On k-generalized ψ-Hilfer Impulsive Boundary Value Problem with Retarded and Advanced Arguments

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Abstract

This paper deals with the existence and uniqueness results for a class of impulsive boundary value problem for implicit nonlinear fractional differential equations and k-Generalized ψ-Hilfer fractional derivative involving both retarded and advanced arguments. Our results are based on some suitable fixed point theorems. Suitable illustrative examples are provided.

Key words and phrases: ψ-Hilfer fractional derivative, k-generalized ψ-Hilfer fractional derivative, impulsions, retarded arguments, advanced arguments, existence, uniqueness.
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1 Introduction

The fractional calculus has long been an attractive research topic in functional space theory due to its applicability in the modeling and scientific understanding of natural phenomena. Indeed, several applications in viscoelasticity and electrochemistry have been investigated. Non-integer derivatives of fractional order have been successfully used to generalize the fundamental laws of nature. For more details, we recommend [1–3, 8, 12, 14, 19–23]. The authors of [6, 7, 13, 15] explored the existence, stability and uniqueness of solutions for various problems with fractional differential equation and inclusions concerning retarded or advanced arguments. Recently in [10], Diaz presented the definitions of the special functions k-gamma and k-beta. Several findings and generalizations for various fractional integrals and derivatives based on the properties of the these special functions can be found in [9, 16, 17]. In [26], Sousa *et al.* introduce another so-called ψ-Hilfer fractional derivative with respect to another function and gave some important properties concerning this type of fractional operators. We direct readers to the papers [4, 5, 24, 25] and the references therein for further results based on this operator.

Recently in [22], we established existence, uniqueness and Ulam stability results to the boundary value problem with nonlinear implicit generalized Hilfer type fractional differential equation with impulses:

$$\begin{cases} \left({}^{\rho}D^{\alpha,\beta}_{t_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}D^{\alpha,\beta}_{t_k^+}u\right)(t)\right); \ t \in J_k, \ k = 0, \dots, m, \\ \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)(t_k^+) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)(t_k^-) + L_k(u(t_k^-)); k = 1, \dots, m, \\ c_1\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) + c_2\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b) = c_3, \end{cases}$$

where ${}^{\rho}D^{\alpha,\beta}_{t_k^+}, {}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}$ are the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha + \beta - \alpha\beta)$ respectively, c_1, c_2, c_3 are reals with $c_1 + c_2 \neq 0$, $J_k := (t_k, t_{k+1}]; k = 0, \dots, m$, $a = t_0 < t_1 < \dots < t_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $L_k : \mathbb{R} \to \mathbb{R}$; $k = 1, \dots, m$ are given continuous functions. The proved results rely on Banach contraction principle, Krasnoselskii and Schaefer fixed point theorems.

In keeping with the spirit of generalizing the previous results, in this paper, we establish existence and uniqueness results to the following k-generalized ψ-Hilfer problem with nonlinear implicit fractional differential equation with impulses involving both retarded and advanced arguments :

$$\left({}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{t_i^+}x\right)(t) = f\left(t, x^t(\cdot), \left({}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{t_i^+}x\right)(t)\right), \quad t \in J_i, \ i = 0, \dots, m, \tag{1}$$

$$\left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_i^+}x\right)(t_i^+) = \left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_{i-1}^+}x\right)(t_i^-) + L_i(x(t_i^-)); i = 1, \dots, m, \tag{2}$$

$$\alpha_1\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(a^+) + \alpha_2\left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_m^+}x\right)(b) = \alpha_3, \tag{3}$$

$$x(t) = \varpi(t), \quad t \in [a-\lambda, a], \ \lambda > 0, \tag{4}$$

$$x(t) = \tilde{\varpi}(t), \quad t \in \left[b, b+\tilde{\lambda}\right], \ \tilde{\lambda} > 0, \tag{5}$$

where ${}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{a+}, \mathcal{J}^{k(1-\xi),k;\psi}_{a+}$ are the k-generalized ψ-Hilfer fractional derivative of order $\vartheta \in (0,k)$ and type $r \in [0,1]$ defined in Section 2, and k-generalized ψ-fractional integral of order $k(1-\xi)$ defined in [18] respectively, where $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, $k>0$, $\varpi \in C\left([a-\lambda, a], \mathbb{R}\right)$, $\tilde{\varpi} \in C\left(\left[b, b+\tilde{\lambda}\right], \mathbb{R}\right)$, $f : [a,b] \times PC_{\xi,k;\psi}\left(\left[-\lambda, \tilde{\lambda}\right]\right) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given appropriate function specified latter, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 \neq 0$, $J_i := (t_i, t_{i+1}]; i = 0, \ldots, m$, $a = t_0 < t_1 < \ldots < t_m < t_{m+1} = b < \infty$, $u(t_i^+) = \lim\limits_{\epsilon \to 0^+} u(t_i + \epsilon)$ and $u(t_i^-) = \lim\limits_{\epsilon \to 0^-} u(t_i + \epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_i$ and $L_i : \mathbb{R} \to \mathbb{R}$; $i = 1, \ldots, m$ are given continuous functions. For each function x defined on $\left[a-\lambda, b+\tilde{\lambda}\right]$ and for any $t \in (a,b]$, we denote by x^t the element defined by

$$x^t(\tau) = x(t+\tau), \quad \tau \in \left[-\lambda, \tilde{\lambda}\right].$$

This paper has the following structure: In Section 2, some notations are introduced and we recall some preliminaries about k-generalized ψ-Hilfer fractional integral, the functions k-Gamma, k-Beta and some auxiliary results. Further, we give the definition of the k-generalized ψ-Hilfer type fractional derivative and some essential theorems and lemmas. In Section 3, we present two existence results for the problem (1)-(5) that are founded on the Banach contraction principle and Schauder fixed point theorem. Finally, in the last section, we give an example to illustrate the applicability of our main result.

2 Preliminaries

First, we present the weighted spaces, notations, definitions, and preliminary facts which are used in this paper. Let $0 < a < b < \infty$, $J = [a,b]$, $\vartheta \in (0,k)$, $r \in [0,1]$, $k > 0$ and $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in J\}.$$

Let $\mathcal{C} = C([a-\lambda, a], \mathbb{R})$ and $\tilde{\mathcal{C}} = C\left(\left[b, b+\tilde{\lambda}\right], \mathbb{R}\right)$ be the spaces endowed, respectively, with the norms

$$\|x\|_{\mathcal{C}} = \sup\{|x(t)| : t \in [a-\lambda, a]\},$$

and

$$\|x\|_{\tilde{\mathcal{C}}} = \sup\left\{|x(t)| : t \in \left[b, b+\tilde{\lambda}\right]\right\}.$$

Consider the weighted Banach space

$$C_{\xi,k;\psi}(J_i) = \left\{x : J_i \to \mathbb{R} : t \to \Psi_\xi^\psi(t, t_i)x(t) \in C([t_i, t_{i+1}], \mathbb{R})\right\},$$

where $\Psi_\xi^\psi(t,t_i) = (\psi(t)-\psi(t_i))^{1-\xi}$ and $i = 0,\dots,m$. And, we consider

$$PC_{\xi,k;\psi}(J) = \Big\{ x:(a,b]\to\mathbb{R} : x\in C_{\xi,k;\psi}(J_i);\, i=0,\dots,m, \text{and there exist}$$
$$x(t_i^-) \text{ and } \left(\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi}x\right)(t_i^+);\, i=1,\dots,m, \text{with } x(t_i^-)=x(t_i)\Big\},$$

with the norm

$$\|x\|_{PC_{\xi,k;\psi}} = \max_{i=0,\dots,m}\left\{\sup_{t\in[t_i,t_{i+1}]}\left|\Psi_\xi^\psi(t,t_i)x(t)\right|\right\}.$$

Consider the weighted Banach space

$$PC_{\xi,k;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right) = \Big\{x:\left[-\lambda,\tilde{\lambda}\right]\to\mathbb{R} : \tau\to\Psi_\xi^\psi(t,t_i)x(\tau)\in C([\tau_i,\tau_{i+1}],\mathbb{R});\, i=0,\dots,m,$$
$$\text{and there exist } x(\tau_i^-) \text{ and } \left(\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi}x\right)(\tau_i^+);\, i=1,\dots,m,$$
$$\text{with } x(\tau_i^-)=x(\tau_i) \text{ and } \tau_i=t_i-t, \text{ for each } t\in J_i\Big\},$$

with the norm

$$\|x^t\|_{[-\lambda,\tilde{\lambda}]} = \max\left\{\max_{i=0,\dots,m}\left\{\sup_{\tau\in[\tau_i,\tau_{i+1}]}\left|\Psi_\xi^\psi(t,t_i)x^t(\tau)\right|\right\}, \sup_{\tau\in[-\lambda,0]}|x^a(\tau)|, \sup_{\tau\in\left[0,\tilde{\lambda}\right]}\left|x^b(\tau)\right|\right\}.$$

Next, we consider the Banach space

$$\mathbb{F} = \left\{x:\left[a-\lambda,b+\tilde{\lambda}\right]\to\mathbb{R} : x|_{[a-\lambda,a]}\in\mathcal{C},\, x|_{\left[b,b+\tilde{\lambda}\right]}\in\tilde{\mathcal{C}} \text{ and } x|_{(a,b]}\in PC_{\xi,k;\psi}(J)\right\}$$

with the norm

$$\|x\|_{\mathbb{F}} = \max\left\{\|x\|_{\mathcal{C}}, \|x\|_{\tilde{\mathcal{C}}}, \|x\|_{PC_{\xi,k;\psi}}\right\}.$$

Consider the space $X_\psi^p(a,b)$, $(c\in\mathbb{R},\ 1\le p\le\infty)$ of those real-valued Lebesgue measurable functions g on $[a,b]$ for which $\|g\|_{X_\psi^p}<\infty$, where the norm is defined by

$$\|g\|_{X_\psi^p} = \left(\int_a^b \psi'(t)|g(t)|^p dt\right)^{\frac{1}{p}},$$

where ψ is an increasing and positive function on $[a,b]$ such that ψ' is continuous on $[a,b]$ with $\psi(0)=0$. In particular, when $\psi(x)=x$, the space $X_\psi^p(a,b)$ coincides with the $L_p(a,b)$ space.

Definition 2.1. *([10]) The k-gamma function is defined by*

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \alpha > 0.$$

When $k \to 1$ *then* $\Gamma(\alpha) = \Gamma_k(\alpha)$*, we have also some useful following relations* $\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right)$*,* $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$ *and* $\Gamma_k(k) = \Gamma(1) = 1$*. Furthermore k-beta function is defined as follows*

$$B_k(\alpha, \beta) = \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1} dt$$

so that $B_k(\alpha, \beta) = \frac{1}{k}B\left(\frac{\alpha}{k}, \frac{\beta}{k}\right)$ *and* $B_k(\alpha, \beta) = \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}$.

Now, we give all the definitions to the different fractional operators used throughout this paper.

Definition 2.2. *([18]) (k-Generalized ψ-fractional Integral) Let* $g \in X^p_\psi(a, b)$ *and* $[a, b]$ *be a finite or infinite interval on the real axis* $\mathbb{R} = (-\infty, \infty)$*,* $\psi(t) > 0$ *be an increasing function on* $(a, b]$ *and* $\psi'(t) > 0$ *be continuous on* (a, b) *and* $\vartheta > 0$*. The generalized k-fractional integral operators of a function f (left-sided and right-sided) of order* ϑ *are defined by*

$$\mathcal{J}^{\vartheta,k;\psi}_{a+} g(t) = \int_a^t \bar{\Psi}^{k,\psi}_\vartheta(t, s)\psi'(s)g(s)ds,$$
$$\mathcal{J}^{\vartheta,k;\psi}_{b-} g(t) = \int_t^b \bar{\Psi}^{k,\psi}_\vartheta(s, t)\psi'(s)g(s)ds,$$

with $k > 0$ *and* $\bar{\Psi}^{k,\psi}_\vartheta(t, s) = \dfrac{(\psi(t) - \psi(s))^{\frac{\vartheta}{k}-1}}{k\Gamma_k(\vartheta)}$.

Also in [17], Nápoles Valdés gave a more generalized fractional integral operators defined by

$$\mathcal{J}^{\vartheta,k;\psi}_{G,a+} g(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\psi'(s)g(s)ds}{G(\psi(t) - \psi(s), \frac{\vartheta}{k})},$$
$$\mathcal{J}^{\vartheta,k;\psi}_{G,b-} g(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_t^b \frac{\psi'(s)g(s)ds}{G(\psi(s) - \psi(t), \frac{\vartheta}{k})},$$

where $G(z, \vartheta) \in AC[a, b]$; the space of absolutely continuous functions defined on $[a, b]$.

Theorem 2.3. *([17]) Let* $g : [a, b] \to \mathbb{R}$ *be an integrable function, and take* $\vartheta > 0$ *and* $k > 0$*. Then* $\mathcal{J}^{\vartheta,k;\psi}_{G,a+} g$ *exists for all* $t \in [a, b]$.

Theorem 2.4. *([17]) Let* $g \in X^p_\psi(a, b)$ *and take* $\vartheta > 0$ *and* $k > 0$*. Then* $\mathcal{J}^{\vartheta,k;\psi}_{G,a+} g \in C([a, b], \mathbb{R})$.

Lemma 2.5. *Let $\vartheta > 0$, $r > 0$ and $k > 0$. Then, we have the following semigroup property given by*

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}\mathcal{J}_{a+}^{r,k;\psi}f(t) = \mathcal{J}_{a+}^{\vartheta+r,k;\psi}f(t) = \mathcal{J}_{a+}^{r,k;\psi}\mathcal{J}_{a+}^{\vartheta,k;\psi}f(t)$$

and

$$\mathcal{J}_{b-}^{\vartheta,k;\psi}\mathcal{J}_{b-}^{r,k;\psi}f(t) = \mathcal{J}_{b-}^{\vartheta+r,k;\psi}f(t) = \mathcal{J}_{b-}^{r,k;\psi}\mathcal{J}_{b-}^{\vartheta,k;\psi}f(t)$$

Proof. By Lemma 1 in [26] and the property of k-gamma function, for $\vartheta > 0$, $r > 0$ and $k > 0$, we get

$$\begin{aligned}
\mathcal{J}_{a+}^{\vartheta,k;\psi}\mathcal{J}_{a+}^{r,k;\psi}f(t) &= \frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I_{a+}^{\frac{\vartheta}{k};\psi}I_{a+}^{\frac{r}{k};\psi}f(t)\\
&= \frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2k^{\frac{\vartheta}{k}-1}\Gamma(\frac{\vartheta}{k})k^{\frac{r}{k}-1}\Gamma(\frac{r}{k})}I_{a+}^{\frac{\vartheta}{k};\psi}I_{a+}^{\frac{r}{k};\psi}f(t)\\
&= \frac{1}{k^{\frac{\vartheta+r}{k}}}I_{a+}^{\frac{\vartheta+r}{k};\psi}f(t)\\
&= \mathcal{J}_{a+}^{\vartheta+r,k;\psi}f(t),
\end{aligned}$$

where $I_{a+}^{\vartheta;\psi}$ is the fractional integral defined in [26], we have also,

$$\begin{aligned}
\mathcal{J}_{a+}^{\vartheta,k;\psi}\mathcal{J}_{a+}^{r,k;\psi}f(t) &= \frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I_{a+}^{\frac{\vartheta}{k};\psi}I_{a+}^{\frac{r}{k};\psi}f(t)\\
&= \frac{\Gamma(\frac{\vartheta}{k})\Gamma(\frac{r}{k})}{k^2\Gamma_k(\vartheta)\Gamma_k(r)}I_{a+}^{\frac{r}{k};\psi}I_{a+}^{\frac{\vartheta}{k};\psi}f(t)\\
&= \mathcal{J}_{a+}^{r,k;\psi}\mathcal{J}_{a+}^{\vartheta,k;\psi}f(t).
\end{aligned}$$

□

Lemma 2.6. *Let $\vartheta, r > 0$ and $k > 0$. Then, we have*

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(t,a) = \bar{\Psi}_{\vartheta+r}^{k,\psi}(t,a)$$

and

$$\mathcal{J}_{b-}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(b,t) = \bar{\Psi}_{\vartheta+r}^{k,\psi}(b,t).$$

Proof. By Definition 2.2 and using the change of variable

$$\mu = \frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}, \quad t > a,$$

we get

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(t,a) = \int_a^t \bar{\Psi}_\vartheta^{k,\psi}(t,s)\psi'(s)\bar{\Psi}_r^{k,\psi}(s,a)ds$$

$$= \int_a^t \bar{\Psi}_\vartheta^{k,\psi}(t,a)\left[1 - \frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}\right]^{\frac{\vartheta}{k}-1} \psi'(s)\bar{\Psi}_r^{k,\psi}(s,a)ds$$
$$= \bar{\Psi}_\vartheta^{k,\psi}(t,a)\bar{\Psi}_r^{k,\psi}(t,a)\int_0^1 [1-\mu]^{\frac{\vartheta}{k}-1}\mu^{\frac{r}{k}-1}d\mu.$$

Using the Definition 2.1 of k-beta function and the relation with gamma function, we have

$$\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_r^{k,\psi}(t,a) = \bar{\Psi}_{\vartheta+r}^{k,\psi}(t,a).$$

□

Theorem 2.7. *Let* $0 < a < b < \infty, \vartheta > 0, 0 \le \xi < 1$, $k > 0$ *and* $x \in C_{\xi,k;\psi}(J)$. *If* $\dfrac{\vartheta}{k} > 1-\xi$, *then*

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(a) = \lim_{t\to a^+}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) = 0.$$

Proof. $x \in C_{\xi,k;\psi}(J)$ means that $\Psi_\xi^\psi(t,a)x(t) \in C(J,\mathbb{R})$, then there exists a positive constant R such that for $t \in (a,b]$ we have

$$|\Psi_\xi^\psi(t,a)x(t)| < R,$$

thus,

$$|x(t)| < R\Gamma_k(k\xi)|\bar{\Psi}_{k\xi}^{k,\psi}(t,a)|. \tag{6}$$

Now, we apply the operator $\mathcal{J}_{a+}^{\vartheta,k;\psi}(\cdot)$ on both sides of Equation (6) and using Lemma 2.6, so that we have

$$\left|\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t)\right| < R\Gamma_k(k\xi)\left|\mathcal{J}_{a+}^{\vartheta,k;\psi}\bar{\Psi}_{k\xi}^{k,\psi}(t,a)\right|$$
$$= R\Gamma_k(k\xi)\bar{\Psi}_{\vartheta+k\xi}^{k,\psi}(t,a).$$

Then, we have the right-hand side $\to 0$ as $x \to a$, and

$$\lim_{t\to a^+}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) = \left(\mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(a) = 0.$$

□

We are now able to define the k-generalized ψ-Hilfer derivative as follows.

Definition 2.8. *(k-Generalized ψ-Hilfer Derivative) Let* $n-1 < \dfrac{\vartheta}{k} \le n$ *with* $n \in \mathbb{N}$, $J = [a,b]$ *an interval such that* $-\infty \le a < b \le \infty$ *and* $g,\psi \in C^n([a,b],\mathbb{R})$ *two functions such that* ψ *is increasing and* $\psi'(t) \neq 0$, *for all* $t \in J$. *The k-generalized*

ψ-Hilfer fractional derivatives (left-sided and right-sided) ${}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}(\cdot)$ and ${}^H_k\mathcal{D}^{\vartheta,r;\psi}_{b-}(\cdot)$ of a function g of order ϑ and type $0 \leq r \leq 1$, with $k > 0$ are defined by

$$\begin{aligned}{}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+} g\,(t) &= \left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{a+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\left(k^n\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} g\right)\right)(t)\\ &= \left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{a+}\delta^n_\psi\left(k^n\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{a+} g\right)\right)(t)\end{aligned}$$

and

$$\begin{aligned}{}^H_k\mathcal{D}^{\vartheta,r;\psi}_{b-} g\,(t) &= \left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{b-}\left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\left(k^n\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{b-} g\right)\right)(t)\\ &= \left(\mathcal{J}^{r(kn-\vartheta),k;\psi}_{b-}(-1)^n\delta^n_\psi\left(k^n\mathcal{J}^{(1-r)(kn-\vartheta),k;\psi}_{b-} g\right)\right)(t)\,,\end{aligned}$$

where $\delta^n_\psi = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n$.

Lemma 2.9. *Let $t > a$, $\vartheta > 0, 0 \leq r \leq 1, k > 0$. Then for $0 < \xi < 1; \xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, we have*

$$\left[{}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\left(\Psi^\psi_\xi(s,a)\right)^{-1}\right](t) = 0.$$

Proof. From Definitions 2.2 and 2.8, we have

$$\mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+} k\left(\Psi^\psi_\xi(t,a)\right)^{-1} = \int_a^t k\bar{\Psi}^{k,\psi}_{kX}(t,s)\left(\Psi^\psi_\xi(s,a)\right)^{-1}\psi'(s)ds,$$

where $X = \frac{1}{k}(1-r)(k-\vartheta)$. Now, we make the change of the variable by $\mu = \frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}$ to obtain

$$\mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+} k\left(\Psi^\psi_\xi(t,a)\right)^{-1} = \frac{k\left(\Psi^\psi_{\xi+X}(t,a)\right)^{-1}}{\Gamma_k(kX)}\left[\frac{1}{k}\int_0^1(1-\mu)^{X-1}\mu^{\xi-1}d\mu\right],$$

then, by the definition of k-beta function

$$B_k(\alpha,\beta) = \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}dt = \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)},$$

we have

$$\mathcal{J}^{(1-r)(k-\vartheta),k;\psi}_{a+} k\left(\Psi^\psi_\xi(t,a)\right)^{-1} = \frac{k\Gamma_k(k\xi)}{\Gamma_k(k(X+\xi))} = k\Gamma_k(k\xi),$$

then, we have

$$\delta_\psi^1\left(\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi} k\left(\Psi_\xi^\psi(t,a)\right)^{-1}\right)=0.$$

□

Theorem 2.10. *If $f\in C^n_{\xi,k;\psi}[a,b], n-1<\frac{\vartheta}{k}<n,\ 0\le r\le 1,$ where $n\in\mathbb{N}$ and $k>0$, then*

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}^H_k\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)=f(t)-\sum_{i=1}^{n}\frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i-n}\Gamma_k(k(\xi-i+1))}\left\{\delta_\psi^{n-i}\left(\mathcal{J}_{a+}^{k(n-\xi),k;\psi}f(a)\right)\right\},$$

where

$$\xi=\frac{1}{k}\left(r(kn-\vartheta)+\vartheta\right).$$

In particular, if $n=1$, we have

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}^H_k\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)=f(t)-\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(r(k-\vartheta)+\vartheta)}\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi}f(a).$$

Proof. From Definition 2.8 and Lemma 2.5, we have

$$\begin{aligned}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}^H_k\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)&=\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ \mathcal{J}_{a+}^{r(kn-\vartheta),k;\psi}\delta_\psi^n\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f\right)\right)(t)\\&=\left(\mathcal{J}_{a+}^{r(kn-\vartheta)+\vartheta,k;\psi}\delta_\psi^n\left(k^nI_{a+}^{(1-r)(kn-\vartheta),k;\psi}f\right)\right)(t)\\&=\int_a^t\bar{\Psi}_{k\xi}^{k,\psi}(t,s)\psi'(s)\left\{\delta_\psi^n\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)\right\}ds\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}&\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}^H_k\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)\\&=\frac{-(\psi(t)-\psi(a))^{\xi-1}}{k\Gamma_k(k\xi)}\left\{\delta_\psi^{n-1}\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\&+\frac{\xi-1}{k\Gamma_k(k\xi)}\int_a^t\frac{\psi'(s)}{(\psi(t)-\psi(s))^{2-\xi}}\left\{\delta_\psi^{n-1}\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)\right\}ds\end{aligned}$$

Using the propriety of the functions gamma and k-gamma, we get

$$\begin{aligned}&\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}^H_k\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t)\\&=\frac{-(\psi(t)-\psi(a))^{\xi-1}}{k^\xi\Gamma(\xi)}\left\{\delta_\psi^{n-1}\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\end{aligned}$$

$$+\frac{1}{k^{\xi}\Gamma(\xi-1)}\int_a^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{2-\xi}}\left\{\delta_\psi^{n-1}\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)\right\}ds.$$

So, with integrating by parts n times, we obtain

$$\begin{aligned}\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t) &= -\sum_{i=1}^{n}\frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{\xi}\Gamma(\xi-i+1)}\left\{\delta_\psi^{n-i}\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\ &+\frac{1}{k^{\xi-n}\Gamma(\xi-n)}\int_a^t\frac{\psi'(s)}{(\psi(t)-\psi(s))^{n+1-\xi}}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)ds,\\ &= -\sum_{i=1}^{n}\frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i}\Gamma_k(k(\xi-i+1))}\left\{\delta_\psi^{n-i}\left(k^n\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\ &+\frac{1}{k\Gamma_k(k(\xi-n))}\int_a^t\frac{\psi'(s)}{(\psi(t)-\psi(s))^{n+1-\xi}}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(s)\right)ds,\\ &= -\sum_{i=1}^{n}\frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i-n}\Gamma_k(k(\xi-i+1))}\left\{\delta_\psi^{n-i}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}\\ &+\mathcal{J}_{a+}^{k(\xi-n),k;\psi}I_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(t),\end{aligned}$$

then by using Lemma 2.5, we get

$$\left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\ {}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}f\right)(t) = f(t)-\sum_{i=1}^{n}\frac{(\psi(t)-\psi(a))^{\xi-i}}{k^{i-n}\Gamma_k(k(\xi-i+1))}\left\{\delta_\psi^{n-i}\left(\mathcal{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi}f(a)\right)\right\}.$$

□

Lemma 2.11. *Let $\vartheta>0, 0\le r\le 1$, and $x\in C^1_{\xi,k;\psi}(J)$, where $k>0$, then for $t\in(a,b]$, we have*

$$\left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) = x(t).$$

Proof. We have from Definition 2.8 and Lemma 2.5 that $\xi=\frac{1}{k}(r(k-\vartheta)+\vartheta)$

$$\begin{aligned}\left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t) &= \left(\mathcal{J}_{a+}^{r(k-\vartheta),k;\psi}\delta_\psi^1\left(k\mathcal{J}_{a+}^{(1-r)(k-\vartheta),k;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)\right)(t)\\ &= \left(\mathcal{J}_{a+}^{k\xi-\vartheta,k;\psi}\delta_\psi^1\left(k\mathcal{J}_{a+}^{(1-r)(k-\vartheta)+\vartheta,k;\psi}x\right)\right)(t)\\ &= \left(\mathcal{J}_{a+}^{k\xi-\vartheta,k;\psi}\delta_\psi^1\left(k\mathcal{J}_{a+}^{k-k\xi+\vartheta,k;\psi}x\right)\right)(t),\end{aligned}$$

then, we obtain

$$\left({}_k^H\mathcal{D}_{a+}^{\vartheta,r;\psi}\ \mathcal{J}_{a+}^{\vartheta,k;\psi}x\right)(t)$$

$$= \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)} \int_a^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}} \delta_\psi^1 \left[\int_a^s \frac{\psi'(\tau)x(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}}\right] ds. \tag{7}$$

On other hand by integrating by parts, we have

$$\int_a^s \frac{\psi'(\tau)x(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}} = \frac{1}{1-\xi+\frac{\vartheta}{k}}\left[x(a)\left(\psi(s)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}} + \int_a^s \frac{x'(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-1-\frac{\vartheta}{k}}}\right],$$

then, by applying δ_ψ^1 we get

$$\delta_\psi^1 \int_a^s \frac{\psi'(\tau)x(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}} = x(a)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}} + \int_a^s \frac{x'(\tau)d\tau}{(\psi(s)-\psi(\tau))^{\xi-\frac{\vartheta}{k}}}. \tag{8}$$

Now, replacing (8) into Equation (7), and by Dirichlet's formula and the properties of k-gamma function, we get

$$\begin{aligned}
&\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t)\\
&= \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left[\int_a^t \frac{x(a)\psi'(s)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}} ds}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}}\right.\\
&\left. + \int_a^t x'(t)dt \int_s^t \frac{\psi'(s)d\tau}{(\psi(t)-\psi(s))^{1-\xi+\frac{\vartheta}{k}}\left(\psi(s)-\psi(\tau)\right)^{\xi-\frac{\vartheta}{k}}}\right].
\end{aligned}$$

Making the following change of variables $\mu = \dfrac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}$ in the integral from a to t and similarly changing the variable in the integral from s to t, then we have

$$\begin{aligned}
&\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t)\\
&= \frac{1}{k\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left[\int_a^t x(a)\psi'(s)\left(\psi(s)-\psi(a)\right)^{-\xi+\frac{\vartheta}{k}}\left(\psi(t)-\psi(s)\right)^{\xi-\frac{\vartheta}{k}-1} ds\right.\\
&\left. + \int_a^t x'(t)dt\int_s^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\xi-\frac{\vartheta}{k}-1}\left(\psi(s)-\psi(\tau)\right)^{-\xi+\frac{\vartheta}{k}} d\tau\right]\\
&= \frac{1}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left[\frac{1}{k}\int_0^1 \mu^{-\xi+\frac{\vartheta}{k}}(1-\mu)^{\xi-\frac{\vartheta}{k}-1}d\mu\right]\left(x(a)+\int_a^t x'(t)dt\right)\\
&= \frac{1}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left[\frac{1}{k}\int_0^1 \mu^{(1-(\xi-\frac{\vartheta}{k}))-1}(1-\mu)^{\xi-\frac{\vartheta}{k}-1}d\mu\right]\left(x(a)+\int_a^t x'(t)dt\right),
\end{aligned}$$

then by the definition of k-beta function, we obtain

$$\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}\ \mathcal{J}^{\vartheta,k;\psi}_{a+}x\right)(t) = \frac{[\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)]}{\Gamma_k(k\xi-\vartheta)\Gamma_k(k(1-\xi)+\vartheta)}\left(x(a)+\int_a^t x'(t)dt\right)$$

$$= x(a) + \int_a^t x'(t)dt$$
$$= x(t).$$

□

Theorem 2.12. *Let the function* $\varphi(\cdot) \in C(J, \mathbb{R})$*. Then* $x \in C_{\xi,k;\psi}(J_i)$ *is a solution of the differential equation:*

$$\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{t_i^+} x\right)(t) = \varphi(t), \quad t \in J_i, \ i = 0, \dots, m, \ 0 < \vartheta < k, \ 0 \leq r \leq 1, \tag{9}$$

if and only if x *satisfies the following Volterra integral equation:*

$$x(t) = \frac{\mathcal{J}^{k(1-\xi),k;\psi}_{t_i^+} x(t_i)}{\Psi^\psi_\xi(t, t_i)\Gamma_k(k\xi)} + \left(\mathcal{J}^{\vartheta,k;\psi}_{t_i^+}\varphi\right)(t), \tag{10}$$

where $\xi = \dfrac{r(k-\vartheta)+\vartheta}{k}$, $k > 0$.

Proof. By applying the fractional integral operator $\mathcal{J}^{\vartheta,k;\psi}_{t_i^+}(\cdot)$ on both sides of the fractional equation (9) and using Theorem 2.10, we obtain the equation (10).

Now, applying the fractional derivative operator ${}^H_k\mathcal{D}^{\vartheta,r;\psi}_{t_i^+}(\cdot)$ on both sides of the fractional equation (10), then we get

$$\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{t_i^+} x\right)(t) = {}^H_k\mathcal{D}^{\vartheta,r;\psi}_{t_i^+}\left(\frac{\mathcal{J}^{k(1-\xi),k;\psi}_{t_i^+} x(t_i)}{\Psi^\psi_\xi(t, t_i)\Gamma_k(k\xi)}\right) + \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{t_i^+}\mathcal{J}^{\vartheta,k;\psi}_{t_i^+}\varphi\right)(t).$$

Using the Lemma 2.9 and Lemma 2.11, we obtain equation (9). □

3 Existence of Solutions

We consider the following fractional differential equation

$$\left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{t_i^+} x\right)(t) = \varphi(t), \quad t \in J_i, \ i = 0, \dots, m, \tag{11}$$

where $0 < \vartheta < k, 0 \leq r \leq 1$, with the conditions

$$\left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_i^+} x\right)(t_i^+) = \left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_{i-1}^+} x\right)(t_i^-) + L_i(x(t_i^-)); i = 1, \dots, m, \tag{12}$$

$$\alpha_1\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+} x\right)(a^+) + \alpha_2\left(\mathcal{J}^{k(1-\xi),k;\psi}_{t_m^+} x\right)(b) = \alpha_3, \tag{13}$$

$$x(t) = \varpi(t), \quad t \in [a-\lambda, a], \ \lambda > 0, \tag{14}$$

$$x(t) = \tilde{\varpi}(t), \quad t \in \left[b, b+\tilde{\lambda}\right], \ \tilde{\lambda} > 0, \tag{15}$$

where $\xi = \dfrac{r(k-\vartheta)+\vartheta}{k}$, $k > 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 \neq 0$ and where $\varphi(\cdot) \in C(J, \mathbb{R})$, $\varpi(\cdot) \in \mathcal{C}$ and $\tilde{\varpi}(\cdot) \in \tilde{\mathcal{C}}$.

The following theorem shows that the problem (11)-(15) have a unique solution.

Theorem 3.1. *The function $x(\cdot)$ satisfies (11)-(15) if and only if it satisfies*

$$x(t) = \begin{cases} \dfrac{1}{\Gamma_k(k\xi)\Psi_\xi^\psi(t,a)}\left[\dfrac{\alpha_3}{\alpha_1+\alpha_2} - \dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m} L_j(x(t_j^-)) \right. \\ \left. -\dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\right] + \left(\mathcal{J}_{a+}^{\vartheta,k;\psi}\varphi\right)(t), \quad \text{if } t \in J_0, \\ \\ \dfrac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\left[\dfrac{\alpha_3}{\alpha_1+\alpha_2} - \dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m} L_j(x(t_j^-)) \right. \\ -\dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j) + \displaystyle\sum_{j=1}^{i}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j) \\ \left. +\displaystyle\sum_{j=1}^{i} L_j(x(t_j^-))\right] + \left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi\right)(t), \quad t \in J_i; i = 1, \ldots, m, \\ \\ \varpi(t), \quad t \in [a-\lambda, a], \\ \tilde{\varpi}(t), \quad t \in \left[b, b+\tilde{\lambda}\right]. \end{cases} \tag{16}$$

Proof. Assume x satisfies (11)-(15). If $t \in J_0$, then

$$\left({}_k^H\mathcal{D}_{a^+}^{\vartheta,r;\psi}x\right)(t) = \varphi(t),$$

Theorem 2.12 implies that the solution can be written as

$$x(t) = \frac{\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x(a)}{\Psi_\xi^\psi(t,a)\Gamma_k(k\xi)} + \left(\mathcal{J}_{a^+}^{\vartheta,k;\psi}\varphi\right)(t). \tag{17}$$

If $t \in J_1$, then Theorem 2.12 implies

$$x(t) \quad = \quad \frac{\mathcal{J}_{t_1^+}^{k(1-\xi),k;\psi}x(t_1)}{\Psi_\xi^\psi(t,t_1)\Gamma_k(k\xi)} + \left(\mathcal{J}_{t_1^+}^{\vartheta,k;\psi}\varphi\right)(t)$$

$$
\begin{aligned}
&= \frac{\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(t_1^-) + L_1(x(t_1^-))}{\Psi_\xi^\psi(t,t_1)\Gamma_k(k\xi)} + \left(\mathcal{J}_{t_1^+}^{\vartheta,k;\psi}\varphi\right)(t)\\
&= \frac{\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a) + \left(\mathcal{J}_{a^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_1) + L_1(x(t_1^-))}{\Psi_\xi^\psi(t,t_1)\Gamma_k(k\xi)} + \left(\mathcal{J}_{t_1^+}^{\vartheta,k;\psi}\varphi\right)(t).
\end{aligned}
$$

If $t \in J_2$, then Theorem 2.12 implies

$$
\begin{aligned}
x(t) &= \frac{\mathcal{J}_{t_2^+}^{k(1-\xi),k;\psi}x(t_2)}{\Psi_\xi^\psi(t,t_2)\Gamma_k(k\xi)} + \left(\mathcal{J}_{t_2^+}^{\vartheta,k;\psi}\varphi\right)(t)\\
&= \frac{\left(\mathcal{J}_{t_1^+}^{k(1-\xi),k;\psi}x\right)(t_2^-) + L_2(x(t_2^-))}{\Psi_\xi^\psi(t,t_2)\Gamma_k(k\xi)} + \left(\mathcal{J}_{t_2^+}^{\vartheta,k;\psi}\varphi\right)(t)\\
&= \frac{1}{\Psi_\xi^\psi(t,t_2)\Gamma_k(k\xi)}\Bigg[\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a) + \left(\mathcal{J}_{t_1^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_2) + L_1(x(t_1^-))\\
&\quad + \left(\mathcal{J}_{a^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_1) + L_2(x(t_2^-))\Bigg] + \left(\mathcal{J}_{t_2^+}^{\vartheta,k;\psi}\varphi\right)(t).
\end{aligned}
$$

Repeating the process in this way, the solution $x(t)$ for $t \in J_i$,$i = 1, \ldots, m$, can be written as

$$
\begin{aligned}
x(t) &= \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\Bigg[\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a) + \sum_{j=1}^{i} L_j(x(t_j^-))\\
&+ \sum_{j=1}^{i}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\Bigg] + \left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi\right)(t).
\end{aligned}
\tag{18}
$$

Applying $\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi}$ on both sides of (18), using Lemma 2.6 and taking $t = b$, we obtain

$$
\begin{aligned}
\left(\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi}x\right)(b) &= \left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a) + \sum_{j=1}^{m} L_j(x(t_j^-))\\
&\quad + \sum_{j=1}^{m}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j) + \left(\mathcal{J}_{t_m^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(b).
\end{aligned}
\tag{19}
$$

Multiplying both sides of (19) by α_2 and using condition (13), we obtain

$$
\begin{aligned}
\alpha_3 - \alpha_1\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a) &= \alpha_2\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a) + \alpha_2\sum_{j=1}^{m} L_j(x(t_j^-))\\
&\quad + \alpha_2\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j),
\end{aligned}
$$

which implies that

$$\begin{aligned}&\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a)\\ =&\ \frac{\alpha_3}{\alpha_1+\alpha_2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m}L_j(x(t_j^-))-\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j).\end{aligned} \tag{20}$$

Substituting (20) into (18) and (17) we obtain (16).

Reciprocally, applying $\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi}$ on both sides of (16) and using Lemma 2.6 and Lemma 2.5, we get

$$\left(\mathcal{J}_{t_i^+}^{k(1-\xi),k;\psi}x\right)(t)=\begin{cases}\dfrac{\alpha_3}{\alpha_1+\alpha_2}-\dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\\ -\dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m}L_j(x(t_j^-))+\left(\mathcal{J}_{a+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t), \quad \text{if } t\in J_0,\\ \\ \dfrac{\alpha_3}{\alpha_1+\alpha_2}-\dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m}L_j(x(t_j^-))+\sum_{j=1}^{i}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\\ -\dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)+\sum_{j=1}^{i}L_j(x(t_j^-))\\ +\left(\mathcal{J}_{t_i^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t), \quad t\in J_i;\, i=1,\ldots,m.\end{cases} \tag{21}$$

Next, taking the limit $t\to a^+$ of (21) and using Theorem 2.7, with $k(1-\xi)<k(1-\xi)+\vartheta$, we obtain

$$\begin{aligned}\left(\mathcal{J}_{a^+}^{k(1-\xi),k;\psi}x\right)(a^+) &= \frac{\alpha_3}{\alpha_1+\alpha_2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\\ &\quad -\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{j=1}^{m}L_j(x(t_j^-)).\end{aligned} \tag{22}$$

Now, taking $t=b$ in (21), we get

$$\begin{aligned}&\left(\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi}x\right)(b)\\ =&\ \frac{\alpha_3}{\alpha_1+\alpha_2}+\left(1-\frac{\alpha_2}{\alpha_1+\alpha_2}\right)\left(\sum_{j=1}^{m}L_j(x(t_j^-))+\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)\right).\end{aligned} \tag{23}$$

From (22) and (23), we find that

$$\alpha_1\left(\mathcal{J}_{a+}^{k(1-\xi),k;\psi}x\right)(a^+)+\alpha_2\left(\mathcal{J}_{t_m^+}^{k(1-\xi),k;\psi}x\right)(b)=\alpha_3,$$

which shows that the boundary condition (13) is satisfied. Next, apply operator ${}_{k}^{H}\mathcal{D}_{t_i^+}^{\vartheta,r;\psi}(\cdot)$ on both sides of (16), where $i=0,\ldots,m$. Then, from Lemma 2.9 and Lemma 2.11 we obtain equation (11). Also, we can easily show that x satisfies the equations (12), (14) and (15). This completes the proof. $\square$

As a consequence of Theorem 3.1, we have the following result

Lemma 3.2. *Let* $\xi=\dfrac{r(k-\vartheta)+\vartheta}{k}$ *where* $0<\vartheta<k$ *and* $0\le r\le 1$, *let* $f: J\times PC_{\xi,k;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right)\times\mathbb{R}\to\mathbb{R}$ *be a continuous function,* $\varpi(\cdot)\in\mathcal{C}$ *and* $\tilde{\varpi}(\cdot)\in\tilde{\mathcal{C}}$, *then* $x\in\mathbb{F}$ *satisfies the problem* (1)-(5) *if and only if* x *is the fixed point of the operator* $\mathcal{T}:\mathbb{F}\to\mathbb{F}$ *defined by*

$$
(\mathcal{T}x)(t)=\begin{cases}\dfrac{1}{\Psi_{\xi}^{\psi}(t,t_i)\Gamma_k(k\xi)}\left[\dfrac{\alpha_3}{\alpha_1+\alpha_2}-\dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m}L_j(x(t_j^-))\right.\\ -\dfrac{\alpha_2}{\alpha_1+\alpha_2}\displaystyle\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_j)+\sum_{a<t_i<t}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\varphi\right)(t_i)\\ \left.+\displaystyle\sum_{a<t_i<t}L_i(x(t_i^-))\right]+\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\varphi\right)(t), \quad t\in J_i;\, i=0,\ldots,m,\\ \\ \varpi(t), \quad t\in[a-\lambda,a],\\ \\ \tilde{\varpi}(t), \quad t\in\left[b,b+\tilde{\lambda}\right].\end{cases}
\tag{24}
$$

where φ *be a function satisfying the functional equation*

$$\varphi(t)=f(t,x^t(\cdot),\varphi(t)).$$

By Theorem 2.4, *we have* $\mathcal{T}x\in\mathbb{F}$.

The following hypotheses will be used in the sequel :

($Ax1$) The function $f: J\times PC_{\xi,k;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right)\times\mathbb{R}\to\mathbb{R}$ is continuous.

($Ax2$) There exist constants $\zeta_1>0$ and $0<\zeta_2<1$ such that

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\le\zeta_1\|x_1-x_2\|_{\left[-\lambda,\tilde{\lambda}\right]}+\zeta_2|y_1-y_2|$$

for any $x_1,x_2\in PC_{\xi,k;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right)$, $y_1,y_2\in\mathbb{R}$ and $t\in(a,b]$.

(*Ax3*) There exists a constant $\ell_1 > 0$ such that

$$|L_i(y_1) - L_i(y_2)| \leq \ell_1 \Psi_\xi^\psi(t_i, t_{i-1})|y_1 - y_2|$$

for any $y_1, y_2 \in \mathbb{R}$ and $i = 1, \ldots, m$.

We are now in a position to state and prove our existence result for the problem (1)-(5) based on based on Banach's fixed point theorem [11].

Theorem 3.3. *Assume* (Ax1)-(Ax3) *hold. If*

$$\mathcal{L} = \frac{1}{\Gamma_k(k\xi)}\left[\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}+1\right)\left(m\ell_1 + \frac{m\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(2k-k\xi+\vartheta)}\right)\right.$$
$$\left.+\frac{\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)}\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(2k-k\xi+\vartheta)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\right] < 1, \quad (25)$$

then the problem (1)-(5) *has a unique solution in* $\mathbb{F}$.

Proof. We show that the operator $\mathcal{T}$ defined in (24) has a unique fixed point in $\mathbb{F}$. Let $x, y \in \mathbb{F}$. Then for any $t \in [a-\lambda, a] \cup \left[b, b+\tilde{\lambda}\right]$, we have

$$|\mathcal{T}x(t) - \mathcal{T}y(t)| = 0.$$

Thus

$$\|\mathcal{T}x - \mathcal{T}y\|_{\mathcal{C}} = \|\mathcal{T}x - \mathcal{T}y\|_{\tilde{\mathcal{C}}} = 0. \quad (26)$$

Further, for $t \in (a, b]$ we have

$$|\mathcal{T}x(t) - \mathcal{T}y(t)|$$
$$\leq \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\left[\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m}|L_j(x(t_j^-)) - L_j(y(t_j^-))|\right.$$
$$+\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi_1(s)-\varphi_2(s)|\right)(t_j) + \sum_{a<t_i<t}|L_i(x(t_i^-)) - L_i(x(t_j^-))|$$
$$\left.+\sum_{a<t_i<t}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi_1(s)-\varphi_2(s)|\right)(t_i)\right] + \left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}|\varphi_1(s)-\varphi_2(s)|\right)(t),$$

where φ_1 and φ_1 be functions satisfying the functional equations

$$\varphi_1(t) = f(t, x^t(\cdot), \varphi_1(t)),$$
$$\varphi_2(t) = f(t, y^t(\cdot), \varphi_2(t)).$$

By $(Ax2)$, we have

$$\begin{aligned}|\varphi_1(t)-\varphi_2(t)| &= |f(t,x^t,\varphi_1(t))-f(t,y^t,\varphi_2(t))|\\ &\leq \zeta_1\|x^t-y^t\|_{[-\lambda,\tilde{\lambda}]}+\zeta_2|\varphi_1(t)-\varphi_2(t)|.\end{aligned}$$

Then,

$$|\varphi_1(t)-\varphi_2(t)| \leq \frac{\zeta_1}{1-\zeta_2}\|x^t-y^t\|_{[-\lambda,\tilde{\lambda}]}.$$

Therefore, for each $t\in(a,b]$

$$\begin{aligned}&|\mathcal{T}x(t)-\mathcal{T}y(t)|\\ &\leq \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\left[\frac{\ell_1|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m}\|x-y\|_{PC_{\xi,k;\psi}}+\ell_1\sum_{j=1}^{m}\|x-y\|_{PC_{\xi,k;\psi}}\right.\\ &+\frac{\zeta_1|\alpha_2|}{(1-\zeta_2)|\alpha_1+\alpha_2|}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|x^s-y^s\|_{[-\lambda,\tilde{\lambda}]}\right)(t_j)\\ &\left.+\frac{\zeta_1}{1-\zeta_2}\sum_{j=1}^{m}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}\|x^s-y^s\|_{[-\lambda,\tilde{\lambda}]}\right)(t_j)\right]+\frac{\zeta_1}{1-\zeta_2}\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}\|x^s-y^s\|_{[-\lambda,\tilde{\lambda}]}\right)(t).\end{aligned}$$

Thus

$$\begin{aligned}&|\mathcal{T}x(t)-\mathcal{T}y(t)|\\ &\leq \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\left[\frac{m\ell_1|\alpha_2|}{|\alpha_1+\alpha_2|}+m\ell_1+\frac{\zeta_1|\alpha_2|}{(1-\zeta_2)|\alpha_1+\alpha_2|}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}(1)\right)(t_j)\right.\\ &\left.+\frac{\zeta_1}{1-\zeta_2}\sum_{j=1}^{m}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}(1)\right)(t_j)\right]\|x-y\|_{\mathbb{F}}+\frac{\zeta_1}{1-\zeta_2}\|x-y\|_{\mathbb{F}}\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}(1)\right)(t).\end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned}&|\mathcal{T}x(t)-\mathcal{T}y(t)|\\ &\leq \frac{\|x-y\|_{\mathbb{F}}}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\left[\frac{m\ell_1|\alpha_2|}{|\alpha_1+\alpha_2|}+m\ell_1+\frac{(m+1)\zeta_1|\alpha_2|\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)|\alpha_1+\alpha_2|\Gamma_k(k(1-\xi)+\vartheta+k)}\right.\\ &\left.+\frac{m\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(k(1-\xi)+\vartheta+k)}\right]+\frac{\zeta_1\left(\psi(t)-\psi(t_i)\right)^{\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(\vartheta+k)}\|x-y\|_{\mathbb{F}}.\end{aligned}$$

Hence

$$\left|\Psi_\xi^\psi(t,t_i)\left(\mathcal{T}x(t)-\mathcal{T}y(t)\right)\right|$$

$$\leq \frac{\|x-y\|_{\mathbb{F}}}{\Gamma_k(k\xi)}\Bigg[\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}+1\right)\left(m\ell_1+\frac{m\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(k(1-\xi)+\vartheta+k)}\right)$$
$$+\frac{\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)}\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(k(1-\xi)+\vartheta+k)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\Bigg],$$

which implies that

$$\|\mathcal{T}x-\mathcal{T}y\|_{PC_{\xi,k;\psi}}$$
$$\leq \frac{\|x-y\|_{\mathbb{F}}}{\Gamma_k(k\xi)}\Bigg[\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}+1\right)\left(m\ell_1+\frac{m\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)\Gamma_k(2k-k\xi+\vartheta)}\right)$$
$$+\frac{\zeta_1\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-\zeta_2)}\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(2k-k\xi+\vartheta)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\Bigg].$$

Thus

$$\|\mathcal{T}x-\mathcal{T}y\|_{C_{\xi,k;\psi}}\leq \mathcal{L}\|x-y\|_{\mathbb{F}}. \tag{27}$$

By (26) and (27), we obtain

$$\|\mathcal{T}x-\mathcal{T}y\|_{\mathbb{F}}\leq \mathcal{L}\|x-y\|_{\mathbb{F}}.$$

By (25), the operator $\mathcal{T}$ is a contraction on $\mathbb{F}$. Hence, by Banach's contraction principle, $\mathcal{T}$ has a unique fixed point $x\in\mathbb{F}$, which is a solution to our problem (1)-(5). □

Our next existence result for the problem (1)-(5) is based on Schauder's fixed point theorem [11].

Remark 3.4. *We note that by taking:*

$$\zeta_1=q_2^*,\ \zeta_2=q_3^*,\ \ell_1=\varrho_2,$$
$$q_1(t)=|f(t,0,0)|\ \text{and}\ \varrho_1=|L_i(0)|.$$

hypothesis (Ax2) *implies that*

$$|f(t,x,y)|\leq q_1(t)+q_2(t)\|x\|_{[-\lambda,\tilde{\lambda}]}+q_3(t)|y|$$

and from (Ax3)*, we get*

$$|L_i(y)|\leq \varrho_1+\varrho_2\Psi_\xi^\psi(t_i,t_{i-1})|y|,$$

for $t\in(a,b]$, $x\in PC_{\xi,k;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right)$ *and* $y\in\mathbb{R}$, *where* $\varrho_1,\varrho_2>0$ *and* $q_1,q_2,q_3\in C(J,\mathbb{R}_+)$ *with*

$$q_1^*=\sup_{t\in J}q_1(t),\ \ q_2^*=\sup_{t\in J}q_2(t),\ \ q_3^*=\sup_{t\in J}q_3(t)<1.$$

Theorem 3.5. *Assume* (Ax1)-(Ax3) *hold. If*

$$\ell = \frac{1}{\Gamma_k(k\xi)}\left[\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}+1\right)\left(m\varrho_2+\frac{mq_2^*\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-q_3^*)\Gamma_k(2k-k\xi+\vartheta)}\right)\right.$$
$$\left.+\frac{q_2^*\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-q_3^*)}\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(2k-k\xi+\vartheta)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\right] < 1, \qquad (28)$$

then the problem (1)-(5) *has at least one solution in* $\mathbb{F}$.

Proof. In several steps, we will use Schauder's fixed point theorem to prove that the operator $\mathcal{T}$ defined in (24) has a fixed point.

Step 1: The operator $\mathcal{T}$ is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $\mathbb{F}$. For each $t \in [a-\lambda, a] \cup \left[b, b+\tilde{\lambda}\right]$, we have

$$|\mathcal{T}x_n(t) - \mathcal{T}x(t)| = 0.$$

And for $t \in (a, b]$, we have

$$\begin{aligned}
&|\mathcal{T}x_n(t) - \mathcal{T}x(t)| \\
&\leq \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\left[\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m}|L_j(x_n(t_j^-)) - L_j(x(t_j^-))|\right. \\
&+\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi_n(s)-\varphi(s)|\right)(t_j) + \sum_{a<t_i<t}|L_i(x_n(t_i^-)) - L_i(x(t_i^-))| \\
&\left.+\sum_{a<t_i<t}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi_n(s)-\varphi(s)|\right)(t_i)\right] + \left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}|\varphi_n(s)-\varphi(s)|\right)(t),
\end{aligned}$$

where φ and φ_n be functions satisfying the functional equations

$$\begin{aligned}
\varphi(t) &= f(t, x^t(\cdot), \varphi(t)), \\
\varphi_n(t) &= f(t, x_n^t(\cdot), \varphi_n(t)).
\end{aligned}$$

Since $x_n \to x$, then we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in (a, b]$, and since f and $L_i; i = 0, \ldots, m$ are continuous, then we have

$$\|\mathcal{T}x_n - \mathcal{T}x\|_{\mathbb{F}} \to 0 \text{ as } n \to \infty.$$

Step 2: $\mathcal{T}(B_M) \subset B_M$.
Let M a positive constant such that

$$M \geq \max\left\{\frac{|\alpha_3| + \tilde{\ell}|\alpha_1+\alpha_2|\Gamma_k(k\xi)}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)\,(1-\ell)}, \|\varpi\|_{\mathcal{C}}, \|\tilde{\varpi}\|_{\tilde{\mathcal{C}}}\right\},$$

such that

$$\tilde{\ell} := \frac{1}{\Gamma_k(k\xi)}\Bigg[\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}+1\right)\left(m\varrho_1+\frac{mq_1^*\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-q_3^*)\Gamma_k(2k-k\xi+\vartheta)}\right)$$
$$+\frac{q_1^*\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{(1-q_3^*)}\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(2k-k\xi+\vartheta)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\Bigg].$$

We define the following bounded closed set

$$B_M = \left\{x \in \mathbb{F} : \|x\|_{\mathbb{F}} \leq M\right\}.$$

For each $t \in [a-\lambda, a]$, we have

$$|\mathcal{T}x(t)| \leq \|\varpi\|_{\mathcal{C}},$$

and for each $t \in \left[b, b+\tilde{\lambda}\right]$, we have

$$|\mathcal{T}x(t)| \leq \|\tilde{\varpi}\|_{\tilde{\mathcal{C}}}.$$

Further, for each $t \in (a, b]$, (24) implies that

$$|\mathcal{T}x(t)| \leq \frac{1}{\Psi_\xi^\psi(t,t_i)\Gamma_k(k\xi)}\Bigg[\frac{|\alpha_3|}{|\alpha_1+\alpha_2|}+\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m}|L_j(x(t_j^-))|$$
$$+\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi(s)|\right)(t_j)+\sum_{a<t_i<t}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi(s)|\right)(t_i)$$
$$+\sum_{a<t_i<t}|L_i(x(t_i^-))|\Bigg]+\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}|\varphi(s)|\right)(t). \tag{29}$$

By the hypothesis (*Ax2*) and Remark 3.4, for $t \in (a, b]$, we have

$$\begin{aligned}|\varphi(t)| &= |f(t, x^t, \varphi(t))|\\ &\leq q_1(t)+q_2(t)\|x^t\|_{[-\lambda,\tilde{\lambda}]}+q_3(t)|\varphi(t)|,\end{aligned}$$

which implies that

$$|\varphi(t)| \leq q_1^* + q_2^*M + q_3^*|\varphi(t)|,$$

then

$$|\varphi(t)| \leq \frac{q_1^*+q_2^*M}{1-q_3^*} := \Delta.$$

Thus for $t \in (a,b]$, by hypothesis (*Ax3*), Remark 3.4 and from (29) we get

$$\begin{aligned}|\Psi_\xi^\psi(t,t_i)\mathcal{T}x(t)| \leq \frac{1}{\Gamma_k(k\xi)}\Bigg[&\frac{|\alpha_3|}{|\alpha_1+\alpha_2|}+\frac{m(\varrho_1+\varrho_2 M)|\alpha_2|}{|\alpha_1+\alpha_2|}+m(\varrho_1+\varrho_2 M)\\ &+\frac{\Delta|\alpha_2|}{|\alpha_1+\alpha_2|}\sum_{j=1}^{m+1}\left(\mathcal{J}_{t_{j-1}^+}^{k(1-\xi)+\vartheta,k;\psi}(1)\right)(t_j)+\Delta\sum_{j=1}^{m}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}(1)\right)(t_j)\Bigg]\\ &+\Delta\Psi_\xi^\psi(t,t_i)\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}(1)\right)(t).\end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned}|\Psi_\xi^\psi(t,t_i)\mathcal{T}x(t)| \leq \frac{1}{\Gamma_k(k\xi)}\Bigg[&\frac{|\alpha_3|}{|\alpha_1+\alpha_2|}+\frac{m(\varrho_1+\varrho_2 M)|\alpha_2|}{|\alpha_1+\alpha_2|}+m(\varrho_1+\varrho_2 M)\\ &+\frac{(m+1)\Delta|\alpha_2|\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{|\alpha_1+\alpha_2|\Gamma_k(k(1-\xi)+\vartheta+k)}\\ &+\frac{m\Delta\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{\Gamma_k(k(1-\xi)+\vartheta+k)}\Bigg]+\frac{\Delta\left(\psi(t)-\psi(t_i)\right)^{1-\xi+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}.\end{aligned}$$

Thus

$$\begin{aligned}|\Psi_\xi^\psi(t,t_i)\mathcal{T}x(t)| &\leq \frac{1}{\Gamma_k(k\xi)}\Bigg[\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|}+1\right)\left(m(\varrho_1+\varrho_2 M)+\frac{m\Delta\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}}{\Gamma_k(2k-k\xi+\vartheta)}\right)\\ &\quad+\Delta\left(\psi(b)-\psi(a)\right)^{1-\xi+\frac{\vartheta}{k}}\left(\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(2k-k\xi+\vartheta)}+\frac{\Gamma_k(k\xi)}{\Gamma_k(\vartheta+k)}\right)\Bigg]\\ &\quad+\frac{|\alpha_3|}{|\alpha_1+\alpha_2|\Gamma_k(k\xi)}\\ &\leq M.\end{aligned}$$

Then, for each $t \in \left[a-\lambda, b+\tilde{\lambda}\right]$ we obtain

$$\|\mathcal{T}x\|_{\mathbb{F}} \leq M.$$

Step 3: $\mathcal{T}(B_M)$ is relatively compact.
Let $\tau_1,\tau_2 \in J_i; i=0,\ldots,m$, $\tau_1<\tau_2$ and let $x \in B_M$. Then

$$\begin{aligned}&\left|\Psi_\xi^\psi(\tau_1,t_i)\mathcal{T}x(\tau_1)-\Psi_\xi^\psi(\tau_2,t_i)\mathcal{T}x(\tau_2)\right|\\ &\leq \frac{1}{\Gamma_k(k\xi)}\left[\sum_{\tau_1<t_i<\tau_2}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi(s)|\right)(t_i)+\sum_{\tau_1<t_i<\tau_2}|L_i(x(t_i^-))|\right]\end{aligned}$$

$$
\begin{aligned}
&+\left|\Psi_{\xi}^{\psi}(\tau_1,t_i)\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}|\varphi(s)|\right)(\tau_1)-\Psi_{\xi}^{\psi}(\tau_2,t_i)\left(\mathcal{J}_{t_i^+}^{\vartheta,k;\psi}|\varphi(s)|\right)(\tau_2)\right|\\
&\leq \frac{1}{\Gamma_k(k\xi)}\left[\sum_{\tau_1<t_i<\tau_2}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi(s)|\right)(t_i)+\sum_{\tau_1<t_i<\tau_2}|L_i(x(t_i^-))|\right]\\
&+\int_{t_i}^{\tau_1}\left|\Psi_{\xi}^{\psi}(\tau_1,t_i)\bar{\Psi}_{\vartheta}^{k,\psi}(\tau_1,s)-\Psi_{\xi}^{\psi}(\tau_2,t_i)\bar{\Psi}_{\vartheta}^{k,\psi}(\tau_2,s)\right||\psi'(s)\varphi(s)|ds\\
&+\left|\Psi_{\xi}^{\psi}(\tau_2,t_i)\left(\mathcal{J}_{\tau_1^+}^{\vartheta,k;\psi}|\varphi(s)|\right)(\tau_2)\right|
\end{aligned}
$$

By Lemma 2.6, we get

$$
\begin{aligned}
&\left|\Psi_{\xi}^{\psi}(\tau_1,t_i)\mathcal{T}x(\tau_1)-\Psi_{\xi}^{\psi}(\tau_2,t_i)\mathcal{T}x(\tau_2)\right|\\
&\leq \frac{1}{\Gamma_k(k\xi)}\left[\sum_{\tau_1<t_i<\tau_2}\left(\mathcal{J}_{t_{i-1}^+}^{k(1-\xi)+\vartheta,k;\psi}|\varphi(s)|\right)(t_i)+\sum_{\tau_1<t_i<\tau_2}|L_i(x(t_i^-))|\right]\\
&+\Delta\int_{t_i}^{\tau_1}\left|\Psi_{\xi}^{\psi}(\tau_1,t_i)\bar{\Psi}_{\vartheta}^{k,\psi}(\tau_1,s)-\Psi_{\xi}^{\psi}(\tau_2,t_i)\bar{\Psi}_{\vartheta}^{k,\psi}(\tau_2,s)\right||\psi'(s)|ds\\
&+\frac{\Delta\Psi_{\xi}^{\psi}(\tau_2,t_i)\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}.
\end{aligned}
$$

As $\tau_1\to\tau_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the other cases is obvious, thus we omit the details. From step1 to step3 with Arzela-Ascoli theorem, we conclude that $\mathcal{T}:\mathbb{F}\to\mathbb{F}$ continuous and compact. As a consequence of Schauder's fixed point theorem, we deduce that $\mathcal{T}$ has a fixed point which is a solution of the problem (1)-(5). □

4 An Example

Example 4.1. *Taking $r\to 0$, $\vartheta=\frac{1}{2}$, $k=1$, $J=[1,\pi]$, $\psi(t)=\ln(t)$, $\alpha_1=2$, $\alpha_2=3$, $\alpha_3=4$, $\lambda=\tilde{\lambda}=\pi$ and $\xi=\frac{1}{2}$, we obtain an impulsive boundary value problem which is a particular case of problem (1)-(5) with Hadamard fractional derivative, given by*

$$\left({}_1^H\mathcal{D}_{1+}^{\frac{1}{2},0;\psi}x\right)(t)=\left({}^H\mathbb{D}_{1+}^{\frac{1}{2}}x\right)(t)=f\left(t,x^t(\cdot),\left({}^H\mathbb{D}_{1+}^{\frac{1}{2}}x\right)(t)\right),\quad t\in J_0\cup J_1, \tag{30}$$

$$\left(\mathcal{J}_{e^+}^{k(1-\xi),k;\psi}x\right)(e^+)-\left(\mathcal{J}_{1^+}^{k(1-\xi),k;\psi}x\right)(e^-)=L_1(x(e^-)), \tag{31}$$

$$2\left(\mathcal{J}_{1^+}^{\frac{1}{2},1;\psi}x\right)(1)+3\left(\mathcal{J}_{e^+}^{\frac{1}{2},1;\psi}x\right)(3)=4. \tag{32}$$

$$x(t)=\varpi(t),\quad t\in[1-\pi,1], \tag{33}$$

$$x(t)=\tilde{\varpi}(t), \quad t\in[\pi,2\pi], \tag{34}$$

where $J_0=(1,e]$ *and* $J_1=(e,\pi]$. *Set*

$$f(t,x_1,x_2)=\frac{1}{105+103e^{\pi-t}}\left[1+2|\sin(t)|+\frac{|x_1|}{1+|x_1|}+\frac{|x_2|}{1+|x_2|}\right],$$

and

$$L_1(x_2)=|\cos(t)|+\frac{|x_2|}{|x_2|+1},$$

where $t\in J$, $x_1\in PC_{\xi,k;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right)$ *and* $x_2\in\mathbb{R}$. *Since the function* f *is continuous, then the condition* (Ax1) *is satisfied.*
For each $x_1,y_1\in PC_{\xi,k;\psi}\left(\left[-\lambda,\tilde{\lambda}\right]\right)$, $x_2,y_2\in\mathbb{R}$ *and* $t\in J$, *we have*

$$|f(t,x_1,x_2)-f(t,y_1,y_2)|\leq\frac{1}{105+103e^{\pi-t}}\left(\|x_1-y_1\|_{[-\lambda,\tilde{\lambda}]}+|x_2-y_2|\right),$$

and

$$|L_1(z_1)-L_1(\tilde{z}_1)|\leq|z_1-\tilde{z}_1|,\ z_1,\tilde{z}_1\in\mathbb{R},$$

then, the conditions (Ax2) *and* (Ax3) *are satisfied with*

$$\zeta_1=\zeta_2=\frac{1}{208}\ \text{and}\ \ell_1=1.$$

Also, We have

$$\mathcal{L}=\frac{1}{\sqrt{\pi}}\left[\frac{8}{5}\left(1+\frac{\ln(3)}{207}\right)+\frac{\ln(3)}{207}\left(\frac{3}{5}+2\right)\right]\approx 0.915279505465885\ <1.$$

As all the assumptions of Theorem 3.3 are satisfied, the problem (30)-(34) *has a unique solution in* $\mathbb{F}$.

In order to prove an existence result based on Theorem 3.5, we can easily show that all the conditions are satisfied by using Remark 3.4 and taking

$$q_1(t)=\frac{1+2|\sin(t)|}{105+103e^{\pi-t}},\ q_2(t)=q_3(t)=\frac{1}{105+103e^{\pi-t}},$$

$$\varrho_1=\varrho_2=1,$$

for each $t\in J$.

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