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Original Research Paper

On Interval Valued Hadamard-Hermite Type Inequalities for CW Convex Functions

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Abstract. This work furnishes some Hadamard-Hermite (HH) type inequalities for interval-valued function by using an efficient partial order CW on the space of all compact intervals. It has been shown that the interval-valued HH's inequality proved herein, applied to already determined inequalities for real-valued functions. Specifically, this paper unifies and generalizes HH inequalities in the literature. We illustrate results with numerical examples.

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1 Introduction

Inequalities play a fundamental role in almost every branch of mathematics. The books like "Inequalities" by Beckenbach and Bellman [3],

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"Analytic Inequalities" by Mitrinovic [18] and "Advanced Inequalities" by Anastassiou [7], made noticeable contribution to this field and provide techniques, ideas, and applications. A lot of effort has been devoted to discover new types of inequalities and their applications in many parts of analysis. In recent years, this theory has attracted many researchers, stimulated new research directions, and influenced various aspects of mathematical analysis and applications. Important advances related to inequalities along with many important applications, remained active areas of research in the last few decades. Many researchers have contributed to develop new inequalities in various areas of mathematics such as Fractional Calculus and Quantum Calculus. For the recent development in inequalities and their applications in different fields, we recommend [17, 18, 19, 23, 25, 35, 36] and the references cited therein to the interested readers.

Several mathematical models, such as dynamic and static, linear and non-linear, discrete or continuous, can study the behavior of real-world systems. The knowledge about the parameters of real-world systems is uncertain and inadequate because we can not measure these parameters accurately. Only real numbers can not help us to represent these parameters in such situations. This deficiency is reduced by using interval or fuzzy models. The interval of real numbers is used for the representation of an uncertain variable for such parameters.

Moore's [20] book "Interval Analysis" was the outcome of his Ph.D. thesis and therefore focuses on bounding solutions of initial value problems for ordinary differential equations. In the past twenty years, many new advances in interval analysis have been seen. Hukuhara [13] was the first one who had introduced the Hukuhara difference (H-difference) and Hukuhara derivative for set-valued mappings and started the new research topic for researchers in set differential equations and fuzzy differential equations. As interval analysis is a special case of set-valued analysis, therefore many authors have contributed to the field of interval-valued differential equations with the help of the Hukuhara derivative. H-difference has the drawback that it does not always applicable to any two compact intervals of real numbers. Stefanini [29] generalized the idea of H-difference to generalized Hukuhara difference (gH-difference), to overcome the drawback of H-difference and proposed the concept of gH-derivative. After that, the theory of interval analysis has many recent developments till now.

By using the concept of gH-difference, Younus *et al.* [30] developed the theory of interval-valued fractional q-calculus for interval-valued functions. Their study investigates the generalized Hukuhara difference, fractional q-differentiability, q-integrability, and fractional q-integrability for interval-valued functions defined on the q-geometric set of real numbers. The concept of interval-valued fractional calculus was given by Lupulescu [14]. Many authors have studied interval-valued differential equations based on the concept of gH-difference in recent years [15, 16, 28]. Some partial orders on the set of compact intervals exist in literature, for instance, see, [8, 31, 33], therefore many classical inequalities for real-valued functions were extended to interval-valued functions by using these partial orders, see, for example [2, 6, 26, 27, 32] and [34].

1.1 Convex analysis

Now, we recall some basic properties, definitions, and results on convex analysis, which are used throughout this article. Convex functions play a vital role in the theory of inequalities. A lot of inequalities are established using convex functions, see, for example [1, 4, 5, 9], and [11].

Let $I \subset \mathbb{R}$ and a mapping $\digamma : I \longrightarrow \mathbb{R}$ is called convex:

$$\theta \digamma(a) + (1-\theta)\digamma(b) \geq \digamma(\theta a + (1-\theta) b), \tag{1}$$

$(\forall)$ $a, b \in I$ and $\theta \in (0,1)$.

For $a_1, a_2, a_3 \in I$ such that $a_1 < a_2 < a_3$, then (1) is equivalent to

$$\frac{a_3 - a_2}{a_3 - a_1}\digamma(a_1) + \frac{a_2 - a_1}{a_3 - a_1}\digamma(a_3) \geq \digamma(a_2). \tag{2}$$

Remark 1.1. Let $I \subset \mathbb{R}$ and $\digamma : I \longrightarrow \mathbb{R}$ is second differentiable. Then $\digamma$ is convex $\Leftrightarrow 0 \leq \digamma''(r)$ $\forall$ $r \in I$.

Convex functions satisfy following properties:

(a) If $G, \digamma$ are convex functions and $\alpha, \beta \geq 0$, then $\alpha G + \beta \digamma$ is convex.

(b) The sum of finitely many convex functions is also convex.

The following is Hadamard's inequality (HH inequality) for convex functions.

Let $\mathbb{R} \supset I$. If $\digamma$ be a convex function from I to $\mathbb{R}$ and for all $a, b \in I$ with $a < b$, then we have:

$$\digamma\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \digamma(x)\,dx \leq \frac{\digamma(a)+\digamma(b)}{2}.$$

Hadamard's contribution motivated many researchers to obtain various extensions of Hadamard's inequality. Some of these inequalities are presented here.

Lemma 1.2. *[21] For $p, q > 0$, $a_1 \leq a < b \leq b_1$, and all convex continuous functions $\digamma : [a_1, b_1] \longrightarrow \mathbb{R}$ the following inequalities*

$$\begin{aligned} \digamma(v) &\leq \tfrac{1}{2y}\int_{v-y}^{v+y} \digamma(\theta)\,d\theta \leq \tfrac{1}{2}\left(\digamma(v-y)+\digamma(v+y)\right) \\ &\leq \frac{p\digamma(a)+q\digamma(b)}{p+q}, \end{aligned}$$

hold for $v = \dfrac{pa+qb}{p+q}$, $y > 0 \Leftrightarrow y \leq \dfrac{b-a}{p+q}\min\{p, q\}$.

Lemma 1.3. *[12] Let $\digamma$ be a real-valued convex function defined on $[a, b]$, then we can obtain the following inequality*

$$\begin{aligned} \digamma\left(\tfrac{a+b}{2}\right) &\leq \tfrac{1}{(b-a)^2}\int_a^b\int_a^b \digamma(\theta x+(1-\theta)y)\,dxdy \\ &\leq \tfrac{1}{(b-a)}\int_a^b \digamma(x)\,dx \\ &\leq \tfrac{\digamma(b)+\digamma(a)}{2}, \end{aligned}$$

for all $\theta \in [a, b]$.

The following lemmas are the generalizations and refinements of HH inequality for linear isotonic functional found in [10] and [22].

Lemma 1.4. *Let X be a real vector space and $S \subseteq X$ be a convex set.*

(1) *Then real-valued functional $\digamma : X \to \mathbb{R}$ is convex on S if and only if, the real-valued functional $G_{x,y} : [0, 1] \to \mathbb{R}$ defined as $G_{x,y} := \digamma(\theta x + (1-\theta)y)$ is convex on $[0, 1]$, $\forall\ x, y \in S$.*

(2) *If $\digamma$ is real-valued convex functional on $S \subseteq X$, then $\forall\ x, y \in S$, the real-valued mapping $G_{x,y}$ on $[0,1]$, defined by*

$$G_{x,y}(\theta) := \left[\frac{1}{2}\digamma(\theta x + (1-\theta) y)\right] + \left[\frac{1}{2}\digamma((1-\theta) x + \theta y)\right]$$

is also convex on $[0,1]$. Moreover, the following inequality is also satisfied

$$\frac{\digamma(x) + \digamma(y)}{2} \geq G_{x,y}(\theta) \geq \digamma\left(\frac{x+y}{2}\right)$$

$\forall\ x, y$ *in S and $\theta \in [0,1]$.*

1.2 Arithmetic for interval valued functions

The set denoted by $\mathcal{K}_C$ is defined as follows:

$$\mathcal{K}_C = \left\{A = \left[a^-, a^+\right] : a^+, a^- \in \mathbb{R} \text{ and } a^+ \geq a^-\right\}.$$

If $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathcal{K}_C$, then the scalar multiplication and Minkowski's addition are defined as:

$$\lambda A = \begin{cases} \lambda\left[a^-, a^+\right] & \text{if } \lambda > 0 \\ \{0\} & \text{if } \lambda = 0 \\ \lambda\left[a^+, a^-\right] & \text{if } \lambda < 0 \end{cases}$$

for $\lambda \in \mathbb{R}$, and

$$A + B = \left[a^- + b^-, a^+ + b^+\right],$$

respectively.

The definition of difference for interval-valued function is given as: $A - B = A + (-1) B$. The set $\mathcal{K}_C$ is not a vector space because in general $A - A \neq \{0\}$. Hukuhara introduced a H-difference to overcome the above drawback which is defined as

$$A = B + C,$$

and denoted by $A -_H B$. The H-difference only exists for intervals A and B for which the $len(A) \geq len(B)$, where $len(A) = a^+ - a^-$. Stefanini generalized H-difference to reduce this deficiency, for any two compact intervals $A, B \in \mathcal{K}_C$, $A \ominus_{gH} B$ is given as follows

$$C = A \ominus_{gH} B \text{ iff } \begin{cases} (1) & B + C = A \\ (2) & A + (-1)\, C = B. \end{cases}$$

In case (1), the gH-difference is similar as H-difference. Thus the gH-difference is the extension of H-difference. In general, gH-difference for two arbitrary closed intervals is given as:

$$A \ominus_{gH} B = \left[\min\left\{a^- - b^-, a^+ - b^+\right\}, \max\left\{a^- - b^-, a^+ - b^+\right\}\right].$$

If $A \in \mathcal{K}_C$, then norm $\|A\|$ is defined as: $\|A\| = \max\{|a^-|, |a^+|\}$.

The Hausdorff metric D_H on $\mathcal{K}_C$ for $A, B \in \mathcal{K}_C$ is given as:

$$D_H(A, B) = \max\left\{\left|a^- - b^-\right|, \left|a^+ - b^+\right|\right\}.$$

Next, we will go through some fundamentals of interval-valued functions on $\mathcal{K}_C$. Let $I \subset \mathbb{R}$, a mapping $H : I \longrightarrow \mathcal{K}_C$ is called interval-valued, if $H(t) = [h^-(t), h^+(t)]$, where $h^-, h^+ : I \longrightarrow \mathbb{R}$ and $h^-(t) \leq h^+(t)$, for all $t \in I$.

Definition 1.5. Consider $F : [t_1, t_2] \to \mathcal{K}_C$ and $t_0 \in (t_1, t_2)$, then $L \in \mathcal{K}_C$ is limit of F at the point t_0, which is denoted as

$$\lim_{t \to t_0} H(t) = L,$$

if for any given $\varepsilon > 0$ their exist $\delta > 0$ such that,

$$0 < |t - t_0| < \delta \Longrightarrow D_H(H(t), L) < \varepsilon.$$

Remark 1.6. Let $H : [t_1, t_2] \to \mathcal{K}_C$ be an interval-valued function which is defined as $H(t) = [h^-(t), h^+(t)]$ for $t \in [t_1, t_2]$, then the $\lim_{t \to t_0} H(t)$ for $t_0 \in (t_1, t_2)$ exist if and only if both $\lim_{t \to t_0} h^-(t)$ and $\lim_{t \to t_0} h^+(t)$ exists as finite real number. So limit of the function H can be defined as:

$$\lim_{t \to t_0} H(t) = \left[\lim_{t \to t_0} h^-(t), \lim_{t \to t_0} h^+(t)\right].$$

The interval-valued function $H : T \to \mathcal{K}_C$ is continuous at t_0 if and only if $\lim_{t \to t_0} H(t) = H(t_0)$.

Definition 1.7. Let $H : [t_1, t_2] \rightarrow \mathcal{K}_C$ be a function and for any $t \in (t_1, t_2)$, we can define $H^{'}(t) \in \mathcal{K}_C$ as

$$H^{'}(t) = \lim_{h \rightarrow 0} \frac{H(t+h) \ominus_{gH} \ H(t)}{h}.$$

If the limit exists, then H is generalized Hukuhara differentiable at t. Further, H is twice generalized Hukuhara differentiable at t, if

$$H^{''}(t) = \lim_{h \rightarrow 0} \frac{H^{'}(t+h) \ominus_{gH} \ H^{'}(t)}{h}.$$

Theorem 1.8. *Let H be interval-valued function defined on $[t_1, t_2]$ such that $H(t) = [h^-(t), h^+(t)]$ for $t \in [t_1, t_2]$, the function H is gH-differentiable if and only if $h^-(t)$ and $h^+(t)$ are differentiable real-valued functions and*

$$H^{'}(t) = \left[\min\left\{h^{-^{'}}(t), h^{+^{'}}(t)\right\}, \ max\left\{h^{-^{'}}(t), h^{+^{'}}(t)\right\}\right].$$

Remark 1.9. Suppose $H : [t_1, t_2] \rightarrow \mathcal{K}_C$ be a gH-differentiable at any point $t_0 \in (t_1, t_2)$. H is said to be (i) gH-differentiable and (ii) gH-differentiable at t_0 if

$$H^{'}(t) = \left[h^{-^{'}}(t_0), h^{+^{'}}(t_0)\right],$$

and

$$H^{'}(t) = \left[h^{+^{'}}(t_0), h^{-^{'}}(t_0)\right],$$

respectively.

Proposition 1.10. *If $H : [t_1, t_2] \rightarrow \mathcal{K}_C$ is gH-differentiable at $t_0 \in (t_1, t_2)$ then $(h^-(t) + h^+(t))$ is also a differentiable function at t_0.*

2 Interval Valued Inequalities for CW Convex Functions

Since $\mathcal{K}_C$ is not a totally order set. For the comparison of images of interval-valued functions in the context of optimization problems, several partial order relations exist in $\mathcal{K}_C$, see, for example [6, 33].

For a given interval $A = [a^-, a^+] \in \mathcal{K}_C$, the center and half width of interval A can be defined as

$$A^C = \frac{1}{2}\left(a^+ + a^-\right) \text{ and } A^W = \frac{1}{2}\left(a^+ - a^-\right), \tag{3}$$

respectively. For $A, B \in \mathcal{K}_C$, CW partial order on $\mathcal{K}_C$ is defined as following

$$A \preceq_{CW} B \text{ if and only if } A^C \leq B^C \text{ and } A^W \leq B^W \tag{4}$$

and $A \prec_{CW} B$ iff $A \preceq_{CW} B$ and $A \neq B$.

Definition 2.1. Let $H : I \longrightarrow \mathcal{K}_C$ be an interval-valued function defined on a convex subset $I \subseteq \mathbb{R}$ such that $H(a) = [h^-(a), h^+(a)]$ where $h^-(a) \leq h^+(a)$ for all $a \in I$. The function H is CW-convex, if the following relation holds

$$H(\theta a + (1-\theta) b) \preceq_{CW} (1-\theta) H(a) + \theta H(a) \tag{5}$$

for all $\theta \in (0,1)$, and for all $a, b \in I$.

By using (4), H is CW-convex if and only if the following two inequalities hold

$$H^C(\theta a + (1-\theta) b) \leq \theta H^C(a) + (1-\theta) H^C(b) \tag{6}$$

and

$$H^W(\theta a + (1-\theta) b) \leq \theta H^W(a) + (1-\theta) H^W(b), \tag{7}$$

for all $\theta \in (0,1)$ and $a, b \in I$, where $H^C = \frac{1}{2}(h^+ + h^-)$ and $H^W = \frac{1}{2}(h^+ - h^-)$, respectively.

Moreover, by using (3), inequalities (6) and (7) can be written as

$$\begin{aligned} &\tfrac{1}{2}(1-\theta)(h^- + h^+)(b) + \tfrac{1}{2}\theta(h^- + h^+)(a) \\ &\geq \tfrac{1}{2}\left[(h^- + h^+)(\theta a + (1-\theta) b)\right] \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\tfrac{1}{2}(1-\theta)(h^+ - h^-)(b) + \tfrac{1}{2}\theta(h^+ - h^-)(a) \\ &\geq \tfrac{1}{2}\left[(h^+ - h^-)(\theta a + (1-\theta) b)\right], \end{aligned} \tag{9}$$

respectively.

It is illustrated in the following example that CW convexity of interval-valued function does not imply the convexity of real-valued function h^-.

Example 2.2. Let $H : [1, \infty) \longrightarrow \mathcal{K}_C$ such that $H(x) = \left[x^{\frac{1}{2}} - x^2, -x^{\frac{1}{2}} + 2x^2\right]$, where $h^-(x) = x^{\frac{1}{2}} - x^2$, $h^+(x) = -x^{\frac{1}{2}} + 2x^2$. The function h^+ is convex but h^- is not convex function. The addition and subtraction of h^- and h^+ is convex i.e., $h^-(x) + h^+(x) = x^2$ and $h^+(x) - h^-(x) = -2x^{\frac{1}{2}} + 3x^2$ are convex functions. It follows that H^C and H^W are both convex functions which implies H is CW-convex function.

Throughout this section, H is considered to be interval-valued function.

Theorem 2.3. *If $H : [a, b] \longrightarrow \mathcal{K}_C$ is CW-convex mapping, then the following inequality holds*

$$H\left(\frac{a+b}{2}\right) \preceq_{CW} \frac{1}{b-a}\int_a^b H(x)\,dx \preceq_{CW} \frac{1}{2}\left[H(a) + H(b)\right] \tag{10}$$

Proof. *Since H is a CW-convex function. By integrating the inequality (9) with respect to θ on interval $[0,1]$, we get*

$$\begin{aligned}
&\tfrac{1}{2}\int_0^1 \left[\left(h^+ - h^-\right)(\theta a + (1-\theta) b)\right] d\theta \\
&\leq \tfrac{1}{2}\int_0^1 \theta \left(h^+ - h^-\right)(a)\, d\theta + \tfrac{1}{2}\int_0^1 (1-\theta)\left(h^+ - h^-\right)(b)\, d\theta \\
&= \tfrac{1}{4}\left[\left(h^+ - h^-\right)(a) + \left(h^+ - h^-\right)(b)\right].
\end{aligned}$$

We obtain the following inequality by using the definition of convexity

$$\begin{aligned}
H^W\left(\tfrac{a+b}{2}\right) &= H^W\left(\tfrac{a+(\theta a - \theta a)+(\theta b - \theta b)+b}{2}\right) \\
&\leq \tfrac{1}{2}H^W(\theta a + (1-\theta) b) + \tfrac{1}{2}H^W((1-\theta) a + \theta b).
\end{aligned} \tag{11}$$

By using (9) in inequality (11) and integrating it with respect to θ on interval $[0,1]$, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\int_0^1 \left(h^+ - h^-\right)\left(\tfrac{a+b}{2}\right) d\theta \leq \\
&\tfrac{1}{4}\left(\int_0^1 \left(h^+ - h^-\right)(\theta a + (1-\theta) b)\, d\theta + \int_0^1 \left(h^+ - h^-\right)((1-\theta) a + \theta b)\, d\theta\right).
\end{aligned} \tag{12}$$

Now, by replacing $1-\theta=\lambda$ in the following integral, we get

$$\int_0^1 \left(h^+ - h^-\right)\left((1-\theta)\,a + \theta b\right) d\theta = \int_0^1 \left(h^+ - h^-\right)\left(\lambda a + (1-\lambda)\,b\right) d\theta. \tag{13}$$

Using (13) in (12), we have

$$\frac{1}{2}\int_0^1 \left(h^+ - h^-\right)\left(\frac{a+b}{2}\right) d\theta \leq \frac{1}{2}\int_0^1 \left(h^+ - h^-\right)\left(\theta a + (1-\theta)\,b\right) d\theta \tag{14}$$

By taking $\theta a + (1-\theta)\,b = x$ in inequality (14), it follows that

$$\begin{aligned} \left(h^+ - h^-\right)\left(\tfrac{a+b}{2}\right) &\leq \left(\tfrac{1}{b-a}\right)\int_a^b \left(h^+ - h^-\right)(x)\,dx \\ &\leq \tfrac{1}{2}\left(\left(h^+ - h^-\right)(a) + \left(h^+ - h^-\right)(b)\right), \end{aligned} \tag{15}$$

which implies that

$$H^W\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b H^W(x)\,dx \leq \frac{H^W(a) + H^W(b)}{2}. \tag{16}$$

Similarly, for H^C, we obtain the following inequality

$$\begin{aligned} \left(h^+ + h^-\right)\left(\tfrac{a+b}{2}\right) &\leq \tfrac{1}{(b-a)}\int_a^b \left(h^+ + h^-\right)(x)\,dx \\ &\leq \tfrac{1}{2}\left(\left(h^+ + h^-\right)(a) + \left(h^+ + h^-\right)(b)\right), \end{aligned}$$

which yields

$$H^C\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b H^C(x)\,dx \leq \frac{H^C(a) + H^C(b)}{2}. \tag{17}$$

We get inequality (10) from (16) and (17). □

Example 2.4. Let $H : [2,10] \longrightarrow \mathcal{K}_C$ such that $H(x) = \left[x^{\frac{1}{2}} - x^2, -x^{\frac{1}{2}} + 2x^2\right]$, where $h^-(x) = x^{\frac{1}{2}} - x^2$, $h^+(x) = -x^{\frac{1}{2}} + 2x^2$. The interval-valued function H is CW-convex. According to Theorem 2.3, we have $H(6) = [-33.5, 69.5]$, $H^C(6) = 18$ and $H^W(6) = 51.5$. For $A := \frac{1}{8}\int_2^{10} H(x)\,dx =$

$[-38.9, 80.2]$, we have $A^C = 20.6$ and $A^W = 59.6$. Finally, for the right hand side of inequality (10), we have $H(2) = [-2.6, 6.6]$, $H(10) = [-96.8, 196.8]$, $B := \dfrac{H(2) + H(10)}{2} = [-49.7, 101.7]$. The center and half width of interval B is $B^C = 25.9$ and $B^W = 75.7$.
Since $H^C(6) < A^C < B^C$ and $H^W(6) < A^W < B^W$. Therefore, $H(6) \prec_{CW} A \prec_{CW} B$.

Theorem 2.5. *Suppose $H : [a, b] \longrightarrow \mathcal{K}_C$ be a CW-convex function, for all $p, q > 0$, and $v = \dfrac{pa + qb}{p + q}$, we have the following inequalities*

$$\begin{aligned} H\left(\tfrac{pa+qb}{p+q}\right) &\preceq_{CW} \tfrac{1}{2y}\int_{v-y}^{v+y} H(\theta)\, d\theta \preceq_{CW} \tfrac{1}{2}[H(v-y) + H(v+y)] \\ &\preceq_{CW} \tfrac{pH(a)+qH(b)}{p+q}. \end{aligned} \tag{18}$$

Proof. *Since H be a CW-convex, so that H^C, H^W are convex. From Lemma 1.2, if*

$$0 < y \le \min\{p, q\}\left(\frac{b-a}{p+q}\right)$$

then it implies that $a \le v - y < v + y \le b$. Therefore, H is CW-convex as well as defined on $[v - y, v + y]$. Now, by applying inequality (10) for $a = v - y$ and $b = v + y$, we have

$$H(v) \preceq_{CW} \frac{1}{2y}\int_{v-y}^{v+y} H(\theta)\, d\theta \preceq_{CW} \frac{H(v-y) + H(v+y)}{2}.$$

First, we observe that

$$\begin{aligned} &\tfrac{1}{2}\left(\left(h^- + h^+\right)(v)\right) \le \tfrac{1}{4y}\int_{v-y}^{v+y} \left(h^- + h^+\right)(\theta)\, d\theta \\ &\le \tfrac{1}{4}\left(\left(h^- + h^+\right)(v-y) + \left(h^- + h^+\right)(v+y)\right). \end{aligned} \tag{19}$$

Since H^C is convex and by applying (2) for $x_1 = a$, $x_3 = b$ and $x_2 = v - y$, we have

$$\begin{aligned} &\tfrac{1}{2}\left(\left(h^- + h^+\right)(v-y)\right) \\ &= \tfrac{1}{2}\left(\left(\tfrac{b-(v-y)}{b-a}\right)\left(h^- + h^+\right)(a) + \left(\tfrac{(v-y)-a}{b-a}\right)\left(h^- + h^+\right)(b)\right). \end{aligned} \tag{20}$$

Similarly, for $x_2 = v + y$, *we have*

$$\begin{aligned} &\tfrac{1}{2}\left(\left(h^- + h^+\right)(v+y)\right) \\ &= \tfrac{1}{2}\left(\left(\tfrac{b-(v+y)}{b-a}\right)\left(h^- + h^+\right)(a) + \left(\tfrac{(v+y)-a}{b-a}\right)\left(h^- + h^+\right)(b)\right). \end{aligned} \tag{21}$$

respectively. Using (20) and (21) in (19), we obtain

$$\begin{aligned} &\tfrac{1}{4y}\int_{v-y}^{v+y}\left(h^- + h^+\right)(\theta)\,d\theta \\ &\leq \tfrac{1}{4}\left(\left(h^- + h^+\right)(v-y) + \left(h^- + h^+\right)(v+y)\right) \\ &= \tfrac{1}{2}\left(\left(\tfrac{b-v}{b-a}\right)\left(h^- + h^+\right)(a) + \left(\tfrac{v-a}{b-a}\right)\left(h^- + h^+\right)(b)\right) \\ &= \tfrac{1}{2(b-a)}\left((b-v)\left(h^- + h^+\right)(a) + (v-a)\left(h^- + h^+\right)(b)\right). \end{aligned}$$

By putting $v = \dfrac{pa+qb}{p+q}$ *in above expression, we get*

$$\frac{1}{4y}\int_{v-y}^{v+y}\left(h^- + h^+\right)(\theta)\,d\theta \leq \frac{1}{2}\left(\frac{p\left(h^- + h^+\right)(a) + q\left(h^- + h^+\right)(b)}{p+q}\right). \tag{22}$$

Now, by using inequalities (19) and (22), we obtain

$$\begin{aligned} H^C\left(\tfrac{pa+qb}{p+q}\right) &\leq \tfrac{1}{2y}\int_{v-y}^{v+y} H^C(\theta)\,d\theta \leq \tfrac{1}{2}\left(H^C(v-y) + H^C(v+y)\right) \\ &\leq \tfrac{pH^C(a)+qH^C(b)}{p+q}. \end{aligned} \tag{23}$$

Similarly for H^W, *we get*

$$\begin{aligned} H^W\left(\tfrac{pa+qb}{p+q}\right) &\leq \tfrac{1}{2y}\int_{v-y}^{v+y} H^W(\theta)\,d\theta \\ &\leq \tfrac{1}{2}\left(H^W(v-y) + H^W(v+y)\right) \\ &\leq \frac{pH^W(a) + qH^W(b)}{p+q}. \end{aligned} \tag{24}$$

We get inequality (18) from (23) and (24). □

Theorem 2.6. *Suppose* $H : [a,b] \longrightarrow \mathcal{K}_C$ *be a CW-convex mapping. We have the following inequalities*

$$\begin{aligned} H\left(\tfrac{a+b}{2}\right) &\preceq_{CW} \tfrac{1}{(b-a)^2}\int_a^b\int_a^b H(\theta x + (1-\theta)y)\,dxdy \\ &\preceq_{CW} \tfrac{1}{b-a}\int_a^b H(x)\,dx \preceq_{CW} \tfrac{H(a)+H(b)}{2} \end{aligned} \tag{25}$$

for all $\theta \in [a,b]$.

Proof. *Since* H *be a* CW*-convex function. Therefore,* H^C *and* H^W *are convex. For* $\theta \in [0,1]$, *and for all* $x, y \in [a,b]$, *integrating inequality (8) with respect to* x, y *on* $[a,b] \times [a,b]$, *we get*

$$\begin{aligned} &\tfrac{1}{2}\int_a^b \int_a^b \left(h^- + h^+\right)\left(\theta x + (1-\theta) y\right) dxdy \\ &\leq \tfrac{1}{2}\int_a^b \int_a^b \left(\theta\left(h^- + h^+\right)(x) + (1-\theta)\left(h^- + h^+\right)(y)\right) dxdy \\ &= \tfrac{(b-a)}{2}\left[\theta \int_a^b \left(h^- + h^+\right)(x)\, dx + (1-\theta)\int_a^b \left(h^- + h^+\right)(x)\, dx\right] \\ &= \tfrac{(b-a)}{2}\int_a^b \left(h^- + h^+\right)(x)\, dx \end{aligned} \tag{26}$$

By using inequality (26) and right half of (15) from Theorem 2.3 , it follows that

$$\begin{aligned} &\tfrac{1}{2(b-a)^2}\int_a^b \int_a^b \left(h^- + h^+\right)\left(\theta x + (1-\theta) y\right) dxdy \\ &\leq \tfrac{1}{2(b-a)}\int_a^b \left(h^- + h^+\right)(x)\, dx \\ &\leq \tfrac{1}{4}\left[\left(h^- + h^+\right)(a) + \left(h^- + h^+\right)(b)\right]. \end{aligned} \tag{27}$$

By Jensen's inequality for double integrals, we obtain

$$\begin{aligned} &\tfrac{1}{2}\left(h^- + h^+\right)\left(\tfrac{1}{(b-a)^2}\int_a^b \int_a^b \left(\theta x + (1-\theta) y\right) dxdy\right) \\ &\leq \tfrac{1}{2(b-a)^2}\int_a^b \int_a^b \left(h^- + h^+\right)\left(\theta x + (1-\theta) y\right) dxdy. \end{aligned} \tag{28}$$

Evaluating the following integral, we have

$$\frac{1}{(b-a)^2}\int_a^b \int_a^b \left(\theta x + (1-\theta) y\right) dxdy = \frac{a+b}{2} \tag{29}$$

By using (29) in (28), we get

$$\begin{aligned} &\tfrac{1}{2}\left(h^- + h^+\right)\left(\tfrac{a+b}{2}\right) \\ &\leq \tfrac{1}{2(b-a)^2}\int_a^b \int_a^b \left(h^- + h^+\right)\left(\theta x + (1-\theta) y\right) dxdy. \end{aligned} \tag{30}$$

By using (27) and (30), we have

$$(h^- + h^+)\left(\tfrac{a+b}{2}\right) \le \tfrac{1}{(b-a)^2}\int_a^b\int_a^b (h^- + h^+)(\theta x + (1-\theta)y)\,dxdy$$
$$\le \tfrac{1}{(b-a)}\int_a^b (h^- + h^+)(x)\,dx \le \tfrac{1}{2}\left[(h^- + h^+)(a) + (h^- + h^+)(b)\right],$$

which yields

$$\begin{aligned} &H^C\left(\tfrac{a+b}{2}\right) \le \tfrac{1}{(b-a)^2}\int_a^b\int_a^b H^C(\theta x + (1-\theta)y)\,dxdy \\ &\le \tfrac{1}{b-a}\int_a^b H^C(x)\,dx \le \tfrac{H^C(a)+H^C(b)}{2}. \end{aligned} \tag{31}$$

Similarly, for H^W, we obtain the following inequality

$$\begin{aligned} &H^W\left(\tfrac{a+b}{2}\right) \le \tfrac{1}{(b-a)^2}\int_a^b\int_a^b H^W(\theta x + (1-\theta)y)\,dxdy \\ &\le \tfrac{1}{b-a}\int_a^b H^W(x)\,dx \le \tfrac{H^W(a)+H^W(b)}{2}. \end{aligned} \tag{32}$$

By using (31) and (32), we have (25). □

Corollary 2.7. *Let $H : [a,b] \longrightarrow \mathcal{K}_C$ be an interval valued CW convex mapping. Then, for all $\theta \in [a,b]$, we have the following inequalities*

$$\begin{aligned} H\left(\tfrac{a+b}{2}\right) &\preceq_{CW} \tfrac{1}{(b-a)^2}\int_a^b\int_a^b H\left(\tfrac{x+y}{2}\right)dxdy \\ &\preceq_{CW} \tfrac{1}{b-a}\int_a^b H(x)\,dx \\ &\preceq_{CW} \tfrac{H(a)+H(b)}{2}. \end{aligned} \tag{33}$$

Proof. *By using $\theta = \frac{1}{2}$ in Theorem 2.6, we have inequality (33).* □

Theorem 2.8. *If $H : [a,b] \longrightarrow \mathcal{K}_C$ is CW convex and gH-differentiable, then the following inequality*

$$\frac{2}{(b-a)}\int_{(3a+b)/4}^{(a+3b)/4} H(x)\,dx \preceq_{CW} \frac{1}{(b-a)}\int_a^b H(x)\,dx, \tag{34}$$

is valid. In addition, if $(h^+ - h^-)(t)$ *is differentiable, then the following inequality holds*

$$\begin{aligned} &\tfrac{1}{b-a}\left(\int_a^b H(x)\,dx + \tfrac{1}{2}\int_a^b H(x)\,dx\right) \\ &\preceq_{CW} \tfrac{1}{2}\tfrac{H(a)+H(b)}{2} + \tfrac{2}{(b-a)}\int_{(3a+b)/4}^{(a+3b)/4} H(x)\,dx. \end{aligned} \tag{35}$$

Proof. Since F is CW convex on $[a,b]$, so H^C, H^W are also convex on $[a,b]$, Moreover, by using inequality (8) for $\theta = \frac{1}{2}$ and integrating it on $[a,b]$, for all x, y in $[a,b]$, we have

$$\begin{aligned} &\tfrac{1}{b-a}\int_a^b (h^+ + h^-)\left(\tfrac{1}{2}x + \left(\tfrac{1}{2}\right)\tfrac{a+b}{2}\right)dx \\ &\leq \tfrac{1}{2(b-a)}\int_a^b (h^+ + h^-)(x)\,dx + \left(\tfrac{1}{2}\right)(h^+ + h^-)\left(\tfrac{a+b}{2}\right). \end{aligned}$$

From (10), we have

$$\frac{1}{b-a}\int_a^b \left(h^+ + h^-\right)\left(\frac{1}{2}x + \frac{1}{2}\frac{a+b}{2}\right)dx \leq \frac{1}{b-a}\int_a^b \left(h^+ + h^-\right)(x)\,dx. \tag{36}$$

Now, consider integral of left side and put $\left(\frac{1}{2}x + \left(\frac{1}{2}\right)\frac{a+b}{2}\right) = y$, implies

$$\begin{aligned} &\tfrac{1}{b-a}\int_a^b (h^+ + h^-)\left(\tfrac{1}{2}x + \left(\tfrac{1}{2}\right)\tfrac{a+b}{2}\right)dx \\ &= \tfrac{2}{(b-a)}\int_{(3a+b)/4}^{(a+3b)/4} (h^+ + h^-)(y)\,dy \end{aligned} \tag{37}$$

By using inequalities (36) and (37), we get

$$\frac{2}{(b-a)}\int_{(3a+b)/4}^{(a+3b)/4} \left(h^+ + h^-\right)(x)\,dx \leq \frac{1}{b-a}\int_a^b \left(h^+ + h^-\right)(x)\,dx. \tag{38}$$

Similarly for H^W, we obtain

$$\frac{2}{(b-a)}\int_{(3a+b)/4}^{(a+3b)/4} \left(h^+ - h^-\right)(x)\,dx \leq \frac{1}{b-a}\int_a^b \left(h^+ - h^-\right)(x)\,dx. \tag{39}$$

Combining inequalities (38) and (39), we get (34).
Given function $(h^{+}-h^{-})$ being differentiable and convex on $[a,b]$, we get

$$\begin{aligned}&(h^{+}-h^{-})\left(\tfrac{1}{2}x+\left(\tfrac{1}{2}\right)\tfrac{a+b}{2}\right)-(h^{+}-h^{-})(x)\\&\geq\left(\tfrac{1}{2}\right)\left(\tfrac{a+b}{2}-x\right)(h^{+}-h^{-})'(x),\end{aligned}$$

$\forall x\in[a,b]$.
Multiplying by $\frac{1}{b-a}$ on both side of the above inequality and integrating it with respect to x on $[a,b]$, we get

$$\begin{aligned}&\tfrac{1}{b-a}\int_a^b(h^{+}-h^{-})\left(\tfrac{1}{2}x+\left(\tfrac{1}{2}\right)\tfrac{a+b}{2}\right)dx-\tfrac{1}{b-a}\int_a^b(h^{+}-h^{-})(x)\,dx\\&\geq\left(\tfrac{\frac{1}{2}}{b-a}\right)\int_a^b\left(\tfrac{a+b}{2}-x\right)(h^{+}-h^{-})'(x)\,dx.\end{aligned}\tag{40}$$

Now, we compute the integral

$$\begin{aligned}&\int_a^b\left(\tfrac{a+b}{2}-x\right)(h^{+}-h^{-})'(x)\,dx\\&=\tfrac{a+b}{2}\int_a^b(h^{+}-h^{-})'(x)\,dx-\int_a^b x\,(h^{+}-h^{-})'(x)\,dx\\&=\tfrac{a+b}{2}\left[(h^{+}-h^{-})(b)-(h^{+}-h^{-})(a)\right]-\left[b\,(h^{+}-h^{-})(b)-a\,(h^{+}-h^{-})(a)\right]\\&+\int_a^b(h^{+}-h^{-})(x)\,dx\\&=\int_a^b(h^{+}-h^{-})(x)\,dx-\left[\begin{array}{c}\frac{a}{2}(h^{+}-h^{-})(b)+\frac{b}{2}(h^{+}-h^{-})(b)-\frac{a}{2}(h^{+}-h^{-})(a)\\-\frac{b}{2}(h^{+}-h^{-})(a)-b\,(h^{+}-h^{-})(b)+a\,(h^{+}-h^{-})(a)\end{array}\right]\\&=\int_a^b(h^{+}-h^{-})(x)\,dx+\left[(h^{+}-h^{-})(a)\left\{a-\tfrac{a}{2}-\tfrac{b}{2}\right\}+(h^{+}-h^{-})(b)\left\{-b+\tfrac{a}{2}+\tfrac{b}{2}\right\}\right]\\&=\int_a^b(h^{+}-h^{-})(x)\,dx-(b-a)\left[\frac{(h^{+}-h^{-})(a)+(h^{+}-h^{-})(b)}{2}\right].\end{aligned}$$

Inequality (37) and (40) yields

$$\begin{aligned}&\tfrac{1}{b-a}\left[\int_a^b(h^{+}-h^{-})(x)\,dx+\tfrac{1}{2}\int_a^b(h^{+}-h^{-})(x)\,dx\right]\\&\leq\tfrac{1}{2}\left[\frac{(h^{+}-h^{-})(a)+(h^{+}-h^{-})(b)}{2}\right]+\tfrac{2}{(b-a)}\int_{(3a+b)/4}^{(a+3b)/4}(h^{+}-h^{-})(x)\,dx.\end{aligned}\tag{41}$$

The function $(h^{+}+h^{-})$ is differentiable by the Proposition 1.10. Similarly, we have

$$\frac{1}{b-a}\left[\int_a^b \left(h^{+}+h^{-}\right)(x)\,dx+\frac{1}{2}\int_a^b \left(h^{+}+h^{-}\right)(x)\,dx\right]$$
$$\leq \frac{1}{2}\left[\frac{\left(h^{+}+h^{-}\right)(a)+\left(h^{+}+h^{-}\right)(b)}{2}\right]+\frac{2}{(b-a)}\int_{(3a+b)/4}^{(a+3b)/4}\left(h^{+}+h^{-}\right)(x)\,dx. \tag{42}$$

By using inequalities (42) and (41), we obtain (35). □

Theorem 2.9. *If X be a real linear space and $A\subseteq X$, $H:X\longrightarrow\mathcal{K}_C$, then the mapping $H:X\longrightarrow\mathcal{K}_C$ is CW-convex on A if and only if $\forall$ $x,y\in A$, the mapping $G_{x,y}:[0,1]\longrightarrow\mathcal{K}_C$ defined as*

$$G_{x,y}=H\left(\theta x+(1-\theta)\,y\right)$$

is CW-convex on $[0,1]$.

Proof. *Consider H is CW-convex on A, that is, inequalities (5), (6) and (7) are satisfied for all $x,y\in A$ and $\alpha\in(0,1)$. Let $\theta_1,\theta_2\in[0,1]$ and $\lambda_1,\lambda_2\geq 0$, with $\lambda_1+\lambda_2=1$. First we have to show that the mapping $G_{x,y}:[0,1]\longrightarrow\mathcal{K}_C$ is CW convex, i.e., $G^{C}_{x,y}(\theta)$ and $G^{W}_{x,y}(\theta)$ are convex functions. More explicitly, we have to show that*

$$G^{C}_{x,y}(\theta)=\frac{1}{2}\left(h^{-}+h^{+}\right)\left(\theta x+(1-\theta)\,y\right) \tag{43}$$

and

$$G^{W}_{x,y}(\theta)=\frac{1}{2}\left(h^{-}-h^{+}\right)\left(\theta x+(1-\theta)\,y\right)$$

are convex on $[0,1]$. By using (43), it follows

$$\begin{aligned}
&G^{C}_{x,y}\left(\lambda_1\theta_1+\lambda_2\theta_2\right)\\
&=\tfrac{1}{2}\left(h^{-}+h^{+}\right)\left(\lambda_1\theta_1+\lambda_2\theta_2\right)x+\left(1-\lambda_1\theta_1-\lambda_2\theta_2\right)y\\
&=\tfrac{1}{2}\left(h^{-}+h^{+}\right)\left[\left(\lambda_1\theta_1+\lambda_2\theta_2\right)x+\left(\lambda_1+\lambda_2-\lambda_1\theta_1-\lambda_2\theta_2\right)y\right]\\
&=\tfrac{1}{2}\left(h^{-}+h^{+}\right)\left[\lambda_1\theta_1x+\lambda_1\left(1-\theta_1\right)y+\lambda_2\theta_2x+\lambda_2\left(1-\theta_2\right)y\right]\\
&=\tfrac{1}{2}\left(h^{-}+h^{+}\right)\left[\lambda_1\left(\theta_1x+\left(1-\theta_1\right)y\right)+\lambda_2\left(\theta_2x+\left(1-\theta_2\right)y\right)\right]\\
&\leq\lambda_1\tfrac{1}{2}\left(h^{-}+h^{+}\right)\left(\theta_1x+\left(1-\theta_1\right)y\right)+\lambda_2\tfrac{1}{2}\left(h^{-}+h^{+}\right)\left(\theta_2x+\left(1-\theta_2\right)y\right)\\
&\leq\lambda_1G^{C}_{x,y}\left(\theta_1\right)+\lambda_2G^{C}_{x,y}\left(\theta_2\right).
\end{aligned}$$

Hence, $G_{x,y}^{C}$ *is convex on* $[0,1]$.
Conversely, now we have to show that H *is* CW*-convex. Let* $x,y\in A$ *and* $\lambda_1,\lambda_2\geq 0,\ \lambda_1+\lambda_2=1,$ *then by (43), we have*

$$\begin{aligned}\left(h^{-}+h^{+}\right)\left(\lambda_1 x+\lambda_2 y\right) &= \left(h^{-}+h^{+}\right)\left(\lambda_1 x+\left(1-\lambda_1\right)y\right)\\ &= G_{x,y}^{C}\left(\lambda_1\right)\\ &\leq \lambda_1 G_{x,y}^{C}\left(1\right)+\lambda_2 G_{x,y}\left(0\right)\\ &= \lambda_1\left(h^{-}+h^{+}\right)\left(x\right)+\lambda_2\left(h^{-}+h^{+}\right)\left(y\right).\end{aligned}$$

We observe that H^{C} *is convex. Similarly,* H^{W} *is also convex. It completes our proof.* □

Theorem 2.10. *Suppose* X *is a real linear space and* $A\subseteq X$, $H:X\longrightarrow\mathcal{K}_C$. *If the mapping* $H:X\longrightarrow\mathcal{K}_C$ *is* CW*-convex on* A*, then* $\forall\ x,y\in A$*, the interval valued mapping* $G_{x,y}$ *on* $[0,1]$, *given by*

$$G_{x,y}\left(\theta\right)=\frac{1}{2}\left[H\left(\theta x+\left(1-\theta\right)y\right)+H\left(\left(1-\theta\right)x+\theta y\right)\right] \tag{44}$$

is CW*-convex. In addition, the following inequality is satisfied*

$$H\left(\frac{x+y}{2}\right)\preceq_{CW} G_{x,y}\left(\theta\right)\preceq_{CW}\frac{H\left(x\right)+H\left(y\right)}{2}, \tag{45}$$

for all $x,y\in A$ *and* $\theta\in[0,1]$.
Proof. *Let* $x,y\in A$, $\theta_1,\theta_2\in[0,1]$ *and* $\lambda_1,\lambda_2\geq 0,$ *with* $\lambda_1+\lambda_2=1.$ *To show that the mapping* $G_{x,y}$ *be* CW*-convex on* $[0,1]$. *Let*

$$G_{x,y}^{C}\left(\theta\right)=\frac{1}{2}\left[\left(H^{C}\right)\left(\theta x+\left(1-\theta\right)y\right)+\left(H^{C}\right)\left(\left(1-\theta\right)x+\theta y\right)\right] \tag{46}$$

Working on the same steps as in Theorem 2.9 and by using (46) it *follows,*

$$\begin{aligned}
&G^C_{x,y}(\lambda_1\theta_1+\lambda_2\theta_2)= \\
&\tfrac{1}{2}\left[H^C((\lambda_1\theta_1+\lambda_2\theta_2)x+(1-\lambda_1\theta_1-\lambda_2\theta_2)y)\right. \\
&\left.+H^C((1-\lambda_1\theta_1-\lambda_2\theta_2)x+(\lambda_1\theta_1+\lambda_2\theta_2)y)\right] \\
&=\tfrac{1}{4}\left[(h^-+h^+)\{(\lambda_1\theta_1+\lambda_2\theta_2)x+(\lambda_1+\lambda_2-\lambda_1\theta_1-\lambda_2\theta_2)y\}\right. \\
&\left.+(h^-+h^+)\{(\lambda_1+\lambda_2-\lambda_1\theta_1-\lambda_2\theta_2)x+(\lambda_1\theta_1+\lambda_2\theta_2)y\}\right] \\
&=\tfrac{1}{4}\left[(h^-+h^+)\{\lambda_1(\theta_1x+(1-\theta_1)y)+\lambda_2(\theta_2x+(1-\theta_2)y)\}\right. \\
&\left.+(h^-+h^+)\{\lambda_1((1-\theta_1)x+\theta_1y)+\lambda_2((1-\theta_2)x+\theta_2y)\}\right] \\
&\le\tfrac{1}{4}\left[\lambda_1(h^-+h^+)(\theta_1x+(1-\theta_1)y)+\lambda_2(h^-+h^+)(\theta_2x+(1-\theta_2)y)\right. \\
&\left.+\lambda_1(h^-+h^+)((1-\theta_1)x+\theta_1y)+\lambda_2(h^-+h^+)((1-\theta_2)x+\theta_2y)\right] \\
&\le\lambda_1\left[\tfrac{1}{4}\{(h^-+h^+)\theta_1x+(1-\theta_1)y+(h^-+h^+)((1-\theta_1)x+\theta_1y)\}\right] \\
&+\lambda_2\left[\tfrac{1}{4}\{(h^-+h^+)(\theta_2x+(1-\theta_2)y)+(h^-+h^+)((1-\theta_2)x+\theta_2y)\}\right] \\
&=\lambda_1G^C_{x,y}(\theta_1)+\lambda_2G^C_{x,y}(\theta_2),
\end{aligned}$$

Thus, $G^C_{x,y}$ *is convex on* $[0,1]$. *Similarly* $G^W_{x,y}$ *defined as*

$$G^W_{x,y}=\frac{1}{2}\left[H^W(\theta x+(1-\theta)y)+H^W((1-\theta)x+\theta y)\right]$$

is also convex function on $[0,1]$ *and we have (44).*
By the CW*-convexity of* H, *we can define*

$$G_{x,y}(\theta)\ge H\left(\frac{x+y}{2}\right)=H\left(\frac{1}{2}(\theta x+(1-\theta)y+(1-\theta)x+\theta y)\right)$$

and

$$\begin{aligned}
G_{x,y}(\theta) &\le \tfrac{1}{2}\left[\theta H(x)+(1-\theta)H(y)+\theta H(y)+(1-\theta)H(x)\right] \\
&= \tfrac{H(x)+H(y)}{2},
\end{aligned}$$

for all $\theta\in[0,1]$. *Hence, we obtain (45).* □

3 Conclusion

In this work, using the partial order CW on the space $\mathcal{K}_C$ of non-empty convex and compact subsets of $\mathbb{R}$, we have defined CW convex interval-valued functions. We have shown that if the interval-valued function is CW convex, then the endpoint real-valued functions used in it don't need to be convex. We established some new Hadamard-Hermite type inequalities for interval-valued functions by using CW convexity. The results are helpful for future research in the generalization of inequalities involving fractional calculus and quantum calculus for fuzzy and interval-valued functions.

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