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A New Algorithm Based on the Homotopy Perturbation Method For a Class of Singularly Perturbed Boundary Value Problems

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Abstract. In this paper, a new algorithm is presented to approximate the solution of a singularly perturbed boundary value problem with left-layer based on the homotopy perturbation technique and applying the Laplace transformation. The convergence theorem and the error bound of the proposed method are proved. The method is examined by solving two examples. The results demonstrate the reliability and efficiency of the proposed method.

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1. Introduction

We consider a class of singularly perturbed two-point singular boundary value problem of the form

$$\epsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0, 1], \tag{1}$$

subject to the boundary conditions:

$$y(0) = \alpha, \quad y(1) = \beta, \quad \alpha, \beta \in \Re \tag{2}$$

where ϵ is a small positive parameter and $a(x) > 0$, $b(x)$ and $f(x)$ are bounded continuous functions.

If $a(x), b(x), f(x) \in C[0, 1]$, then the problem (1) with the boundary conditions (2) posses unique solution $y(x) \in C[0, 1]$. In general, as ϵ tends to zero, the solution $y(x)$ may exhibit exponential boundary layers at left-end of the interval $[0, 1]$.

These problems arise frequently in many areas of science and engineering such as heat transfer problem with large Peclet numbers, Navier-Stokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, reaction-diffusion process, quantum mechanics, optimal control etc. Due to the variation in the width of the layer with respect to the small perturbation parameter ϵ, several difficulties are experienced in solving the singular perturbation problems using standard numerical methods [28].

Several numerical methods have been developed for the numerical solution of singularly perturbed boundary value problems [6],[7], in particular to the problems having the boundary layers at one or both ends of the interval. Boglaev [5], Schatz and Wahlbin [29] used the finite element technique to solve such types of problems. Miller [20] gave sufficient conditions for the first-order uniform convergence of three-point difference scheme. Cen et al. [6] presented hybrid finite difference scheme with Shishkin mesh for solving a system of singularly perturbed initial value problems. While Stojanovic [30] gave an optimal difference scheme by considering the quadratic interpolating splines instead of piecewise constants on each subinterval $[x_{i-1}, x_i]$ as an approximation for the coefficient $f(x)$. Surla and Jerkovic [31] considered the spline collocation method for the solution of singularly perturbed boundary

value problems. Rao and Kumar [23] gave an optimal B-spline collocation method for solving singularly perturbed boundary value problems. Loghmani and Ahmadinia [8] develop a numerical technique for singularly perturbed boundary value problems using B-spline functions and least square method. Dua and Kong [7] used the new Liouville-Green transform to solve a singularly perturbed second-order ordinary differential equation. Attili [3] used Pade approximation to obtain the solution of singularly perturbed two point boundary value problems. Also, Surla etal. [32] used tension spline to solve the singularly perturbed boundary value problems.
The concept of replacing singularly perturbed two-point boundary value problem by an initial value problem is presented by Reddy et al. [15], [25], [26]. Reddy and Chakravarthy [27] have extended boundary value technique to solve general singularly perturbed two-point boundary value problems using trapezoidal formula integration in the forward direction with left-layer boundary problems and in backward direction with right-layer boundary problems, and both formulas for interior or two boundary layers, where, their method is iterative on the deviating argument.
In this paper, we present a new algorithm based on HPM to find the approximate solution of singularly perturbed two-point boundary value problems with left-layer. The HPM was first proposed by He [9], [10], [4], [16], for solving functional equations. The method is based on homotopy in topology and provides an analytical approximate solution for functional equations. In recent years, this method has been efficiently employed to solve a wide range of linear and nonlinear problems in applied sciences [1].

The traditional perturbation techniques are based on the assumption that a small parameter must exist, which is too over-strict to find wide application, for most nonlinear equations have no small parameter at all. Some new techniques have been proposed to eliminate the small parameter assumption[12], such as the homotopy analysis method (HAM) proposed by Liao [17, 18].

In [13], a comparison of HPM and HAM was made, such that the HPM was revealed more powerful than the HAM. Furthermore, the HPM was further developed by applying the modern perturbation methods. The HAM, different from the perturbation methods, can be categorized into

a generalized Taylor expansion method. He in [13] explained that the HPM is different from the HAM and the difference between them is clear just as the Taylor series method is different from the perturbation methods. He mentioned that it is clear HAM is a generalized Taylor series method, searching for an infinite series solution, in order to enlarge the convergence region but HPM is clearly a new perturbation method, searching an asymptotic solution with few terms and no convergence theory is needed. But, Liao proved that the HPM is a special case of the HAM by special choice of the free parameters in this method.
This paper is organized as follows: In Section 2, we remind the classical homotopy perturbation method. In Section 3, we present a new algorithm to solve the above singularly perturbed boundary value problem based on the HPM and applying the Laplace transformation. The convergence and error bound of the proposed method are proved in Section 4. Finally, in Section 5, the method is applied to solve two sample examples to show the efficiency and importance of the method. Conclusion remarks are mentioned in Section 6.

2. Homotopy Perturbation Method

To illustrate the basic ideas of HPM, we consider the following equation:

$$A(y) - f(x) = 0, \qquad x \in \Omega, \tag{3}$$

with boundary condition(s):

$$B\left(y, \frac{\partial y}{\partial n}\right) = 0, \qquad x \in \Gamma, \tag{4}$$

where A is a general differential operator, B a boundary operator, $f(x)$ a known analytical function and Γ is the boundary of the domain Ω. The operator A can be generally decomposed into two parts L and N, where L is linear, while N is nonlinear. Therefore, Equation (3) can be written as follows:

$$L(y) + N(y) - f(x) = 0. \tag{5}$$

We construct a homotopy of (3), $u(x,p) : \Omega \times [0,1] \longrightarrow \Re$, which satisfies

$$H(y,p) = (1-p)[L(y) - L(y_0)] + p[A(y) - f(x)] = 0, \; x \in \Omega, \tag{6}$$

which is equivalent to

$$H(y,p) = L(y) - L(y_0) + pL(y_0) + p[N(y) - f(x)] = 0, \tag{7}$$

where $p \in [0,1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary condition(s) (4). It follows from (7) that

$$H(y,0) = L(y) - L(y_0) = 0 \quad \textit{and} \quad H(y,1) = A(y) - f(x) = 0. \tag{8}$$

Thus, the changing process of p from 0 to 1 is just that of $y(x,p)$ from $y_0(x)$ to $y(x)$. In topology, this is called deformation and $L(y) - L(y_0)$ and $A(y) - f(x)$ are called homotopic. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; further it can be considered as a small parameter for $0 \leqslant p \leqslant 1$. So it is very natural to assume that the solution of (6) can be expressed as

$$y(x,p) = u_0(x) + pu_1(x) + p^2u_2(x) + \cdots . \tag{9}$$

By substituting (9) into (7) and rearranging the resultant in terms of ascending powers of p, an infinite number of differential equations, is achieved. This set of almost simple differential equations with proper initial conditions is then solved. Finally an approximate solution of (3) is written as:

$$y \approx \sum_{i=0}^{m} u_i = u_0 + u_1 + \cdots + u_m. \tag{10}$$

The convergence of series (9) as $p \to 1$ has been considered by He in [9] and [14].

3. Main Idea

In this section, we shall introduce a reliable new algorithm to solve singularly perturbed boundary value problem (1) by using HPM and

Laplace transformations. The HPM will be applied in a straightforward manner, but with a new choice for the differential operator $L = \epsilon y'' + a^\star y'$ where $a^\star = a(0)$.
Now, we construct a homotopy which satisfies in (7) and

$$N(y) = (a(x) - a^\star)\, y' + b(x)y. \tag{11}$$

Suppose the solution of (1) has the form,

$$y(x) = u_0(x) + pu_1(x) + p^2 u_2(x) + \cdots . \tag{12}$$

Substituting (12) into (7) and collecting terms with the same powers of p, following sets of linear differential equations can be obtained:

$$\begin{aligned}
p^0 &: \ \epsilon u_0'' + a^\star u_0' - L(y_0) = 0, \ \ u_0(0) = \alpha, \ \ u_0'(0) = A, && (13)\\
p^1 &: \ \epsilon u_1'' + a^\star u_1' + L(y_0) + N(u_0) - f(x) = 0, \quad u_1(0) = u_1'(0) = 0, && (14)\\
p^i &: \ \epsilon u_i'' + a^\star u_i' + N(u_{i-1}) = 0, \quad u_i(0) = u_i'(0) = 0, \quad i = 2, 3, \cdots . && (15)
\end{aligned}$$

The initial approximation $y_0(x)$ can be freely chosen. Here we set $y_0(x) = \alpha + Ax$, from (13), we have

$$u_0(x) = \alpha + Ax, \tag{16}$$

where A is an unknown real number.
Solving linear equations (14) and (15), by using Laplace transformations, we obtain the following results:

$$\begin{aligned}
u_1(x) &= \int_0^x (A + N(u_0)(t) - f(t)) \left(e^{-\frac{a^\star (x-t)}{\epsilon}} - 1 \right) dt, && (17)\\
u_i(x) &= \int_0^x N(u_{i-1})(t) \left(e^{-\frac{a^\star (x-t)}{\epsilon}} - 1 \right) dt, \quad i = 2, 3, \cdots . && (18)
\end{aligned}$$

By using the relations (11) and (18) for $i = 2, 3, \cdots$

$$u_i(x) = \int_0^x (a(t) - a^\star)\left(e^{-\frac{a^\star(x-t)}{\epsilon}} - 1\right) u'_{i-1}(t)dt$$
$$+ \int_0^x b(t)\left(e^{-\frac{a^\star(x-t)}{\epsilon}} - 1\right) u_{i-1}(t)dt. \qquad (19)$$

Integration by parts gives

$$\begin{aligned} u_i(x) &= (a(t) - a^\star)\left(e^{-\frac{a^\star(x-t)}{\epsilon}} - 1\right) u_{i-1}(t)|_0^x \\ &+ \frac{a^\star}{\epsilon}\int_0^x (a(t) - a^\star)e^{-\frac{a^\star(x-t)}{\epsilon}} u_{i-1}(t)dt \\ &- \int_0^x a'(t)\left(e^{-\frac{a^\star(x-t)}{\epsilon}} - 1\right) u_{i-1}(t)dt \\ &+ \int_0^x b(t)\left(e^{-\frac{a^\star(x-t)}{\epsilon}} - 1\right) u_{i-1}(t)dt. \end{aligned} \qquad (20)$$

Since

$$(a(t) - a^\star)\left(e^{-\frac{a^\star(x-t)}{\epsilon}} - 1\right) u_{i-1}(t)|_0^x = 0,$$

then, for $i = 2, 3, \cdots$

$$u_i(x) = \frac{a^\star}{\epsilon}\int_0^x (a(t) - a^\star)e^{-\frac{a^\star(x-t)}{\epsilon}} u_{i-1}(t)dt$$
$$+ \int_0^x \big(b(t) - a'(t)\big)\left(e^{-\frac{a^\star(x-t)}{\epsilon}} - 1\right) u_{i-1}(t)dt. \qquad (21)$$

We can approximate the solution $y(x)$ by the finit series:

$$\phi_n(x) = \sum_{i=0}^{n-1} u_i(x). \qquad (22)$$

In most cases, 15 or 20 terms of the series give a satisfactory approximation of the solution. Consider the components $u_i(x), i \geqslant 0$ are determined, the solution in a series form is constructed using (22), where the constant $A = y'(0)$ is as yet undetermined. We should choose A, such that ϕ_n satisfies the boundary conditions. So, to determine the unknown constant A, we impose the boundary condition at $x = 1$ on ϕ_n. This will lead to linear algebraic equation for each approximant ϕ_n, which we solve by means of computational Maple Package.

4. Convergence of the Method

In this section, the convergence conditions of the proposed method for the differential equation (1) is analyzed. For any function u_i, we define the following norm

$$\|u_i\| = \max_{0\leqslant x\leqslant 1} |u_i(x)|.$$

Theorem 4.1. *Let* $\max_{0\leqslant x\leqslant 1} |a(x)-a^\star| = M_1$, $\max_{0\leqslant x\leqslant 1} |b(x)-a'(x)| = M_2$ *and* $M^\star = \max\{M_1, M_2\}$, *for all* $x \in [0,1]$ *and* $i = 3, 4, \cdots$
(a) if $M_1 = 0$ *and* $M_2 \leqslant 1$ *then* $\|u_i\| \leqslant \lambda_\epsilon \|u_{i-1}\|$ *where* $0 < \lambda_\epsilon < 1$ *for any* $0 < \epsilon < a^\star$.
(b) if $M_1 \neq 0$ *and* $M^\star \leqslant \frac{1}{2}$ *then* $\|u_i\| \leqslant \lambda^\star_\epsilon \|u_{i-1}\|$ *where* $0 < \lambda^\star_\epsilon < 1$ *for any* $0 < \epsilon < a^\star$.

Proof. For $i = 3, 4, \cdots$, from (21), the following inequalitis holds

$$\begin{aligned}
|u_i(x)| &\leqslant \Big|\frac{a^\star}{\epsilon}\int_0^x (a(t)-a^\star) e^{-\frac{a^\star(x-t)}{\epsilon}} u_{i-1}(t)dt\Big| \\
&+ \Big|\int_0^x \big(b(t)-a'(t)\big)\left(e^{-\frac{a^\star(x-t)}{\epsilon}}-1\right) u_{i-1}(t)dt\Big| \\
&\leqslant \frac{a^\star}{\epsilon}\int_0^x |(a(t)-a^\star)|\,|u_{i-1}(t)|\, e^{-\frac{a^\star(x-t)}{\epsilon}} dt \\
&+ \int_0^x \big|\big(b(t)-a'(t)\big)\big|\,|u_{i-1}(t)|\left(1-e^{-\frac{a^\star(x-t)}{\epsilon}}\right) dt \\
&\leqslant \frac{a^\star}{\epsilon} M_1 \|u_{i-1}\| \int_0^x e^{-\frac{a^\star(x-t)}{\epsilon}} dt \\
&+ M_2 \|u_{i-1}\| \int_0^x \left(1-e^{-\frac{a^\star(x-t)}{\epsilon}}\right) dt \\
&= \|u_{i-1}\| \left\{ M_1\left(1-e^{-\frac{a^\star x}{\epsilon}}\right) + \frac{M_2}{a^\star}\left(\epsilon e^{-\frac{a^\star x}{\epsilon}} + a^\star x - \epsilon\right)\right\} \\
&\leqslant \|u_{i-1}\| \left\{ M_1\left(1-e^{-\frac{a^\star}{\epsilon}}\right) + M_2\left(1+\frac{\epsilon}{a^\star}e^{-\frac{a^\star}{\epsilon}} - \frac{\epsilon}{a^\star}\right)\right\}. \qquad (23)
\end{aligned}$$

(a) If $M_1 = 0$ and $M_2 \leqslant 1$ then from (23), $|u_i(x)|$ satisfy the following bound

$$|u_i(x)| \leqslant \|u_{i-1}\| \left(1+\frac{\epsilon}{a^\star}e^{-\frac{a^\star}{\epsilon}} - \frac{\epsilon}{a^\star}\right).$$

Set $\lambda_\epsilon = 1 + \frac{\epsilon}{a^\star} e^{-\frac{a^\star}{\epsilon}} - \frac{\epsilon}{a^\star}$, since $0 < \epsilon < a^\star$ then it is clear $0 < \lambda_\epsilon < 1$. This completes the proof of case (a).

(*b*) If $M_1 \neq 0$ then from (23), $|u_i(x)|$ satisfy the following bound

$$|u_i(x)| \leqslant M^\star \|u_{i-1}\| \left\{ 2 + \left(\frac{\epsilon}{a^\star} - 1 \right) e^{-\frac{a^\star}{\epsilon}} - \frac{\epsilon}{a^\star} \right\}.$$

Set $\lambda_\epsilon^\star = M^\star \left\{ 2 + \left(\frac{\epsilon}{a^\star} - 1 \right) e^{-\frac{a^\star}{\epsilon}} - \frac{\epsilon}{a^\star} \right\}$, Since $0 < \epsilon < a^\star$ and $M^\star \leqslant \frac{1}{2}$, then it is clear $0 < \lambda_\epsilon^\star < 1$. This completes the proof of case (b). □

Theorem 4.2. *The series solution (22) of problem (1) using the HPM converges if one of the cases (a) or (b) in Theorem 4.1 is satisfied.*

Proof. Denote as $(C[0,1], \|.\|)$ the Banach space of all continuous functions on $[0,1]$. Define the sequence of partial sums $\{S_n\}$. We show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space if the case (a) in Theorem 4.1 is satisfied. For this purpose, we have

$$\|S_{n+1} - S_n\| = \|u_{n+1}\| \leqslant \lambda_\epsilon \|u_n\| \leqslant \lambda_\epsilon^2 \|u_{n-1}\| \leqslant \cdots \leqslant \lambda_\epsilon^{n-1} \|u_2\|, \quad n > 2. \tag{24}$$

Therefore, for every $n, j \in N, n > j > 2$,

$$\begin{aligned} \|S_n - S_j\| &= \|\,(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_{j+1} - S_j)\,\| \\ &\leqslant \|\,(S_n - S_{n-1})\,\| + \|\,(S_{n-1} - S_{n-2})\,\| + \cdots + \|\,(S_{j+1} - S_j)\,\| \\ &\leqslant \lambda_\epsilon^{n-2} \|u_2\| + \lambda_\epsilon^{n-3} \|u_2\| + \cdots + \lambda_\epsilon^{j-1} \|u_2\| \\ &= \frac{1 - \lambda_\epsilon^{n-j}}{1 - \lambda_\epsilon} \lambda_\epsilon^{j-1} \|u_2\|. \end{aligned} \tag{25}$$

Since $0 < \lambda_\epsilon < 1$, we have $\left(1 - \lambda_\epsilon^{n-j}\right) < 1$, then

$$\|S_n - S_j\| \leqslant \frac{\lambda_\epsilon^{j-1}}{1 - \lambda_\epsilon} \|u_2\|. \tag{26}$$

Since $u_1(x), (a(x) - a^\star)$ and $(b(x) - a'(x))$ are bounded for $x \in [0,1]$, then $\|u_2\| < \infty$, so, as $j \longrightarrow \infty$, then $\|S_n - S_j\| \longrightarrow 0$. We conclude that $\{S_n\}$ is a Cauchy sequence in $C[0,1]$, so the series converges and the proof is complete. □

Lemma 4.3. *The maximum absolute truncation error of the series solution (22) to problem (1) is estimated to be* $\|y(x) - \phi_j(x)\| = \|y(x) - \sum_{k=0}^{j-1} u_k(x)\| \leqslant \frac{\lambda_\epsilon^{j-2}}{1-\lambda_\epsilon} \|u_2\|$ *for* $j > 2$.

Proof. From inequality (26) for $j > 2$ we have

$$\|S_n - \phi_j\| = \|S_n - S_{j-1}\| \leqslant \frac{\lambda_\epsilon^{j-2}}{1-\lambda_\epsilon}\|u_2\|. \tag{27}$$

As $n \longrightarrow \infty$ then $S_n \longrightarrow y(x)$, so

$$\|y(x) - \phi_j(x)\| \leqslant \frac{\lambda_\epsilon^{j-2}}{1-\lambda_\epsilon}\|u_2\|. \tag{28}$$

This completes the proof. $\square$

Remark 4.4. *As $\epsilon \to 0$, λ_ϵ and $\lambda_\epsilon^\star$ tend to one, then the rate of convergence series (22) is decreased. This lemma will be illustrated obviously in the Example (5.2) in the next section.*

5. Numerical Examples

To demonstrate the applicability of the proposed method, two linear singular perturbation problems with left-end boundary layer are solved. The following examples have been chosen because they have been widely discussed in the literature and the approximate solutions are available for comparison. The proposed method in section 3. can be done by using the following algorithm:

Algorithm 1:
Step 1. Set $u_0(x) = \alpha + Ax$.
Step 2. calculate the $u_1(x)$, by applying the Eq. (17).
Step 3. for i from 2 to n do
calculate the $u_i(x)$, by applying the Eq. (21),
end do.
Step 4. Set $\phi_n(x) = \sum_{i=0}^{n-1} u_i(x)$ as the approximate of the exact solution.
Step 5. Calculate A, by solving $\phi_n(1) = \beta$.
The algorithm 1 is performed by Maple 13 with 500 digits precision.

Example 5.1. Consider the following (SPBVP) [2], [4], [8], [21]

$$\epsilon y''(x) + y\prime(x) - y(x) = 0, \quad x \in [0,1] \tag{29}$$

with boundary conditions $y(0) = 1$ and $y(1) = 1$. The exact solution is given by

$$y(x) = \frac{(e^{\eta_2} - 1)\, e^{\eta_1 x} + (1 - e^{\eta_1})\, e^{\eta_2 x}}{e^{\eta_2} - e^{\eta_1}} \tag{30}$$

where $\eta_1 = \frac{-1+\sqrt{1+4\epsilon}}{2\epsilon}$ and $\eta_2 = \frac{-1-\sqrt{1+4\epsilon}}{2\epsilon}$.

Solution: In order to solve the equation (29) using algorithm 1, we set

$$a^\star = 1, \quad N(y) = -y, \quad f(x) = 0, \quad \alpha = 1, \quad \beta = 1.$$

In this example, $M_1 = 0$ and $M_2 = 1$ according to theorem 4.2, series obtained by algorithm 1 is convergent to the exact solution (30) for each $0 < \epsilon < 1$. In view of steps 1 and 2, we get

$$\begin{aligned} u_0(x) &= 1 + Ax, \\ u_1(x) &= \int_0^x (A - 1 - At)\left(e^{-\frac{(x-t)}{\epsilon}} - 1\right) dt = \epsilon e^{-\frac{x}{\epsilon}}(1 - A - A\epsilon) \\ &+ \epsilon(A + A\epsilon - Ax - 1) \\ &+ x\left(1 + \frac{1}{2}Ax - A\right). \end{aligned} \tag{31}$$

Other components are determined similarly. Further we compute $y(x)$ for various value of ϵ.

$$\begin{aligned} For \quad \epsilon = 10^{-2}, \quad & y(x) \approx l_1 = \phi_{15}(x) = \frac{99}{100} - \frac{1}{100}Ax \\ & + x - \frac{101}{10000}\,\mathrm{e}^{-100\,x}A + \cdots, \\ For \quad \epsilon = 10^{-3}, \quad & y(x) \approx l_2 = \phi_{15}(x) = \frac{999}{1000} - \frac{1}{1000}Ax \\ & + x - \frac{1001}{1000000}\,\mathrm{e}^{-1000\,x}A + \cdots, \\ For \quad \epsilon = 10^{-4}, \quad & y(x) \approx l_3 = \phi_{15}(x) = \frac{9999}{10000} - \frac{1}{10000}Ax \\ & + x - \frac{10001}{100000000}\,\mathrm{e}^{-10000\,x}A + \cdots. \end{aligned} \tag{32}$$

To determine A, we impose the boundary condition at $x = 1$, we find

$$\phi_{15}(1) = 1, \quad then \quad \begin{cases} A = -63.1040528 & for \quad \epsilon = 10^{-2} \\ A = -632.016473 & for \quad \epsilon = 10^{-3} \\ A = -6321.10189 & for \quad \epsilon = 10^{-4} \end{cases} \tag{33}$$

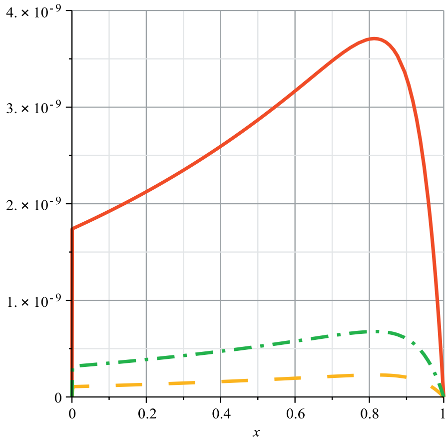

Figure 1: Dash spaced: $|l_1 - y|$, Dash doted : $|l_2 - y|$ and solid line : $|l_3 - y|$ for Example 5.1.

The numerical results are given in Table 1 for $\epsilon = 10^{-3}$ and $\epsilon = 10^{-4}$ and also it shows the numerical results in [8] obtained by using finite differences method (FDM).The comparison between the results shows the efficiency and accuracy of the proposed method. In Fig. 1, we plot the absolute error of the approximate solutions l_1, l_2, l_3 and y.

Table 1: Numerical results of Example 5.1.

x	$\lvert l_2 - y\rvert$ by proposed method	absolute error of the FDM in [8], $\epsilon = 10^{-3}$, $h = 10^{-2}$	$\lvert l_3 - y\rvert$ by proposed method	absolute error of the FDM in [8], $\epsilon = 10^{-4}, h = 10^{-2}$
0.1	1.57e-10	1.45e-3	1.92e-9	1.78e-3
0.2	1.74e-10	1.43e-3	2.12e-9	1.76e-3
0.4	2.12e-10	1.31e-3	2.59e-9	1.61e-3
0.6	2.59e-10	1.06e-3	3.12e-9	1.30e-3
0.8	3.03e-10	6.50e-4	3.70e-9	7.97e-4
0.9	2.72e-10	3.58e-4	3.31e-9	4.41e-4

Example 5.2. Consider the variable coefficient (SPBVP) [2], [8], [16]

$$\epsilon y''(x) + \left(1 - \frac{x}{2}\right) y\prime(x) - \frac{1}{2} y(x) = 0, \quad x \in [0, 1] \tag{34}$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. The exact solution is given by

$$y(x) = \frac{e^{\frac{(x-1)(x-3)}{4\epsilon}} \left(erf\left(\frac{x-2}{2\sqrt{\epsilon}}\right) + erf\left(\frac{1}{\sqrt{\epsilon}}\right)\right)}{erf\left(\frac{1}{\sqrt{\epsilon}}\right) - erf\left(\frac{1}{2\sqrt{\epsilon}}\right)}. \tag{35}$$

Solution: Algorithm 1 is used to solve this equation too. From (34), we set

$$a^\star = 1, \quad N(y) = -\frac{1}{2}xy' - \frac{1}{2}y, \quad f(x) = 0, \quad \alpha = 0, \quad \beta = 1.$$

In this example, $M^\star = \frac{1}{2}$, and according to theorem 4.2, series obtained by algorithm 1 is convergent to the exact solution (35) for every $0 < \epsilon < 1$. The numerical results are given in table 2 for $\epsilon_0 = 2^{-3}$, $\epsilon_1 = 2^{-7}$ and $\epsilon_2 = 2^{-10}$.

Table 2: Maximum norm of the errors for Example 5.2.

	$\epsilon_0 = 2^{-3}$	$\epsilon_1 = 2^{-7}$	$\epsilon_2 = 2^{-10}$
n	$\Vert\phi_n(x) - y(x)\Vert_\infty$	$\Vert\phi_n(x) - y(x)\Vert_\infty$	$\Vert\phi_n(x) - y(x)\Vert_\infty$
10	1.0119171533131E-5	4.380242442735E-2	2.739613363280E-1
20	7.888343074869E-15	1.927221406334E-5	6.289384083300E-4
30	5.177465259961E-26	2.815776213199E-9	5.267458188788E-7
40	1.384504734416E-38	1.802919238601E-13	3.748724821379E-10

In addition, we have

$$\begin{aligned} \lambda^\star_{\epsilon_0} &= 0.9373532351, \\ \lambda^\star_{\epsilon_1} &= 0.9960937500, \\ \lambda^\star_{\epsilon_2} &= 0.9995117190. \end{aligned} \tag{36}$$

By comparison between the obtained results in (36), we conclude that the rate of convergence of the proposed method, for ϵ_0 is higher than ϵ_1 and ϵ_2.

6. Conclusions

In this paper, the singularly perturbed two-point boundary layer problems have been considered by means of the homotopy perturbation technique and Laplace transformation. The success of the method has later been tested by applying it to two singularly perturbed cases taken from the literature. Under some conditions, the convergence of the proposed method, guaranteed by a mathematical proof of convergence. The presented approach has clearly shown its advantage over the recently introduced conventional numerical methods for the singularly perturbed boundary value problems.

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