to a problem easier by using the aforementioned matrices. Some numerical examples were considered to show the accuracy of the proposed method. Moreover, the numerical results were compared with the exact solutions. It is apparent from the obtained solutions that the numerical approximations are confirmed in excellent agreement with the exact solutions.

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