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Construction Methods and Sum of c-K-g-Frames in Hilbert Spaces

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Abstract. This paper was aimed at studying some novel methods of constructing new c-K-g-frames in a Hilbert space H. Some necessary and sufficient conditions were given for some bounded operators on H under which new c-K-g-frames were obtained from the existing ones. Also, the sum of c-K-g-frames were discussed, some of their characterizations were identified, and some bounded operators offered to construct new c-K-g-frames from the old ones.

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1 Introduction

A frame for a Hilbert space H is a sequence of elements in H which provides a linear combination for each element in H, but the elements are not necessarily linear independent. Indeed, a frame can be thought of as a basis to which one has added more elements.

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K-frames in Hilbert spaces were introduced by Gavruta to investigate atomic decomposition systems, stating some properties of them [5, 11, 12]. After that, K-g-frames have been introduced in [13] and some new results and characterizations of K-g-frames have been studied in [10, 14, 18]. Furthermore, the notion of continuous K-g-frames is presented in [3] and some properties of them have been studied in [4, 15].
Throughout this paper, (Ω, μ) is a measure space with positive measure μ, H, H_1, H_2 and H_ω are separable Hilbert spaces and $B(H, H_\omega)$ is the set of all bounded linear operators from H into H_ω, $\omega \in \Omega$. Also, $B(H)$ is the set of all bounded linear operators on H. We will use the symbols $R(U)$ and $N(U)$ for the range and null space of an operator $U \in B(H_1, H_2)$, respectively.

Definition 1.1. ([8]) The operator $U \in B(H)$ is called a bounded below operator if there exists a positive number α such that

$$\alpha \|f\| \leq \|U(f)\|, \quad f \in H.$$

A bounded operator $U : H \longrightarrow H$ is called self-adjoint if $U = U^*$. For a self-adjoint operator U, the inner product $\langle Uf, f\rangle$ is real for each $f \in H$ ([6]). Also, the partial order $U \leq V$ for the self-adjoint operators U and V is defined by

$$U \leq V \Leftrightarrow \langle Uf, f\rangle \leq \langle Vf, f\rangle, \quad f \in H.$$

Lemma 1.2. ([6]) *Let $U \in B(H_1, H_2)$. Then the following holds:*

1. *$R(U)$ is closed in H_2 if and only if $R(U^*)$ is closed in H_1.*
2. *$(U^*)^\dagger = (U^\dagger)^*$.*
3. *The orthogonal projection of H_2 onto $R(U)$ is given by $UU^\dagger$.*
4. *The orthogonal projection of H_1 onto $R(U^\dagger)$ is given by $U^\dagger U$.*
5. *$N(U^\dagger) = R^\perp(U)$ and $R(U^\dagger) = N^\perp(U)$.*
6. *U is surjective if and only if there exists a constant $\delta > 0$ such that $\|U^* f\| \geq \delta \|f\|, \quad \forall f \in H_1$.*

Lemma 1.3. ([7]) *Let $L_1 \in B(H_1, H)$ and $L_2 \in B(H_2, H)$. Then the following assertions are equivalent:*

(i) $R(L_1) \subseteq R(L_2)$.

(ii) $L_1 L_1^* \leq \lambda L_2 L_2^*$ *for some* $\lambda > 0$.

(iii) There exists an operator $U \in B(H_1, H_2)$ *such that* $L_1 = L_2 U$.

Moreover, if (i), (ii) and (iii) are valid, then there exists a unique operator U such that

1. $\|U\|^2 = inf\{\mu : L_1 L_1^* \leq \mu L_2 L_2^*\}$,
2. $N(L_1) = N(U)$,
3. $R(U) \subseteq \overline{R(L_2)^*}$.

Definition 1.4. ([1]) Let $\varphi \in \Pi_{\omega\in\Omega} H_\omega$. We call that φ is strongly measurable if φ as a mapping of Ω to $\oplus_{\omega\in\Omega} H_\omega$ is measurable, where

$$\Pi_{\omega\in\Omega} H_\omega = \big\{ f : \Omega \longrightarrow \cup_{\omega\in\Omega} H_\omega \ ; \ f(\omega) \in H_\omega \big\}.$$

Definition 1.5. Choose the set

$$\big(\oplus_{\omega\in\Omega} H_\omega, \mu \big)_{L^2} = \big\{ F \in \Pi_{\omega\in\Omega} H_\omega | \ F \textit{ is strongly measurable}, \\ \int_\Omega \|F(\omega)\|^2 d\mu(\omega) < \infty \big\},$$

with inner product given by

$$\langle F, G \rangle = \int_\Omega \langle F(\omega), G(\omega) \rangle d\mu(\omega).$$

It can be proved that $\Big(\oplus_{\omega\in\Omega} H_\omega, \mu \Big)_{L^2}$ is a Hilbert space ([1]). We will show the norm of $F \in \Big(\oplus_{\omega\in\Omega} H_\omega, \mu \Big)_{L^2}$ by $\|F\|_2$.

Now, the definition of continuous g-frames is reviewed.

Definition 1.6. The family of operators $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is called a continuous generalized frame, or simply a cg-frame, for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if:

(i) for each $f \in H$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,

(ii) there are two positive constants A and B such that

$$A\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \quad f \in H. \tag{1}$$

A and B are called the lower and upper cg-frame bounds, respectively. If A, B can be chosen such that $A = B$, then $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight cg-frame and if $A = B = 1$, it is called a Parseval cg-frame. A family $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is called cg-Bessel family if the second inequality in (1) holds.

Theorem 1.7. ([1]) *Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a cg-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with bound B. Then the mapping T_Λ of $\Big(\oplus_{\omega\in\Omega} H_\omega, \mu\Big)_{L^2}$ to H weakly defined by*

$$\langle T_\Lambda F, g\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle d\mu(\omega), \quad F \in \Big(\oplus_{\omega\in\Omega} H_\omega, \mu\Big)_{L^2}, \ g \in H,$$

is linear and bounded with $\|T_\Lambda\| \leq \sqrt{B}$. Furthermore for each $g \in H$ and $\omega \in \Omega$,

$$T_\Lambda^*(g)(\omega) = \Lambda_\omega g.$$

The operator T_Λ is called the synthesis operator of $\{\Lambda_\omega\}_{\omega\in\Omega}$ and its adjoint T_Λ^* is called the analysis operator of $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$.

The continuous K-g-frames have been introduced in [3] as following:

Definition 1.8. Let $K \in B(H)$. A family $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is called a continuous K-g-frame, or c-K-g-frame, for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if:

(i) $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable for each $f \in H$,

(ii) there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^* f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \leq B\|f\|^2, \quad f \in H. \tag{2}$$

The constants A, B are called lower and upper c-K-g-frame bounds, respectively. If $A = B$, then $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-K-g-frame and if $A = B = 1$, it is called a Parseval c-K-g-frame. The family $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is called a c-g-Bessel family if the right hand inequality in (2) holds. In this case, B is called the Bessel constant.

Now, assume that $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with frame bounds A, B. The c-K-g-frame operator $S_\Lambda : H \longrightarrow H$ is weakly defined by

$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle f, \Lambda_\omega^* \Lambda_\omega g\rangle \, d\mu(\omega), \quad f, g \in H.$$

Therefore

$$AKK^* \leq S_\Lambda \leq BI.$$

Lemma 1.9. ([3]) *Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a cg-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Then $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if and only if there exists a constant $A > 0$ such that $S_\Lambda \geq AKK^*$, where S_Λ is the frame operator of $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$.*

Duals of c-K-g-frames have been indicated in [4] as following:

Definition 1.10. Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. A cg-Bessel family $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ for H is called a dual c-K-g-Bessel family of Λ if for each $f, h \in H$,

$$\langle Kf, h\rangle = \int_\Omega \langle \Lambda_\omega^* \Gamma_\omega f, h\rangle \, d\mu(\omega).$$

2 Constructing new c-K-g-frames

In this section, we construct new c-K-g-frames by using of linear bounded operators.

The following theorem, for a given c-K-g-frame $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ of H, provides a new c-K-g-frame for H by applying a linear bounded operator.

Theorem 2.1. *Let $K \in B(H)$, $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, with bounds A and B and $U \in B(H)$ be a* *closed range operator such that $UK = KU$. If $R(K^*) \cap N(U^*) = \{0\}$, Then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with the lower bound $A\|U^\dagger\|^{-2}$ and the upper bound $B\|U\|^2$.*

Proof. It is easy to check that $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a cg-Bessel family with upper bound $B\|U\|^2$. Since $UK = KU$, we have $K^*U^* = U^*K^*$. U has closed range and $R(K^*) \cap N(U^*) = \{0\}$, by Lemma 1.2, for each $f \in H$, we have

$$\begin{aligned}\|K^*f\|^2 &= \|UU^\dagger K^*f\|^2 = \|(U^\dagger)^*U^*K^*f\|^2 = \|(U^\dagger)^*K^*U^*f\|^2 \\ &\leq \|U^\dagger\|^2\|K^*U^*f\|^2.\end{aligned}$$

Then for each $f \in H$, we have

$$\int_\Omega \|\Lambda_\omega U^*f\|^2 d\,\mu(\omega) \geq A\|K^*U^*f\|^2 \geq A\|U^\dagger\|^{-2}\|K^*f\|^2.$$

This proves the theorem. □

Corollary 2.2. *Suppose that $K \in B(H)$ is with dense range, $U \in B(H)$ has closed range and $UK = KU$. If $\{\Lambda_\omega U\}_{\omega\in\Omega}$ and $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ are both c-K-g-frames for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, then $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Proof. Since $\overline{R(K)} = H$, so $N(K^*)^\perp = H$ and $N(K^*) = \{0\}$. For each $f \in H$, we have

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega U^*f\|^2 d\,\mu(\omega),$$

so $N(U^*) \subseteq N(K^*)$, which implies

$$H = N(K^*)^\perp \subseteq N(U^*)^\perp = R(U).$$

So U is surjective. Also, For each $f \in H$, we have

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega Uf\|^2 d\,\mu(\omega),$$

so $N(U) \subseteq N(K^*) = \{0\}$. That is, U is one to one. Therefore U is invertible. Since $UK = KU, U^{-1}K = KU^{-1}$, $R(K^*) \cap N((U^{-1})^*) = \{0\}$, and $\{\Lambda_\omega\}_{\omega\in\Omega} = \{(\Lambda_\omega U^*)(U^{-1})^*\}_{\omega\in\Omega}$, so by Theorem 2.1, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. □

For a given tight c-K-g-frame $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ of H, we can obtain another c-K-g-frame for H. The following theorem presents us necessary and sufficient conditions on $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ to be a c-K-g-frame for H.

Theorem 2.3. *Let $K, U \in B(H)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a D-tight c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. If K^* is bounded below and $UK = KU$, then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if and only if U is surjective.*

Proof. If U is surjective, then Theorem 2.1 implies the first part of proof. For the other implication, we prove that U is surjective. Assume that $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with bounds A and B. Then, for all $f \in H$, we have:

$$A\|K^*f\|^2 \le \int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu(\omega) \le B\|f\|^2. \tag{3}$$

Also for each $g \in H$, we have

$$D\|K^*g\|^2 = \int_\Omega \|\Lambda_\omega g\|^2 d\,\mu(\omega).$$

By $U^*K^* = K^*U^*$, we obtain

$$D\|U^*K^*f\|^2 = D\|K^*U^*f\|^2 = \int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu(\omega), \quad f \in H. \tag{4}$$

So by (3) and (4), we have

$$\|U^*K^*f\|^2 = D^{-1}\int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu(\omega) \ge D^{-1}A\|K^*f\|^2, \quad f \in H. \tag{5}$$

Sine K^* is bounded below, so there exist $\alpha > 0$ such that $\|K^*f\| \ge \alpha\|f\|$, for each $f \in H$. Thus, from (2.3), we conclude that for each $f \in H$,

$$\|U^*K^*f\| \ge \alpha\|f\|.$$

Therefore U^*K^* is bounded below, thus by Lemma 1.2, KU is surjective and $KU = UK$ implies that U is surjective. □

Suppose that operators $T, U \in B(H)$ and T^* preserves a c-K-g-frame for $R(T)$. In the following theorem, we state some conditions on K, U and T such that U^* can also preserve the same c-K-g-frame for $R(U)$.

Theorem 2.4. *Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Suppose that $T, U \in B(H)$ are closed ranged, and $N(T) = N(U)$ with $KUT^\dagger = UT^\dagger K$ and $R(K^*) \cap N(U^*) = \{0\}$. If $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K-g-frame for $R(T)$ with respect to $\{H_\omega\}_{\omega\in\Omega}$, then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for $R(U)$ with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Proof. We only need to show that $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ has the lower frame condition. We define

$$L : R(T) \longrightarrow R(U),$$

by $Lf = UT^\dagger f, \qquad f \in R(T)$. By the assumptions, $KL = LK$. Since $N(T) = N(U)$, we have $R(T^\dagger) = R(U^\dagger)$. Hence by Lemma (1.2), $N(L) = N(UT^\dagger) = N(TT^\dagger) = (R(T))^\perp$, which implies

$$N(L) = N(UT^\dagger) \cap R(T) = (R(T))^\perp \cap R(T) = \{0\}.$$

So, L is invertible on $R(T)$. By Lemma 1.2, $T^\dagger T = P_{R(T^\dagger)} = P_{R(U^\dagger)} = U^\dagger U$. Therefore

$$LT = UT^\dagger T = UU^\dagger U = U \tag{6}$$

Now, let C, D be the frame bounds of $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$, then for each $f \in R(U)$, from (6), we have

$$\begin{aligned}\int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu(\omega) &= \int_\Omega \|\Lambda_\omega T^* L^* f\|^2 d\,\mu(\omega) \geq C\|K^* L^* f\|^2\\ &= C\|L^* K^* f\|^2 \geq C\|L^{-1}\|^{-2}\|K^* f\|^2.\end{aligned}$$

Furthermore for each $f \in R(U)$,

$$\int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu(\omega) = \int_\Omega \|\Lambda_\omega T^* L^* f\|^2 d\,\mu(\omega) \leq B\|L^* f\|^2 = B\|L\|^2\|f\|^2.$$

So $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for $R(U)$. □

Theorem 2.5. *Let $K \in B(H)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-g-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Suppose that T_Λ is the synthesis operator of Λ. Then, the following conditions are equivalent:*

(i) $R(K) = R(T_\Lambda)$.

(ii) There exist two constants $C, D > 0$, such that for each $f \in H$,

$$C\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega) \leq D\|K^*f\|^2. \tag{7}$$

(iii) $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ *is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and there exists a c-g-Bessel family $\{\Gamma_\omega\}_{\omega\in\Omega}$ for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ such that $\Lambda_\omega = \Gamma_\omega K^*$ for each $\omega \in \Omega$.*

Proof. *(i)* $\Rightarrow$ *(ii)* By Lemma 1.3, there exist $C, D > 0$, such that $CKK^* \leq T_\Lambda T_\Lambda^* \leq DKK^*$. Thus, for each $f \in H$,

$$C\|K^*f\|^2 \leq \|T_\Lambda^* f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega) \leq D\|K^*f\|^2$$

(ii) $\Rightarrow$ *(iii)* It suffices to show the second part of the result. The right-hand inequity in (7) is equivalent to $T_\Lambda T_\Lambda^* \leq DKK^*$. By Lemma 1.3, there exists an operator $Q \in B\Big((\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}, H\Big)$ such that $T_\Lambda = KQ$ and $T_\Lambda^* = Q^*K^*$. We define for each $g \in H$ and for almost all $\omega \in \Omega$,

$$\Gamma_\omega g = (Q^*g)(\omega).$$

Therefore we have

$$\{\Lambda_\omega(g)\}_{\omega\in\Omega} = \{(Q^*(K^*g)(\omega)\}_{\omega\in\Omega} = \{\Gamma_\omega(K^*g)\}_{\omega\in\Omega},$$

which implies that $\Lambda_\omega = \Gamma_\omega K^*$ for almost all $\omega \in \Omega$. So for each $g \in H$,

$$\int_\Omega \|\Gamma_\omega g\|^2 d\,\mu(\omega) = \int_\Omega \|(Q^*g)(\omega)\|^2 d\,\mu(\omega) = \|(Q^*g)\|_2^2 \leq \|Q\|_2^2\|g\|^2.$$

Hence, $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a c-g-Bessel family for H.

(iii) $\Rightarrow$ *(i)* For each $f \in H$, we have

$$C_\Lambda\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega = \int_\Omega \|\Gamma_\omega K^* f\|^2 d\,\mu(\omega) \leq D_\Gamma\|K^*f\|^2.$$

Thus, $C_\Lambda KK^* \leq T_\Lambda T_\Lambda^* \leq D_\Gamma KK^*$, by Lemma (1.3), $R(K) = R(T_\Lambda)$. □

The following theorem is applied to construct c-K-g-frames by given some linear bounded operators and some c-K-g-frames.

Theorem 2.6. *Suppose that $K_1, K_2 \in B(H)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

(i) If $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is also a c-K_2-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, then it is a c-$(K_1 + K_2)$-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.

(ii) If, in addition, $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is A-tight c-K_1-g-frame, then it is a c-K_2-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if and only if $R(K_2) \subseteq R(K_1)$.

Proof. (i) Since $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame and also c-K_2-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, so for each $f \in H$, we have

$$A_1\|K_1^* f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega) \tag{8}$$

and

$$A_2\|K_2^* f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega) \tag{9}$$

By (8) and (9), we have

$$(\frac{A_1}{2}\|K_1^* f\|^2 + \frac{A_1}{2}\|K_2^* f\|^2) \leq \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega) \tag{10}$$

Now, by taking $\lambda = min\{\frac{A_1}{2}, \frac{A_2}{2}\}$ in (10) , we obtain

$$\lambda\|(K_1 + K_2)^* f\|^2 \leq (A_1\|K_1^* f\|^2 + A_2\|K_2^* f\|^2) \leq 2\int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega),$$

that is $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-$(K_1 + K_2)$-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. (ii) By the assumptions, $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is A-tight c-K_1-g-frame and c-K_2-g-frame, there exists a $D > 0$, such that for each $f \in H$, we have

$$A\|K_1^* f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega) \geq D\|K_2^* f\|^2.$$

Hence, $K_2K_2^* \leq \frac{A}{D}K_1K_1^*$ and by Lemma 1.3, $R(K_2) \subseteq R(K_1)$.
For the opposite implication, by Lemma 1.3, there exists $\gamma > 0$, such that $K_2K_2^* \leq \gamma K_1K_1^*$. Hence for each $f \in H$, we have

$$\|K_2^*f\|^2 \leq \gamma\|K_1^*f\|^2 = \frac{\gamma}{A}\int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega),$$

therefore

$$\frac{A}{\gamma}\|K_2^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu(\omega) = A\|K_1^*f\|^2 \leq A\|K\|^2\|f\|^2.$$

So $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. □

3 Sum of c-K-g-frames

In this section, we suppose that Λ and Γ are arbitrary c-K-g-frames and we study the sum of these frames.

Theorem 3.1. *Suppose that $K_1, K_2 \in B(H)$ are closed range operators, $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ are c-K_1-g-frame and c-g-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, respectively.*

(i) If $K_1 \geq 0$ and $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-dual for $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$, then the family $\{\Lambda_\omega+\Gamma_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.

(ii) If $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ is c-K_2-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $T_\Lambda T_\Gamma^* = 0$, then $\{\Lambda_\omega + \Gamma_\omega\}_{\omega\in\Omega}$ is a c-$(K_1 + K_2)$-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.

Proof. (i) Since $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$, for each $f \in H$, we have

$$\begin{aligned}\langle K_1^*f, h\rangle = \langle f, K_1h\rangle = \overline{\langle K_1h, f\rangle} &= \overline{\int_\Omega \left\langle \Lambda_\omega^*\Gamma_\omega h, f\right\rangle d\mu(\omega)} \\ &= \int_\Omega \left\langle \Gamma_\omega^*\Lambda_\omega f, h\right\rangle d\mu(\omega).\end{aligned}$$

We denote by $S_{\Lambda+\Gamma}$, the c-g-frame operator of $\{\Lambda_\omega + \Gamma_\omega\}_{\omega\in\Omega}$. So for each $f, h \in H$,

$$\begin{aligned}
\langle S_{\Lambda+\Gamma} f, h\rangle &= \int_\Omega \big\langle f, (\Lambda_\omega + \Gamma_\omega)^*(\Lambda_\omega + \Gamma_\omega)h\big\rangle \, d\mu(\omega) \\
&= \int_\Omega \big\langle (\Lambda_\omega + \Gamma_\omega)^*(\Lambda_\omega + \Gamma_\omega)f, h\big\rangle \, d\mu(\omega) \\
&= \int_\Omega \big\langle \Lambda_\omega^*\Lambda_\omega f, h\big\rangle \, d\mu(\omega) + \int_\Omega \big\langle \Gamma_\omega^*\Gamma_\omega f, h\big\rangle \, d\mu(\omega) \\
&+ \int_\Omega \big\langle \Lambda_\omega^*\Gamma_\omega f, h\big\rangle \, d\mu(\omega) + \int_\Omega \big\langle \Gamma_\omega^*\Lambda_\omega f, h\big\rangle \, d\mu(\omega) \\
&= \langle S_\Lambda f, h\rangle + \langle S_\Gamma f, h\rangle + \langle K_1 f, h\rangle + \langle K_1^* f, h\rangle,
\end{aligned}$$

therefore

$$\begin{aligned}
\langle S_{\Lambda+\Gamma} f, f\rangle &= \int_\Omega \|(\Lambda_\omega + \Gamma_\omega)f\|^2 \, d\mu(\omega) \geq \langle S_\Lambda f, f\rangle \\
&= \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \geq C_\Lambda \|K_1^* f\|^2.
\end{aligned}$$

This shows that $\{\Lambda_\omega + \Gamma_\omega\}_{\omega\in\Omega}$ has the lower frame condition.
Now, we show $\{\Lambda_\omega + \Gamma_\omega\}_{\omega\in\Omega}$ is a c-g-Bessel family. For each $f \in H$, we have

$$\begin{aligned}
\int_\Omega \|(\Lambda_\omega + \Gamma_\omega)f\|^2 \, d\mu(\omega) &\leq 2\int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) + 2\int_\Omega \|\Gamma_\omega f\|^2 \, d\mu(\omega) \\
&\leq 2B_1\|f\|^2 + 2B_2\|f\|^2 = 2(B_1 + B_2)\|f\|^2.
\end{aligned}$$

(ii) We only need to show that $\{\Lambda_\omega + \Gamma_\omega\}_{\omega\in\Omega}$ has the lower frame condition. Since $T_\Lambda T_\Gamma^* = 0$, for each $f \in H$, we have $\int_\Omega \big\langle \Lambda_\omega^*\Gamma_\omega f, f\big\rangle \, d\mu(\omega) = 0$ and

$$\begin{aligned}
\int_\Omega \|(\Lambda_\omega + \Gamma_\omega)f\|^2 \, d\mu(\omega) &= \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) + \int_\Omega \|\Gamma_\omega f\|^2 \, d\mu(\omega) \\
&\geq A_1\|K_1^* f\|^2 + A_2\|K_2^* f\|^2 \geq \lambda\|(K_1 + K_2)f\|^2,
\end{aligned}$$

where $\lambda = min\{A_1, A_2\}$. This is the desired conclusion. □

The following theorem is the continuous version of Theorem 2.1 in [10].

Theorem 3.2. *Suppose that $K_1 \in B(H_1)$, $K_2 \in B(H_2)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame and $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ is a c-g-Bessel family for H_1. Assume that $U_1, U_2 \in B(H_1, H_2)$ and $U_1T_\Lambda T_\Gamma^* U_2^* + U_2T_\Gamma T_\Lambda^* U_1^* + U_2S_\Gamma U_2^* \geq 0$. If U_1 has closed range with $U_1K_1 = K_2U_1$ and $R(K_2^*) \cap N(U_1^*) = \{0\}$, then $\{\Lambda_\omega U_1^* + \Gamma_\omega U_2^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2 with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Proof. Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$, $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ be a c-K_1-g-frame and c-g-Bessel family for H_1 with bounds A_1, B_1 and B_2, respectively. Similar analysis to the proof of Theorem 3.1, we show that $\{\Lambda_\omega U_1^* + \Gamma_\omega U_2^*\}_{\omega\in\Omega}$ is c-g-Bessel family for H_2 with bound $2B_1\|U_1\|^2 + 2B_2\|U_2\|^2$. Now, for each $g \in H_2$, we have

$$\begin{aligned}\int_\Omega \|(\Lambda_\omega U_1^* + \Gamma_\omega U_2^*)g\|^2 \, d\mu(\omega) &= \int_\Omega \|\Lambda_\omega U_1^* g\|^2 \, d\mu(\omega) + \langle U_2T_\Gamma T_\Lambda^* U_1^* g, g\rangle \\ &\quad + \langle U_1T_\Lambda T_\Gamma^* U_2^* g, g\rangle + \langle U_2T_\Gamma T_\Gamma^* U_2^* g, g\rangle \\ &= \int_\Omega \|\Lambda_\omega U_1^* g\|^2 \, d\mu(\omega) + \langle (U_1T_\Lambda T_\Gamma^* U_2^* \\ &\quad + U_2T_\Gamma T_\Lambda^* U_1^* + U_2S_\Gamma U_2^*)g, g\rangle\end{aligned}$$

By the assumptions, for each $g \in H$ we obtain

$$\begin{aligned}\int_\Omega \|(\Lambda_\omega U_1^* + \Gamma_\omega U_2^*)g\|^2 \, d\mu(\omega) &\geq \int_\Omega \|\Lambda_\omega U_1^* g\|^2 \, d\mu(\omega) \geq A_1\|K_1^* U_1^* g\|^2 \\ &= A_1\|U_1^* K_2^* g\|^2 \geq A_1\|U_1^\dagger\|^{-2}\|K_2^* g\|^2.\end{aligned}$$

Therefore, for each $g \in H_2$, we have

$$\begin{aligned}A_1\|U_1^\dagger\|^{-2}\|K_2^* g\|^2 &\leq \int_\Omega \|(\Lambda_\omega U_1^* + \Gamma_\omega U_2^*)g\|^2 \, d\mu(\omega) \\ &\leq (2B_1\|U_1\|^2 + 2B_2\|U_2\|^2)\|g\|^2.\end{aligned}$$

□

Corollary 3.3. *Suppose that $K_1 \in B(H_1)$, $K_2 \in B(H_2)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for H_1 with respect to $\{H_\omega\}_{\omega\in\Omega}$. If $U \in B(H_1, H_2)$ has closed range, $UK_1 = K_2U$ and $R(K_2^*) \cap N(U^*) = \{0\}$, then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2 with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Corollary 3.4. *Let $K, U \in B(H)$. Suppose that $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. If U is positive operator such that , $US_\Lambda = S_\Lambda U$, then $\{\Lambda_\omega + \Lambda_\omega U\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Proof. Since $T_\Lambda T_\Lambda^* U^* + UT_\Lambda T_\Lambda^* + UT_\Lambda T_\Lambda^* U^* = S_\Lambda U + US_\Lambda + US_\Lambda U^*$, by Theorem 3.2, we need only to show that $S_\Lambda U + US_\Lambda + US_\Lambda U^* \geq 0$. By Theorem 4.33 in [8], there exists a unique positive operator V such that $U = V^2$. In addition, since $US_\Lambda = S_\Lambda U$, implies that $VS_\Lambda = S_\Lambda V$. For each $f \in H$, we have

$$\begin{aligned}\langle (S_\Lambda U + US_\Lambda + US_\Lambda U^*)f, f\rangle &= \langle S_\Lambda Uf, f\rangle + \langle US_\Lambda f, f\rangle + \langle US_\Lambda U^* f, f\rangle \\ &= 2\langle US_\Lambda f, f\rangle + \langle UT_\Lambda T_\Lambda^* U^* f, f\rangle \\ &= 2\langle V^2 S_\Lambda f, f\rangle + \|T_\Lambda^* U^* f\|^2 \\ &= 2\langle VS_\Lambda V f, f\rangle + \|T_\Lambda^* U^* f\|^2 \\ &= 2\|T_\Lambda^* V f\|^2 + \|T_\Lambda^* U^* f\|^2 \geq 0.\end{aligned}$$

□

Theorem 3.5. *Let $K_1 \in B(H_1)$ be closed range, $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ be c-K_1-g-frames for H_1 with respect to $\{H_\omega\}_{\omega\in\Omega}$. Suppose that $K_2 \in B(H_2)$, $U_1, U_2 \in B(H_1, H_2)$ and $U_1 T_\Lambda T_\Gamma^* U_2^* + U_2 T_\Gamma T_\Lambda^* U_1^* \geq 0$. If one the following conditions holds, then for each $\alpha_1, \alpha_2 > 0$, $\{\alpha_1 \Lambda_\omega U_1^* + \alpha_2 \Gamma_\omega U_2^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2 with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

(i) $P = \alpha_1 U_1 + \alpha_2 U_2,\ R(P^*) \subseteq R(K_1),\ R(K_2) \subseteq R(P)$.

(ii) $Q = \alpha_1 U_1 - \alpha_2 U_2,\ R(Q^*) \subseteq R(K_1),\ R(K_2) \subseteq R(Q)$.

Proof. Let A_1, B_1 and A_2, B_2 be frame bounds of Λ and Γ, respectively. Similar to proof of Theorem 3.2, $\alpha_1, \alpha_2 > 0$, $\{\alpha_1 \Lambda_\omega U_1^* + \alpha_2 \Gamma_\omega U_2^*\}_{\omega\in\Omega}$ is a c-g-Bessel family for H_2 with respect to $\{H_\omega\}_{\omega\in\Omega}$, with bound $2\alpha_1 B_1 \|U_1\|^2 + 2\alpha_2 B_2 \|U_2\|^2$ and for each $g \in H$, we have

$$\begin{aligned}\int_\Omega \|(\alpha_1 \Lambda_\omega U_1^* + \alpha_2 \Gamma_\omega U_2^*)g\|^2 \, d\mu(\omega) &= \alpha_1^2 \int_\Omega \|\Lambda_\omega U_1^* g\|^2 \, d\mu(\omega) \\ &\quad + 2\alpha_1\alpha_2 \langle (U_2 T_\Gamma T_\Lambda^* U_1^* + U_1 T_\Lambda T_\Gamma^* U_2^*)g \\ &\quad , g\rangle + \alpha_2^2 \int_\Omega \|\Gamma_\omega U_2^* g\|^2 \, d\mu(\omega) \\ &\quad \geq \alpha_1^2 A_1 \|K_1^* U_1^* g\|^2 + \alpha_2^2 A_2 \|K_1^* U_2^* g\|^2.\end{aligned}$$

Without loss of generality, suppose that condition (ii) holds. Set

$$\lambda = min\{A_1, A_2\},$$

by the parallelogram law, for each $g \in H_2$, we have

$$\begin{aligned}\alpha_1^2 A_1 \|K_1^* U_1^* g\|^2 + \alpha_2^2 A_2 \|K_1^* U_2^* g\|^2 &\geq \lambda(\|\alpha_1 K_1^* U_1^* g\|^2 + \|\alpha_2 K_1^* U_2^* g\|^2) \\ &= \frac{\lambda}{2}\Big(\|K_1^*(\alpha_1 U_1 + \alpha_2 U_2)^* g\|^2 \\ &+ \|K_1^*(\alpha_1 U_1 - \alpha_2 U_2)^* g\|^2\Big) \\ &\geq \frac{\lambda}{2}\|K_1^* Q^* g\|^2 \geq \frac{\lambda}{2}\|K_1^\dagger\|^{-2}\|Q^* g\|^2.\end{aligned}$$

Since $R(K_2) \subseteq R(Q)$, so by the Lemma 1.3, there exists $\alpha > 0$ such that $K_2 K_2^* \leq \alpha Q Q^*$. It follows that for each $g \in H_2$, $\alpha^{-1}\|K_2^* g\|^2 \leq \|Q^* g\|^2$. Therefore, for each $g \in H_2$, we have

$$\begin{aligned}\frac{\lambda}{2}\alpha^{-1}\|K_1^\dagger\|^{-2}\|K_2^* g\|^2 &\leq \int_\Omega \|(\alpha_1 \Lambda_\omega U_1^* + \alpha_2 \Gamma_\omega U_2^*) g\|^2 \, d\mu(\omega) \\ &\leq (2\alpha_1^2 B_1 \|U_1\|^2 + 2\alpha_2^2 B_2 \|U_2\|^2)\|g\|^2.\end{aligned}$$

□

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