Journal of Mathematical Extension
Vol. 7, No. 4, (2013), 1-9

A Study of Regular Grouplikes

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Abstract. Recently, a new algebraic structure, namely grouplike, has been introduced and studied by the first author. A grouplike is something between semigroup and group with so close relations to groups. The grouplike axioms are generalizations of the four group axioms. In this paper, we study regular and irregular grouplikes as a class of semigroups. The motivation of this study lies in some interesting properties of regular proper grouplikes.

AMS Subject Classification: 20M17; 20M99
Keywords and Phrases: Grouplike, identity-like, regular semigroup

1. Introduction

In the great range of special classes of semigroups, regular semigroups take a central position from the point of view of richness of their structural "regularity". The principal classes of regular semigroups are inverse semigroups and completely regular semigroups with a great diversity of their various generalizations (see [10]). These statements are corroborated amply by the semigroup literature and are reflected somewhat by the books on semigroups (see [1,8]). Recently a class of semigroups namely "Grouplikes" has been introduced and studied in [3]. A grouplike is something between semigroup and group and its axioms are

Received: January 2013; Accepted: November 2013
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generalization of the four group axioms. The first ideas of grouplikes come from b-parts and b-addition of real numbers, introduced and studied in [4,6] and were generalized (for semigroups and groups) in [5]. Also, "Homogroup" was introduced by G. Thierrin, that is a semigroup containing an ideal subgroup (see [7,2]). We observe that every grouplike is a homogroup with a unique central idempotent.

2. Grouplikes

Now, we consider an algebraic structure that is something between semigroup and group (introduced and studied in [3]).

Definition 2.1. *A semigroup Γ is called* grouplike *if it satisfies the following axioms:*
(1) *There exists $\varepsilon \in \Gamma$ such that*

$$\varepsilon x = \varepsilon^2 x = x\varepsilon^2 = x\varepsilon \ : \ \ \forall x \in \Gamma,$$

(2) *For every ε satisfying* (1) *and every $x \in \Gamma$, there exists $y \in \Gamma$ such that*

$$xy = yx = \varepsilon^2.$$

Every $\varepsilon \in \Gamma$ satisfying the axioms (1) *and* (2) *is called an* identity-like*. If $(\Gamma, \cdot)$ is a grouplike and it is not group, then we call it* proper grouplike*. If a semigroup satisfies the axiom* (1)*, then we call it* monoidlike*. Note that a semigroup S is monoidlike if and only if it contains a central idempotent. It is interesting to know that every grouplike is a semigroup containing the least ideal that is also a maximal subgroup but the converse is not valid. Recall that for every semigroup $(S, \cdot)$, $Z(S)$ and $It(S) = E(S)$ are the center and the set of all idempotent elements of S, respectively (it may be empty). Also, by $Iz(S)$ we denote the set of all identity-likes and put $Zt(S) = Z(S) \cap It(S)$. If S is a grouplike, then $Zt(S)$ is singleton.*

The following lemma states an important basic property of grouplikes.

Lemma 2.2. ([3, Lemma 2.2]) *Every grouplike contains a unique idempotent identity-like element.*

Definition 2.3. *Let Γ be a grouplike and let e be its unique idempotent identity-like. Then, we call e standard identity-like and use the notation $(\Gamma, \cdot, e)$. Γ is a standard grouplike if e is the only idempotent of Γ. Γ is a zero grouplike if e is a zero of Γ. Every y that is corresponded to x in axiom (2) is called inverse-like of x and is denoted by x'_e or x' (so $x'x = xx' = e$). By $Inv(x)$ we mean the set of all inverse-likes of x (the set of all x').*

Regarding to the conception inverse-like in the above definition, note that every identity-like ε satisfies $\varepsilon^2 = e = e^2$, by Lemma 2.2. So y is an inverse-like of x (for a given identity-like ε) if and only if $xy = yx = e$.
For every grouplike $(\Gamma, \cdot, e)$, the subset $e\Gamma$ is its unique ideal subgroup (the same least ideal that is also a maximal subgroup).
There are some examples for finite and infinite grouplike (as one can see in Example 2.4, [3]). Also, note that every group is standard group-like and every monoid is monoidlike.
It is interesting to know that a grouplike also can be axiomatized by the following conditions that is very similar to the four groups axioms:
(i) Closure,
(ii) Associativity,
(iii) There exists a unique element $e \in \Gamma$ such that $ex = xe$, $e^2 = e$, for all $x \in X$.
(iv) For every $x \in \Gamma$, there exists $y \in \Gamma$ such that $xy = yx = e$.

In [3], the following hypothesises for grouplikes are stated:

($\mathbf{H_1}$) (The identity-like hypothesis) $exy = xy$, for every $x, y \in \Gamma$.

($\mathbf{H_2}$) (The inverse-like hypothesis) $Inv(e) = Iz(\Gamma)$.
Obviously, (H_2) is equivalent to $Inv(\varepsilon) \subseteq Iz(\Gamma)$ for some [every] $\varepsilon \in Iz(\Gamma)$.
The following theorem shows the relation between (H_1) and (H_2).

Theorem 2.4. ([3, Theorem 2.7]) *In every grouplike the identity-like hypothesis implies the inverse-like hypothesis.*
But it is an unsolved problem that "Does (H_2) imply (H_1)?"
There is an important class of grouplikes which plays important roles in the theory of grouplikes. For constructing them, note that we call $\mathcal{G}$

a class group *if $\mathcal{G}$ is a group for which all its elements are nonempty disjoint sets. Also, every function $\Psi : \cup\mathcal{G} \to \mathcal{G}$ is called a class function if $x \in \Psi(x)$, for every $x \in \cup\mathcal{G}$. Because of our assumption for $\mathcal{G}$, always the surjective class function Ψ exists and it is unique. We use the notation $\Psi(x) = A_x$, when $A \in \mathcal{G}$ and $x \in A = \Psi(x)$. Now, if φ is a choice function from $\mathcal{G}$ to $\cup\mathcal{G}$ ($\varphi(A) \in A$), then it is injective and $\Psi = \Psi\varphi\Psi$ or equivalently $A_x = A_{\varphi(A_x)}$, for every $x \in \cup\mathcal{G}$.*
Now, let E be the identity element of the class group $(\mathcal{G}, \circ)$ and define the binary operation $\centerdot = \centerdot^{\varphi}$ in $\cup\mathcal{G}$ by

$$x \centerdot^{\varphi} y = x \centerdot y := \varphi(\Psi(x) \circ \Psi(y)) = \varphi(A_x \circ A_y) \quad : \quad \forall x, y \in \cup\mathcal{G},$$

where Ψ is the unique class function and φ is an arbitrary choice function.

Definition 2.5. *Let $(\mathcal{G}, \circ)$ be a class group with the identity element E and $\Psi_{\mathcal{G}} : \cup\mathcal{G} \to \mathcal{G}$ the unique class function and φ a given choice function from $\mathcal{G}$. We call the algebraic structure $(\mathcal{G}, \centerdot^{\varphi}, \varphi(E))$ φ-*class united grouplike. *Also, we say a grouplike $(\Gamma, \cdot)$ is* class united *if there exists a class group $(\mathcal{G}, \circ)$ and a choice function φ such that $\cup\mathcal{G} = \Gamma$ and $\centerdot^{\varphi} = \cdot$.*
The following fundamental theorem characterizes the class united grouplikes.

Theorem 2.6. ([3, Theorem 2.11])
(A) A grouplike is class united if and only if satisfies the hypothesis (H_1).
(B) (General form of grouplikes satisfying the identity-like hypothesis) A binary system $(\Gamma, \cdot)$ is a grouplike with the identity-like hypothesis if and only if there exists a class group $\mathcal{G}$ and a choice function $\varphi : \mathcal{G} \to \cup\mathcal{G}$ such that $\Gamma = \cup\mathcal{G}$ and $\cdot = \centerdot^{\varphi}$.
Recall that an epigroup is a semigroup in which every element has a power that belongs to a subgroup (see [9]). Every class united grouplike is a unipotent epigroup.
In [3] a semigroup congruence is defined as follows. For every $x, y \in \Gamma$, put $x \sim_e y$ if and only if $ex = ey$. Also, denote by $\overline{\Gamma}_e$ (or simply $\overline{\Gamma}$) the set of all equivalent classes that are gotten from $\sim_e$. Then, $\sim_e$ is a semigroup congruence and $\overline{\Gamma} = \Gamma / \sim_e$ is a quotient semigroup that is

isomorphic to the group $e\Gamma$.
Now, we prove a relation between the equivalent class of a' *and* $Inv(a)$.

Proposition 2.7. *In every grouplike* Γ *we have*

$$eInv(a) \subseteq Inv(a) \subseteq Inv(ea) = \bar{a'}$$

for all $a \in \Gamma$ *and every its inverse-like* $a' \in Inv(a)$.

Proof. The identities $a(ea') = (ea')a = e^2 = e$ implies $eInv(a) \subseteq Inv(a)$. Now, let $b \in Inv(a)$, then $ba = ab = e$. Multiplying the equality by a' we have $be = ea'$ and so $b \in \bar{a'}$. Thus, $Inv(a) \subseteq \bar{a'}$. Also, if $b \in \bar{a'}$, then $eb = ea'$ and

$$a(ea') = (ea')a = e = a(eb) = (eb)a$$

So, $(ea)b = b(ea) = e$ and so $b \in Inv(ea)$. Conversely, if $b \in Inv(ea)$ then $b(ea) = e$, thus $eb = ea'$ and so $b \in \bar{a'}$. Therefore, $\bar{a'} = Inv(ea)$ and the proof is complete. $\square$

Note. Considering the above proposition, each of the conditions $ex \in eInv(a)$, $ex \in Inv(a)$ or $ex \in Inv(ea)$ imply $x \in Inv(ea)$. Because each of them implies $ex = ea'$, for some inverse-like a', and so $x(ea) = (ex)a = ea'a = e^2 = e = (ea)x$.

By the above proposition, we have $Inv(a) \subseteq Inv(ea)$ but its converse is not true.

Example 2.8. Consider $\Gamma = \{a, b, c\}$ with the following multiplication table

$\cdot$	a	b	c
a	a	a	a
b	a	b	b
c	a	a	a

It is easy to see that $(\Gamma, \cdot, a = e)$ is a grouplike, $Reg(\Gamma) = \{a, b\}$ and $e\Gamma = \{a\}$. Hence $Reg(\Gamma) \nsubseteq e\Gamma$. Also, we have $Inv(ea) \nsubseteq Inv(a)$, because $Inv(eb) = Inv(a) = \{a, b, c\}$ but $Inv(b) = \{a\}$.

3. Regular Grouplikes

Recall that an element a of a semigroup S is called regular [resp. completely regular] if $a = axa$ [resp. $a = axa$ and $ax = xa$], for some $x \in S$. It is easy to check that if a is regular, then both ax and xa are idempotent elements of S so $\{ax, xa\} \subseteq It(S)$. By $Reg(S)$ [resp. $Gr(S)$], we denote the set of all regular [resp. completely regular] elements of S. A semigroup S is called regular [resp. completely regular] if every its element is regular [resp. completely regular] (i.e. $Reg(S) = S$ [resp. $Gr(S) = S$]).
Since every grouplike is a semigroup, then we can discuss regular elements of an arbitrary grouplike and also regular grouplikes. We now give an example of a regular grouplike.

Example 3.1. The set $\Gamma = \{a, b, c\}$ with the following binary operation " $\cdot$ " is a regular (proper) grouplike

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

The following lemma states a basic property for regular elements of a grouplike.

Lemma 3.2. *In every grouplike* Γ *we have*

$$e\Gamma \subseteq Gr(\Gamma) \subseteq Reg(\Gamma) \neq \emptyset\ ,\ Iz(\Gamma) \cap Reg(\Gamma) = Iz(\Gamma) \cap Gr(\Gamma) = \{e\}.$$

Proof. Let $b = ea$ be an element of $e\Gamma$, then putting $x = a^{'}$, we have

$$bxb = eaa^{'}ea = eaa^{'}a = e^2a = ea = b.$$

So $e\Gamma \subseteq Reg(\Gamma)$.
Also, obviously $e \in Iz(\Gamma) \cap Reg(\Gamma)$ and if $\varepsilon \in Iz(\Gamma) \cap Reg(\Gamma)$, then there exists $x \in \Gamma$ such that $\varepsilon = \varepsilon x \varepsilon$ and so $\varepsilon = ex$. On the other hand, by $\varepsilon\varepsilon = \varepsilon(\varepsilon x \varepsilon)$, we have $e = ex$, hence $\varepsilon = e$. □

Lemma 3.3. *If a is a regular element of Γ, then there exists $x \in Inv(ea) = \bar{a'}$ such that $ex \in eInv(a)$ $(\subseteq Inv(a))$ and $\{ax, xa\} \subseteq It(\Gamma)(\subseteq Inv(e))$.*

Proof. Since $a \in \Gamma$ is regular, then there exists $x \in \Gamma$ such that $a = axa$, so $a'a = a'axa$ and so $e = exa = xae$. Also, $aa' = axaa'$ and $e = axe = eax$. Therefore, $xa \in Inv(e)$ and $ax \in Inv(e)$. On the other hand, we have $a'aa' = a'axaa'$, so

$$ea' = exe = e^2x = ex$$

hence $ea' = ex$, $a' \sim_e x$ and $x \in \bar{a'}$. Finally, considering the above identities we have

$$a(ex) = (ax)e = e = (ex)a.$$

Therefore, $ex \in Inv(a)$ and the proof is complete. □

Corollary 3.4. *If Γ is a regular grouplike, then for every $a \in \Gamma$ there exists $x \in Inv(ea) = \bar{a'}$ such that $ex \in eInv(a)$ and $\{ax, xa\} \subseteq Inv(e) \cap It(\Gamma)$.*

Proof. This is clear from the Lemma 3.3. □

Theorem 3.5. *A grouplike Γ is standard if and only if $Reg(\Gamma) = e\Gamma$.*

Proof. If Γ is a standard grouplike, then e is the only idempotent in Γ. Let $a \in \Gamma$ be a regular element, so $a = axa$, for some $x \in \Gamma$, and ax is an idempotent in Γ. Thus, $ax = e$ and $a = ea \in e\Gamma$. Therefore, $Reg(\Gamma) \subseteq e\Gamma$ and so $Reg(\Gamma) = e\Gamma$ (by lemma 3.2). Conversly, if $Reg(\Gamma) = e\Gamma$ then we show that $It(\Gamma) = e$. If $t^2 = t$, then $t \in Reg(\Gamma) = e\Gamma$ and $t = ey$, for some $y \in \Gamma$. Thus, $t'(t^2) = t't$ and $e = et = e^2y = ey = t$. Therefore, e is the only idempotent of Γ and so Γ is standard grouplike. □

Corollary 3.6. *If Γ is a grouplike satisfying (H_1) or (H_2), then*

$$Gr(\Gamma) = Reg(\Gamma) = e\Gamma.$$

So, in every class united grouplike all elements of $\Gamma \setminus e\Gamma$ are irregular.

Proof. Since every grouplike with the hypothesis H_1 or H_2 is standard, then we get the result from the above theorem. Also, we can conclude it from Lemma 3.3. Because, if $a = axa$ then

$$\{ax, xa\} \subseteq Inv(e) \cap It(\Gamma) = Iz(\Gamma) \cap It(\Gamma) = Zt(\Gamma) = \{e\},$$

thus $ax = xa = e$ and $a = ea \in e\Gamma$. □

Corollary 3.7. *For every grouplike Γ, the following statements are equivalent:*
(i) Γ is standard grouplike,
(ii) $Reg(\Gamma) = e\Gamma$,
(iii) $Reg(\Gamma) \subseteq e\Gamma$,
(iv) $Gr(\Gamma) = e\Gamma$,
(v) $Gr(\Gamma) \subseteq e\Gamma$,
(vi) $Gr(\Gamma) = Reg(\Gamma) = e\Gamma$.

Note that $Reg(\Gamma) \nsubseteq e\Gamma$, in general. Now, we close this paper with an example of regular grouplike with a completely regular element and we show that e is only element of Γ that satisfies in corollary 3.6 conditions.

Example 3.8. Let $\Gamma = \{a, e, \eta\}$ be a grouplike and define a binary operation "$\cdot$" by the following multiplication table

$\cdot$	a	e	η
a	a	e	η
e	e	e	e
η	a	e	a

It is easy to see that $(\Gamma, \cdot, e)$ is a regular grouplike and only for $e \in \{a, e, \eta\}$ we have $ex = xe$ for every $x \in \{a, e, \eta\}$ and therefore e is the only completely regular element of regular grouplike Γ.

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