Journal of Mathematical Extension
Vol. 8, No. 4, (2014), 75-92

Some Grüss Type Inequalities for n-Tuples of Vectors in Semi-Inner Modules

A. G. Ghazanfari*
Lorestan University

B. Ghazanfari
Lorestan University

Abstract. Some Grüss type inequalities in semi-inner product modules over C^*-algebras and H^*-algebras for n-tuples of vectors are established. Also we give their natural applications for the approximation of the discrete Fourier and the Melin transforms in such modules.

AMS Subject Classification: 46L08; 46H25; 46C99; 26D99
Keywords and Phrases: The Grüss inequality, semi-inner product C^*-modules, semi-inner product H^*-modules

1. Introduction

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt\frac{1}{b-a}\int_a^b g(t)dt.$$

In 1934, G. Grüss [5] showed that

$$|T(f,g)| \leqslant \frac{1}{4}(M-m)(N-n), \tag{1}$$

provided m, M, n, N are real numbers with the property $-\infty < m \leqslant f \leqslant M < \infty$ and $-\infty < n \leqslant g \leqslant N < \infty$ a.e. on $[a, b]$. The constant $\frac{1}{4}$ is best possible

Received: May 2014; Accepted: August 2014
*Corresponding author

in the sense that it cannot be replaced by a smaller quantity and is achieved for

$$f(x) = g(x) = sgn\Big(x - \frac{a+b}{2}\Big).$$

The discrete version of (1) states that: If $a \leqslant a_i \leqslant A,\ b \leqslant b_i \leqslant B,\ (i = 1, ..., n)$ where a, A, b, B, a_i, b_i are real numbers, then

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i . \frac{1}{n} \sum_{i=1}^{n} b_i \right| \leqslant \frac{1}{4}(A-a)(B-b), \tag{2}$$

where the constant $\frac{1}{4}$ is the best possible for an arbitrary $n \geqslant 1$. Some refinements of the discrete version of Grüss inequality (2) are given in [7].

In the recent years, this inequality has been investigated, applied and generalized by many authors in different areas of mathematics, among others in inner product spaces [2], in the approximation of integral transforms [8] and the references therein, in semi-inner $*$-modules for positive linear functionals and C^*-seminorms [3], for positive maps [11].

A good example of how Grüss type inequalities can cross mathematical categories is provided by the development of the Grüss type inequalities in inner product modules over H^*-algebras and C^*-algebras [4, 6]. For an entire chapter devoted to the history of this inequality see [9] where further references are given.

We recall some of the most important Grüss type discrete inequalities for inner product spaces that are available in [1].

Theorem 1.1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$; $(\mathbb{K} = \mathbb{C}, \mathbb{R})$, x_i, $y_i \in H$, $p_i \geqslant 0$ $(i = 1, ..., n)$ $(n \geqslant 2)$ with $\sum_{i=1}^{n} p_i = 1$. If $x, X, y, Y \in H$ are such that*

$$Re\,\langle X - x_i, x_i - x\rangle \geqslant 0 \quad and \quad Re\,\langle Y - y_i, y_i - y\rangle \geqslant 0,$$

for all $i \in \{1, ..., n\}$, or, equivalently,

$$\left\| x_i - \frac{x+X}{2} \right\| \leqslant \frac{1}{2}\|X - x\| \quad and \quad \left\| y_i - \frac{y+Y}{2} \right\| \leqslant \frac{1}{2}\|Y - y\|,$$

for all $i \in \{1, ..., n\}$, then the following inequality holds

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leqslant \frac{1}{4}\|X - x\|\|Y - y\|.$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Theorem 1.2. *Let $(H; \langle\cdot,\cdot\rangle)$ and $\mathbb{K}$ be as above and $\overline{x} = (x_1, ..., x_n) \in H^n$, $\overline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$ and $\overline{p} = (p_1, ..., p_n)$ a probability vector. If $x, X \in H$ are such that*

$$Re\,\langle X - x_i, x_i - x\rangle \geqslant 0 \ for\ all\ i \in \{1, ..., n\},$$

or, equivalently,

$$\left\| x_i - \frac{x+X}{2} \right\| \leqslant \frac{1}{2}\|X - x\| \ for\ all\ i \in \{1, ..., n\},$$

holds, then the following inequality holds

$$\begin{aligned} \left\| \sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i \right\| &\leqslant \frac{1}{2}\|X - x\| \sum_{i=1}^{n} p_i \left| \alpha_i - \sum_{j=1}^{n} p_j\alpha_j \right| \\ &\leqslant \frac{1}{2}\|X - x\| \left[\sum_{i=1}^{n} p_i|\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i\alpha_i \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

The constant $\frac{1}{2}$ in the first and second inequalities is best possible.
Motivated by the above results we establish some new Grüss type inequalities in semi-inner product modules over C^*-algebras and H^*-algebras for n-tuples of vectors, which are generalizations of Theorem 1.1 and Theorem 1.2. We also give some their applications for the approximation of the discrete Fourier and Melin transforms. In order to do that we need the following preliminary definitions and results.

2. Preliminaries

Hilbert C^*-modules are used as the framework for Kasparov's bivariant K-theory and form the technical underpinning for the C^*-algebraic approach to quantum groups. Hilbert C^*-modules are very useful in the following research areas: operator K-theory, index theory for operator-valued conditional expectations, group representation theory, the theory of AW^*-algebras, noncommutative geometry, and others. Hilbert C^*-modules form a category in between Banach spaces and Hilbert spaces and obey the same axioms as a Hilbert space except that the inner product takes values in a general C^*-algebra rather than in the complex number $\mathbb{C}$. This simple generalization gives a lot of trouble. Fundamental and familiar Hilbert space properties like Pythagoras' equality, self-duality and decomposition into orthogonal complements must be given up. Moreover, a bounded module map between Hilbert C^*-modules does not

need to have an adjoint; not every adjointable operator needs to have a polar decomposition. Hence to get its applications, we have to use it with great care.
A proper H^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|.\|)$ where the underlying Banach space is a Hilbert space with respect to the inner product $\langle .,. \rangle$ satisfying the properties $\langle ab, c\rangle = \langle b, a^*c\rangle = \langle a, cb^*\rangle$ for all $a, b, c \in \mathcal{A}$. A C^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|.\|)$ such that $\|a^*a\| = \|a\|^2$ for every $a \in \mathcal{A}$. If $\mathcal{A}$ is a proper H^*-algebra or a C^*-algebra and $a \in \mathcal{A}$ is such that $\mathcal{A}a = 0$ or $a\mathcal{A} = 0$ then $a = 0$. An element a in a proper H^*-algebra $\mathcal{A}$ is called positive ($a \geqslant 0$) if $\langle ax, x\rangle \geqslant 0$ for every $x \in \mathcal{A}$. Every positive element a in a proper H^*-algebra is self-adjoint (that is $a^* = a$). An element a in a C^*-algebra $\mathcal{A}$ is called positive ($a \geqslant 0$) if it is self-adjoint and has positive spectrum. An element a^*a is positive for every $a \in \mathcal{A}$, in both structures.
For a proper H^*-algebra $\mathcal{A}$, the trace class associated with $\mathcal{A}$ is $\tau(\mathcal{A}) = \{ab : a, b \in \mathcal{A}\}$. It is a self-adjoint two-sided ideal of $\mathcal{A}$ which is dense in $\mathcal{A}$. For every positive $a \in \tau(\mathcal{A})$ there exists the square root of a, that is, a unique positive $a^{\frac{1}{2}} \in \mathcal{A}$ such that $\left(a^{\frac{1}{2}}\right)^2 = a$, the square root of a^*a is denoted by $|a|$. There are a positive linear functional tr on $\tau(\mathcal{A})$ and a norm τ on $\tau(\mathcal{A})$, related to the norm of $\mathcal{A}$ by the equality $tr(a^*a) = \tau(a^*a) = \|a\|^2$ for every $a \in \mathcal{A}$. The trace-class is a Banach $*$-algebra with respect to the norm $\tau(.)$ defined by $\tau(a) = tr(|a|)$. Let us mention that $|tr(a)| \leqslant \tau(a)$ and $\|a\| \leqslant \tau(a)$ for every $a \in \tau(\mathcal{A})$.
Let $\mathcal{A}$ be a proper H^*-algebra or a C^*-algebra. A semi-inner product module over $\mathcal{A}$ is a right module X over $\mathcal{A}$ together with a generalized semi-inner product, that is, with a mapping $\langle .,. \rangle$ on $X \times X$, which is $\tau(\mathcal{A})$-valued if $\mathcal{A}$ is a proper H^*-algebra, or $\mathcal{A}$-valued if $\mathcal{A}$ is a C^*-algebra, has the following properties:

(i) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in X$,

(ii) $\langle x, ya\rangle = \langle x, y\rangle\, a$ for $x, y \in X, a \in \mathcal{A}$,

(iii) $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in X$,

(iv) $\langle x, x\rangle \geqslant 0$ for $x \in X$.

We will say that X is a semi-inner product H^*-module if $\mathcal{A}$ is a proper H^*-algebra and that X is a semi-inner product C^*-module if $\mathcal{A}$ is a C^*-algebra.
The absolute value of $x \in X$ is defined as the square root of $\langle x, x\rangle$, and it is denoted by $|x|$.
If, in addition,

(v) $\langle x, x\rangle = 0$ implies $x = 0$,

then $\langle .,. \rangle$ is called a generalized inner product and X is called an inner product module over $\mathcal{A}$. We will say that X is a (semi-)inner product H^*-module if it is a (semi-)inner product module over a proper H^*-algebra, and that X is a (semi-)inner product C^*-module if it is a (semi-)inner product module over a C^*-algebra.

As we can see, an inner product module obeys the same axioms as an ordinary inner product space, except that the inner product takes values in a more general structure rather than in the field of complex numbers.

If $\mathcal{A}$ is a C^*-algebra and X is a semi-inner product $\mathcal{A}$-module, then the following Schwarz inequality holds:

$$\|\langle x,y\rangle\|^2 \leqslant \|\langle x,x\rangle\|\|\langle y,y\rangle\| \ (x,y\in X) \tag{3}$$

($e.g.$ [13, Lemma 15.1.3]).

It follows from the Schwarz inequality (3) that $\|x\| := \|\langle x,x\rangle\|^{\frac{1}{2}} \quad (x\in X)$, is a semi-norm on X.

If X is a semi-inner product H^*-module, then there are two forms of the Schwarz inequality: for every $x,y\in X$

$$(tr\langle x,y\rangle)^2 \leqslant tr\langle x,x\rangle tr\langle y,y\rangle \qquad \text{(the weak Schwarz inequality)}, \tag{4}$$

$$(\tau\langle x,y\rangle)^2 \leqslant tr\langle x,x\rangle tr\langle y,y\rangle \qquad \text{(the strong Schwarz inequality)}. \tag{5}$$

First Saworotnow in [12] proved the strong Schwarz inequality, but the direct proof of that for a semi-inner product H^*-module can be found in [10].

Weak Schwarz inequality (4) implies that $\big\||x|\big\| = \big(tr\langle x,x\rangle\big)^{\frac{1}{2}}$ $(x\in X)$, is a semi-norm on X.

Now let $\mathcal{A}$ be a $*$-algebra, φ a positive linear functional on $\mathcal{A}$, and let X be a semi-inner $\mathcal{A}$-module. We can define a sesquilinear form on $X\times X$ by $\sigma(x,y)=\varphi\left(\langle x,y\rangle\right)$; the Schwarz inequality for σ implies that

$$|\varphi\langle x,y\rangle|^2 \leqslant \varphi\langle x,x\rangle\varphi\langle y,y\rangle.$$

In [3, Proposition 1, Remark 1] the authors present two other forms of the Schwarz inequality in semi-inner $\mathcal{A}$-module X, one for a positive linear functional φ on $\mathcal{A}$:

$$\varphi(\langle x,y\rangle\langle y,x\rangle) \leqslant \varphi\langle x,x\rangle r\langle y,y\rangle, \tag{6}$$

where r is the spectral radius, and another one for a C^*-seminorm γ on $\mathcal{A}$:

$$(\gamma\langle x,y\rangle)^2 \leqslant \gamma\langle x,x\rangle\gamma\langle y,y\rangle. \tag{7}$$

Before stating the main results, let us fix the rest of our notation. We assume that $\mathcal{A}$ is a C^*-algebra or a H^*-algebra, and assume unless stated otherwise, throughout this paper $\overline{p}=(p_1,...,p_n)\in\mathbb{R}^n$ a probability vector i.e.

$p_i \geqslant 0 \quad (i=1,...,n)$ and $\sum_{i=1}^n p_i = 1$. If X is a semi-inner product $\mathcal{A}$-module and $\overline{x}=(x_1,...,x_n), \overline{y}=(y_1,...,y_n) \in X^n$ we put

$$G_{\overline{p}}(\overline{x},\overline{y}) = \sum_{i=1}^n p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle,$$

we use $G_{\overline{p}}(\overline{x})$ instead of $G_{\overline{p}}(\overline{x},\overline{x})$.

Lemma 2.1. *Let X be a semi-inner product C^*-module or a semi-inner H^*-module, $a,b\in X$, $\overline{x}=(x_1,...,x_n), \overline{y}=(y_1,...,y_n)\in X^n, \overline{\alpha}=(\alpha_1,...,\alpha_n)\in\mathbb{K}^n$; $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $\overline{p}=(p_1,...,p_n)\in\mathbb{R}^n$ a probability vector, then*

$$\sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i \sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_i\Big(\alpha_i - \sum_{j=1}^n p_j\alpha_j\Big)(x_i - a), \tag{8}$$

and

$$G_{\overline{p}}(\overline{x},\overline{y}) = \sum_{i=1}^n p_i \left\langle x_i - a, y_i - b\right\rangle - \left\langle \sum_{i=1}^n p_i(x_i-a), \sum_{i=1}^n p_i(y_i-b) \right\rangle. \tag{9}$$

In particular

$$G_{\overline{p}}(\overline{x}) = \sum_{i=1}^n p_i \left|x_i - a\right|^2 - \left|\sum_{i=1}^n p_i x_i - a\right|^2 \leqslant \sum_{i=1}^n p_i\left|x_i-a\right|^2. \tag{10}$$

Proof. For every $a\in X$ a simple calculation shows that

$$\begin{aligned}\sum_{i=1}^n p_i\Big(\alpha_i - \sum_{j=1}^n p_j\alpha_j\Big)(x_i-a) &= \sum_{i=1}^n p_i\alpha_i x_i - \sum_{j=1}^n p_j\alpha_j \sum_{i=1}^n p_i x_i \\ &\quad - a\sum_{i=1}^n p_i\alpha_i + a\sum_{i=1}^n p_i \sum_{j=1}^n p_j\alpha_j \\ &= \sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i\sum_{i=1}^n p_i x_i.\end{aligned}$$

For every $a,b\in X$, a simple calculation shows that

$$\begin{aligned}
\sum_{i=1}^{n} p_i \langle x_i - a, y_i - b\rangle - \left\langle \sum_{i=1}^{n} p_i(x_i - a), \sum_{i=1}^{n} p_i(y_i - b) \right\rangle \\
= \sum_{i=1}^{n} p_i\big(\langle x_i, y_i\rangle - \langle x_i, b\rangle - \langle a, y_i\rangle + \langle a, b\rangle\big) \\
- \left\langle \sum_{i=1}^{n} p_i x_i - a, \sum_{i=1}^{n} p_i y_i - b \right\rangle \\
= \sum_{i=1}^{n} p_i\langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle = G_{\overline{p}}(\overline{x}, \overline{y}).
\end{aligned}$$

In particular for $a = b, x_i = y_i$ we have

$$\begin{aligned}
G_{\overline{p}}(\overline{x}) &= \sum_{i=1}^{n} p_i \langle x_i - a, x_i - a\rangle - \left\langle \sum_{i=1}^{n} p_i(x_i - a), \sum_{i=1}^{n} p_i(x_i - a) \right\rangle \\
&= \sum_{i=1}^{n} p_i |x_i - a|^2 - \left| \sum_{i=1}^{n} p_i(x_i - a) \right|^2 \leqslant \sum_{i=1}^{n} p_i |x_i - a|^2 . \quad \square
\end{aligned}$$

3. Grüss Type Inequalities in Semi-Inner Product C^*-Modules

In the following theorem we give a generalization of Theorem 1.1 for semi-inner product C^*-modules.

Theorem 3.1. *Let X be a semi-inner product C^*-module, $a, b \in X$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$, then the following inequality holds*

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right\|^2 \\
&\quad \leqslant \left\| \sum_{i=1}^{n} p_i |x_i - a|^2 - \left| \sum_{i=1}^{n} p_i x_i - a \right|^2 \right\| \left\| \sum_{i=1}^{n} p_i |y_i - b|^2 - \left| \sum_{i=1}^{n} p_i y_i - b \right|^2 \right\|
\end{aligned}$$

$$\leqslant \qquad \left(\sum_{i=1}^{n} p_i \|x_i - a\|^2\right)\left(\sum_{i=1}^{n} p_i \|y_i - b\|^2\right) \quad (11)$$

Proof. A simple calculation shows that

$$\sum_{i=1}^{n} p_i\langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \langle x_i - x_j, y_i - y_j\rangle ,$$

therefore

$$G_{\overline{p}}(\overline{x}) = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \langle x_i - x_j, x_i - x_j\rangle \geqslant 0.$$

It is easy to show that $G_{\overline{p}}(\cdot,\cdot)$ is an $\mathcal{A}$-value semi-inner product on X^n, so Schwarz inequality holds i.e.,

$$\|G_{\overline{p}}(\overline{x},\overline{y})\|^2 \leqslant \|G_{\overline{p}}(\overline{x})\|\|G_{\overline{p}}(\overline{y})\|.$$

From inequality (10) we get

$$\|G_{\overline{p}}(\overline{x})\| = \left\|\sum_{i=1}^{n} p_i |x_i - a|^2 - \left|\sum_{i=1}^{n} p_i x_i - a\right|^2\right\| \leqslant \sum_{i=1}^{n} p_i \|x_i - a\|^2 .$$

Similarly

$$\|G_{\overline{p}}(\overline{y})\| = \left\|\sum_{i=1}^{n} p_i |y_i - b|^2 - \left|\sum_{i=1}^{n} p_i y_i - b\right|^2\right\| \leqslant \sum_{i=1}^{n} p_i \|y_i - b\|^2 .$$

Therefore we obtain the inequality (11). □

Corollary 3.2. *Let X be a semi-inner product C^*-module, $a, b \in X$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$, $r \geqslant 0, s \geqslant 0$ are such that*

$$\|x_i - a\| \leqslant r, \quad \|y_i - b\| \leqslant s, \text{ for all } i \in \{1, ..., n\}, \tag{12}$$

then the following inequality holds

$$\left\|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle\right\| \leqslant rs. \tag{13}$$

The constant 1 coefficient of rs in the inequality (13) is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. From inequalities (11) and (12) we obtain (13). To prove the sharpness of the constant 1 in the inequality in (13), let us assume that, under the assumptions of the theorem, the inequalities hold with a constant $c > 0$, i.e.,

$$\|G_{\overline{p}}(\overline{x}, \overline{y})\| \leqslant crs. \tag{14}$$

Assume that $n = 2, p_1 = p_2 = \frac{1}{2}$ and e is an element of X such that $\|\langle e, e\rangle\| = 1$. We put

$$x_1 = a + re, \quad y_1 = b + se,$$
$$x_2 = a - re, \quad y_2 = b - se,$$

then, obviously,

$$\|x_i - a\| \leqslant r, \quad \|y_i - b\| \leqslant s, \quad (i = 1, 2),$$

which shows that the condition (12) holds. If we replace $n, p_1, p_2, x_1, x_2, y_1, y_2$ in (14), we obtain

$$\|G_{\overline{p}}(\overline{x}, \overline{y})\| = rs \leqslant crs,$$

from where we deduce that $c \geqslant 1$, which proves the sharpness of the constant 1. $\square$

The following Remark 3.3 (ii) is a generalization of Theorem 1.2 for semi-inner product C^*-modules.

Remark 3.3.

(i) Let $\mathcal{A}$ be a C^*-algebra, and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a, b, a_i, b_i, (i = 1, 2, ..., n) \in \mathcal{A}, r \geqslant 0, s \geqslant 0$ are such that

$$\|a_i - a\| \leqslant r, \quad \|b_i - b\| \leqslant s, \ \textit{for all } i \in \{1, ..., n\},$$

it is known that $\mathcal{A}$ is a Hilbert C^*-module over itself with the inner product defined by $\langle a, b\rangle := a^*b$. In this case (13) implies that

$$\left\|\sum_{i=1}^{n} p_i a_i^* b_i - \sum_{i=1}^{n} p_i a_i^* . \sum_{i=1}^{n} p_i b_i\right\| \leqslant rs.$$

Since

$$\|a_i^* - a^*\| \leqslant r, \quad \textit{for all } i \in \{1, ..., n\},$$

we deduce

$$\left\| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i . \sum_{i=1}^{n} p_i b_i \right\| \leqslant rs.$$

(ii) *Let X be a semi-inner product C^*-module, $a \in X, \overline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $\overline{x} = (x_1, ..., x_n) \in X^n, r \geqslant 0$ are such that*

$$\|x_i - a\| \leqslant r, \text{ for all } i \in \{1, ..., n\},$$

holds, from equality (8) we obtain

$$\left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right\| \leqslant r \sum_{i=1}^{n} p_i \left| \alpha_i - \sum_{i=1}^{n} p_j \alpha_j \right| \tag{15}$$

$$\leqslant r \left[\sum_{i=1}^{n} p_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}.$$

The constant 1 in the first and second inequalities in (15) is best possible.

4. Applications

In this section we give applications of Corollary 3.2 for the approximation of some discrete transforms such as the discrete Fourier and the Melin transforms. Let X be a semi-inner product C^*-module on C^*-algebra $\mathcal{A}$ and $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n$. For a given $\omega \in \mathbb{R}$, define the *discrete Fourier transform*

$$\mathcal{F}_\omega(x)(m) = \sum_{k=1}^{n} \exp(2\omega imk) \times x_k, \quad m = 1, ..., n. \tag{16}$$

The element $\sum_{k=1}^{n} \exp(2\omega imk) \times \langle x_k, y_k \rangle$ of $\mathcal{A}$ is called Fourier transform of the vector $(\langle x_1, y_1 \rangle, ..., \langle x_k, y_k \rangle) \in \mathcal{A}^n$ and will be denoted by

$$\mathcal{F}_\omega(x, y)(m) = \sum_{k=1}^{n} \exp(2\omega imk) \times \langle x_k, y_k \rangle \quad m = 1, ..., n.$$

The following Theorems 4.1, 4.2 and 4.3 are generalizations of [1, Theorems 66, 67 and 68] for semi-inner product C^*-modules respectively.

Theorem 4.1. *Let X be a semi-inner product C^*-module, $a, b \in X$. If $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n, r \geqslant 0, s \geqslant 0$ are such that*

$$\|x_k - a\| \leqslant r, \quad \big\|\exp(2\omega imk)y_k - b\big\| \leqslant s, \text{ for all } k,m \in \{1, ..., n\},$$

then the following inequality holds

$$\left\| \mathcal{F}_\omega(x, y)(m) - \left\langle \frac{1}{n}\sum_{k=1}^{n} x_k, \mathcal{F}_\omega(y)(m) \right\rangle \right\| \leqslant nrs,$$

for all $m \in \{1, ..., n\}$.

Proof. By Corollary 3.2 applied for $p_k = \frac{1}{n}$ and for the vectors x_k and $\exp(2\omega imk)y_k (k = 1, ..., n)$, we get

$$\begin{aligned} &\left\| \mathcal{F}_\omega(x, y)(m) - \left\langle \frac{1}{n}\sum_{k=1}^{n} x_k, \mathcal{F}_\omega(y)(m) \right\rangle \right\| \\ &\quad = n \left\| \sum_{k=1}^{n} \frac{1}{n}\Big\langle x_k, \exp(2\omega imk)y_k \Big\rangle - \left\langle \sum_{k=1}^{n}\frac{1}{n}x_k, \sum_{k=1}^{n}\frac{1}{n}\exp(2\omega imk)y_k \right\rangle \right\| \\ &\quad \leqslant nrs. \quad \square \end{aligned}$$

We can also consider the Mellin transform

$$\mathcal{M}(x)(m) = \sum_{k=1}^{n} k^{m-1} x_k, \quad m = 1, ..., n. \tag{17}$$

of the vector $x = (x_1, ..., x_n) \in X^n$.

The Mellin transform of the vector $(\langle x_1, y_1\rangle, ..., \langle x_k, y_k\rangle) \in \mathcal{A}^n$ is defined by $\sum_{k=1}^{n} k^{m-1}\langle x_k, y_k\rangle$ and will be denoted by

$$\mathcal{M}(x, y)(m) = \sum_{k=1}^{n} k^{m-1}\langle x_k, y_k\rangle.$$

Theorem 4.2. *Let X be a semi-inner product C^*-module, $a, b \in X$. If $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n, r \geqslant 0, s \geqslant 0$ are such that*

$$\|x_k - a\| \leqslant r, \quad \big\|k^{m-1}y_k - b\big\| \leqslant s, \text{ for all } k, \quad m \in \{1, ..., n\},$$

then the inequality

$$\left\| \mathcal{M}(x, y)(m) - \left\langle \frac{1}{n}\sum_{k=1}^{n} x_k, \mathcal{M}(y)(m) \right\rangle \right\| \leqslant nrs,$$

holds for all $m \in \{1, ..., n\}$.

The proof of Theorem 4.2 follows by Corollary 3.2 applied for $p_k = \frac{1}{n}$ and for the vectors x_k and $k^{m-1}y_k (k = 1, ..., n)$. We omit the details.
Another result which connects the Fourier transforms for different parameters ω also holds.

Theorem 4.3. *Let* X *be a semi-inner product* C^**-module,* $a, b \in X$. *If* $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n, r \geqslant 0, s \geqslant 0$ *are such that for all* $k, m \in \{1, ..., n\}$,

$$\left\| \exp(2\omega_1 imk)x_k - a \right\| \leqslant r, \ \left\| \exp(2\omega_2 imk)y_k - b \right\| \leqslant s,$$

then for all $m \in \{1, ..., n\}$, *the following inequality holds*

$$\left\| \frac{1}{n}\mathcal{F}_{\omega_2-\omega_1}(x, y)(m) - \left\langle \frac{1}{n}\mathcal{F}_{\omega_1}(x)(m), \frac{1}{n}\mathcal{F}_{\omega_2}(y)(m) \right\rangle \right\| \leqslant rs.$$

The proof of Theorem 4.3 follows by Theorem 3.1 applied for $p_k = \frac{1}{n}$ and for the vectors $\exp(2\omega_1 imk)x_k$ and $\exp(2\omega_2 imk)y_k$ $(k = 1, ..., n)$. We omit the details.

Let X be a semi-inner product C^*-module, $x = (x_1, ..., x_n), \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$ and $p = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a \in X, r \geqslant 0$, such that

$$\|x_i - a\| \leqslant r, \text{ for all i } \in \{1, ..., n\},$$

holds, from equality (9) in Lemma 2.1 we get

$$\begin{aligned} \left\| \sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i \sum_{i=1}^n p_i x_i \right\| &\leqslant r \sum_{i=1}^n p_i \left| \alpha_i - \sum_{i=1}^n p_j\alpha_j \right| \\ &\leqslant r \left[\sum_{i=1}^n p_i|\alpha_i|^2 - \left| \sum_{i=1}^n p_i\alpha_i \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{18}$$

The constant 1 coefficient of r in the first and second inequalities in (18) is best possible.
The following approximation result for the Fourier transform (16) which is a generalization of [2, Theorem 3] in semi-inner product C^*-modules holds.

Proposition 4.4. *Let* X *be a semi-inner product* C^**-module and* $a \in X$. *If* $x = (x_1, ..., x_n) \in X^n, r \geqslant 0$ *are such that*

$$\|x_i - a\| \leqslant r, \ \textit{for all } i \ \in \{1, ..., n\},$$

then for all $m \in \{1, ..., n\}$ *and* $\omega \in \mathbb{R}, \omega \neq \frac{l}{m}\pi, l \in \mathbb{Z}$ *the following inequality holds*

$$\left\|\mathcal{F}_\omega(x)(m) - \frac{\sin(\omega mn)}{\sin(\omega m)} \exp[\omega(n+1)im] \times \frac{1}{n}\sum_{k=1}^{n} x_k\right\| \leqslant r\left[n^2 - \frac{\sin^2(\omega mn)}{\sin^2(\omega m)}\right]^{\frac{1}{2}}. \quad (19)$$

Proof. From the inequality (18) we can state that,

$$\left\|\frac{1}{n}\sum_{i=1}^{n}\alpha_i x_i - \frac{1}{n}\sum_{i=1}^{n}\alpha_i.\frac{1}{n}\sum_{i=1}^{n}x_i\right\| \leqslant r\left[\frac{1}{n}\sum_{i=1}^{n}|\alpha_i|^2 - \left|\frac{1}{n}\sum_{i=1}^{n}\alpha_i\right|^2\right]^{\frac{1}{2}},$$

for all $\alpha_i \in \mathbb{C},\ x_i \in X\ (i = 1, ..., n)$. Consequently, we conclude that

$$\left\|\sum_{i=1}^{n}\alpha_i x_i - \sum_{i=1}^{n}\alpha_i.\frac{1}{n}\sum_{i=1}^{n}x_i\right\| \leqslant r\left[n\sum_{i=1}^{n}|\alpha_i|^2 - \left|\sum_{i=1}^{n}\alpha_i\right|^2\right]^{\frac{1}{2}}.$$

A simple calculation shows that (see the proof of Theorem 59 in [1]),

$$\sum_{k=1}^{n}\exp(2\omega imk) = \frac{\sin(\omega mn)}{\sin(\omega m)} \times \exp[\omega(n+1)im].$$

Putting $\alpha_k = \exp(2\omega imk)$, we get the desired result (19). $\square$

The following approximation result for the Mellin transform (17) in semi-inner product C^*-modules holds, (see [2, Theorem 4]).

Proposition 4.5. *Let* X *be a semi-inner product* C^**-module and* $a \in X$*. If* $x = (x_1, ..., x_n) \in X^n, r \geqslant 0$ *are such that*

$$\|x_i - a\| \leqslant r, \text{ for all } i \in \{1, ..., n\},$$

then

$$\left\|\mathcal{M}(x)(m) - S_{m-1}(n).\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \leqslant r\left[nS_{2m-2}(n) - S_{m-1}^2(n)\right]^{\frac{1}{2}}, m \in \{1, ..., n\},$$

where $S_p(n), p \in \mathbb{R}, n \in \mathbb{N}$ is the p-powered sum of the first n natural numbers, i.e.,

$$S_p(n) := \sum_{k=1}^{n} k^p.$$

Consider the following particular values of Mellin Transform

$$\mu_1(x) := \sum_{k=1}^{n} k x_k$$

and

$$\mu_2(x) := \sum_{k=1}^{n} k^2 x_k.$$

The following Corollary is a generalization of [2, Corollary 4], furthermore, the quantities in the right hand sides of inequalities (5.5) and (5.6) in [2, Corollary 4] have been corrected by the following inequalities (20) and (21) respectively.

Corollary 4.6. *Let X be a semi-inner product C^*-module, $a \in X$. If $x = (x_1, ..., x_n) \in X^n$ and $r \geqslant 0$ are such that*

$$\|x_i - a\| \leqslant r, \text{ for all } i \in \{1, ..., n\},$$

then

$$\left\| \mu_1(x) - \frac{n+1}{2} . \sum_{k=1}^{n} x_k \right\| \leqslant \frac{rn}{2} \left[\frac{(n-1)(n+1)}{3} \right]^{\frac{1}{2}}, \tag{20}$$

and

$$\left\| \mu_2(x) - \frac{(n+1)(2n+1)}{6} . \sum_{k=1}^{n} x_k \right\| \leqslant \frac{rn}{6\sqrt{5}} \sqrt{(n-1)(n+1)(2n+1)(8n+11)}. \tag{21}$$

Other inequalities related to the Grüss type discrete inequalities for polynomials with coefficients in a Hilbert space such as Theorem 61, Theorem 62, Corollary 52 in [1], have versions that are valid for polynomials with coefficients in a semi-inner C^*-module. However, the details are omitted.

5. Grüss Type Inequalities in Semi-Inner Product H^*-Modules

The following Theorem is a version of Corollary 3.2 for H^*-modules.

Theorem 5.1. *Let X be a semi-inner product H^*-module, $a, b \in X$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n, r \geqslant 0$ and $s \geqslant 0$ are such that*

$$\big\||x_i - a|\big\| \leqslant r, \quad \big\||y_i - b|\big\| \leqslant s, \quad \textit{for all} \ \ i \in \{1, ..., n\},$$

then the following inequality holds

$$\big|\tau(G_{\overline{p}}(\overline{x}, \overline{y}))\big| \leqslant rs. \tag{22}$$

The constant 1 coefficient of rs in the inequalities (22) is sharp.

Proof. By strong Schwarz inequality (5) we have

$$\tau(G_{\overline{p}}(\overline{x}, \overline{y}))^2 \leqslant tr(G_{\overline{p}}(\overline{x}))tr(G_{\overline{p}}(\overline{y})). \tag{23}$$

From inequality (10) in Lemma 2.1 we obtain

$$tr(G_{\overline{p}}(\overline{x})) \leqslant \sum_{i=1}^{n} p_i \big\||x_i - a|\big\|^2 \leqslant r^2, \tag{24}$$

similarly

$$tr(G_{\overline{p}}(\overline{y})) \leqslant \sum_{i=1}^{n} p_i \big\||y_i - b|\big\|^2 \leqslant s^2. \tag{25}$$

Now (23), (24) and (25) imply (22).
The fact that the constant 1 is sharp may be proven in a similar manner to the one embodied in the proof of Corollary 3.2. We omit the details. □

The following companion of the Grüss inequality for H^*-modules.

Theorem 5.2. *Let X be a semi-inner product H^*-module, $a, b \in X$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n, r \geqslant 0, s \geqslant 0$ are such that*

$$\big\||x_i - a|\big\| \leqslant r, \quad \big\||y_i - b|\big\| \leqslant s \textit{ for all } i \in \{1, ..., n\},$$

then the following inequality holds

$$\big|\tau(G_{\overline{p}}(\overline{x}, \overline{y}))\big| \leqslant rs - \left\|\Big|\sum_{i=1}^{n} p_i(x_i - a)\Big|\right\| \left\|\Big|\sum_{i=1}^{n} p_i(y_i - b)\Big|\right\| \leqslant rs. \tag{26}$$

Proof. From equality (9) in Lemma 2.1 for $\overline{y} = \overline{x}$ and $b = a$, we get

$$tr(G_{\overline{p}}(\overline{x})) = \sum_{i=1}^{n} p_i \big\||x_i - a|\big\|^2 - \left\|\Big|\sum_{i=1}^{n} p_i(x_i - a)\Big|\right\|^2.$$

Similarly for every $b \in X$ by substitution $\overline{x}$ with $\overline{y}$, and a with b, we have

$$tr(G_{\overline{p}}(\overline{y})) = \sum_{i=1}^{n} p_i \big\| |y_i - b| \big\|^2 - \left\| \Big| \sum_{i=1}^{n} p_i (y_i - b) \Big| \right\|^2. \tag{27}$$

By strong Schwarz inequality (5) we have

$$\begin{aligned} \big|\tau(G_{\overline{p}}(\overline{x},\overline{y}))\big| &\leqslant \left[\sum_{i=1}^{n} p_i \big\| |x_i - a| \big\|^2 - \left\| \Big| \sum_{i=1}^{n} p_i (x_i - a) \Big| \right\|^2 \right]^{\frac{1}{2}} \\ &\quad \times \left[\sum_{i=1}^{n} p_i \big\| |y_i - b| \big\|^2 - \left\| \Big| \sum_{i=1}^{n} p_i (y_i - b) \Big| \right\|^2 \right]^{\frac{1}{2}} \\ &\leqslant \left[r^2 - \left\| \Big| \sum_{i=1}^{n} p_i (x_i - a) \Big| \right\|^2 \right]^{\frac{1}{2}} \left[s^2 - \left\| \Big| \sum_{i=1}^{n} p_i (y_i - b) \Big| \right\|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Now using the elementary inequality for real numbers

$$(m^2 - n^2)(p^2 - q^2) \leqslant (mp - nq)^2,$$

on

$$\begin{aligned} m &= r, & n &= \left\| \Big| \sum_{i=1}^{n} p_i (x_i - a) \Big| \right\|, \\ p &= s, & q &= \left\| \Big| \sum_{i=1}^{n} p_i (y_i - b) \Big| \right\|, \end{aligned}$$

we get the inequality (26). $\square$

Corollary 5.3. *Let X be a semi-inner product H^*-module, $a \in X$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$ and $r \geqslant 0$ are such that*

$$\big\| |x_i - a| \big\| \leqslant r \text{ for all } i \in \{1, ..., n\}$$

then the following inequality holds

$$\big|\tau(G_{\overline{p}}(\overline{x},\overline{y}))\big| \leqslant r \left[\sum_{i=1}^{n} p_i \big\| |y_i| \big\|^2 - \left\| \Big| \sum_{i=1}^{n} p_i y_i \Big| \right\|^2 \right]^{\frac{1}{2}}. \tag{28}$$

Proof. From the equality (5.) and the condition (5.3) we have

$$tr(G_{\overline{p}}(\overline{x})) = \sum_{i=1}^{n} p_i \big\| |x_i - a| \big\|^2 - \left\| \Big| \sum_{i=1}^{n} p_i (x_i - a) \Big| \right\|^2 \leqslant \sum_{i=1}^{n} p_i r^2 - \left\| \Big| \sum_{i=1}^{n} p_i (x_i - a) \Big| \right\|^2 \leqslant r^2. \tag{29}$$

Using the inequalities (23), (29) and the equality (27) we get

$$\big|\tau(G_{\overline{p}}(\overline{x}, \overline{y}))\big| \leqslant r \left[\sum_{i=1}^{n} p_i \big\| |y_i - b| \big\|^2 - \left\| \Big| \sum_{i=1}^{n} p_i (y_i - b) \Big| \right\|^2 \right]^{\frac{1}{2}},$$

and for $b = 0$ we get the inequality (28). $\square$

There exists a version of Remark 3.3 for semi-inner product H^*-modules and there are applications from theorems and results in this section for the approximation of some discrete transforms in a semi-inner product H^*-module. However, the details are omitted but each of them can be proven in a similar manner as section 4.
Let $\mathcal{A}$ be a Banach $*$-algebra and X be a semi-inner product $\mathcal{A}$-module (see [3]). Utilizing Schwarz inequality (6) or (7) and a version of Lemma 2.1 for semi-inner product $\mathcal{A}$-module X. The technique of the proof of Theorem 3.1 is applicable to semi-inner product Banach $*$-modules as well.

References

[1] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova science puplishers inc., New York, 2005.

[2] S. S. Dragomir, A Grüss type discrete inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, 250 (2000), 494-511.

[3] A. G. Ghazanfari and S. S. Dragomir, Schwarz and Grüss type inequalities for C*-seminorms and positive linear functionals on Banach $*$-modules, *Linear Algebra and Appl.*, 434 (2011), 944-956.

[4] A. G. Ghazanfari and S. S. Dragomir, Bessel and Grüss type inequalities in inner product modules over Banach $*$-algebras, *J. of Inequalities. Applications.*, 2011. doi: 10.1155/562923

[5] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)$ $g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.*, 39 (1934), 215-226.

[6] D. Ilišević and S. Varošanec, Grüss type inequalities in inner product modules, *Proc. Amer. Math. Soc.*, 133 (2005), 3271-3280.

[7] A. I. Kechriniotis and K. K. Delibasis, On generalizations of Grüss inequality in inner product spaces and applications, *J. Inequal. Appl.*, 2010.

[8] X. Li, R. N. Mohapatra and R. S. Rodriguez, Grüss-type inequalities, *J. Math. Anal. Appl.*, 267 (2) (2002), 434-443.

[9] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

[10] L. Molnár, A note on the strong Schwarz inequality in Hilbert A-modules, *Publ. Math. Debrecen,* 40 (1992), 323-325.

[11] M. S. Moslehian and R. Rajić, Gruss inequality for n-positive linear maps, *Linear Algebra Appl.*, 433 (2010), 1555-1560.

[12] P. P. Saworotnow, A generalized Hilbert space, *Duke Math. J.,* 35 (1968), 191-197.

[13] N. E. Wegge-Olsen, *K-theory and C^*-algebras- A Friendly Approach*, Oxford University Press, Oxford, 1993.

Amir Ghasem Ghazanfari
Department of Mathematics
Assistant Professor of Mathematics
Lorestan University
Khoramabad, Iran
E-mail: ghazanfari.amir@gmail.com

Bahman Ghazanfari
Department of Mathematics
Assistant Professor of Mathematics
Lorestan University
Khoramabad, Iran
E-mail: bahman_ghazanfari@yahoo.com