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Original Research Paper

Some New Inequalities Using Conformable Fractional Integral of Order β

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Abstract. In this article, we establish several new inequalities for convex functions in the framework of conformable generalized fractional integral operators of order β, natural generalization of several fractional integrals reported in the literature. The results obtained are generalizations and refinements of some well-known results.

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1 Introduction

The fractional calculus is associated with the integrals and differentiation of arbitrary non integral order. This field has been considered the most effective tool in the last three centuries to characterize anomalous kinetics and its extensive applications in various domains. By using ordinary differential equations for fractional derivatives, various phenomena such as biology, chemistry, engineering, mathematics, physics and statistics can be modelled.

The development of several fractional operators is a noteworthy feature of this investigation (see [3, 10, 51]). A more complete overview of the development of this area with its overlapping with the generalized local calculus can be found at [1, 14].

Definition 1.1. ([22]) *Let $\vartheta \in L_1[a,b]$. The right-sided and left-sided Riemann Liouville fractional integrals of order $\varrho > 0$, with $\Re^{\varrho}_{a^+}\vartheta$ and $\Re^{\varrho}_{b^-}\vartheta$, are defined by:*

$$\Re^{\varrho}_{a^+}\vartheta(\varsigma) = \frac{1}{\Gamma(\varrho)}\int_a^{\varsigma} \vartheta(\zeta)(\varsigma-\zeta)^{\varrho-1}d\zeta, \qquad (\varrho > a),$$

and

$$\Re^{\varrho}_{b^-}\vartheta(\varsigma) = \frac{1}{\Gamma(\varrho)}\int_{\varsigma}^{b} \vartheta(\zeta)(\zeta-\varsigma)^{\varrho-1}d\zeta, \qquad (\varrho < b),$$

respectively, where $\Gamma(\varrho) = \int_0^{\infty} e^{-\wp}\wp^{\varrho-1}d\wp$ is the usual gamma function.

Definition 1.2. ([30]) *Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then the right-sided and left-sided Katugampola fractional integrals of order $\varrho > 0$ of $f \in X^p_c(a,b)$ are defined by:*

$${}^{\rho}I^{\varrho}_{a^+}f(\varsigma) = \frac{\rho^{1-\varrho}}{\Gamma(\varrho)}\int_a^{\varsigma} \frac{\zeta^{\rho-1}f(\zeta)d\zeta}{(\varsigma^{\rho}-\zeta^{\rho})^{1-\varrho}},$$

and

$${}^{\rho}I^{\varrho}_{b^-}f(\varsigma) = \frac{\rho^{1-\varrho}}{\Gamma(\varrho)}\int_{\varsigma}^{b} \frac{\zeta^{\rho-1}f(\zeta)d\zeta}{(\zeta^{\rho}-\varsigma^{\rho})^{1-\varrho}},$$

with $a < \varsigma < b$ and $\rho > 0$, provided the integrals exist.

Definition 1.3. ([34]) *Let (a,b) be the finite interval, where $-\infty < a < b < +\infty$ and $\varrho > 0$. Let Ψ be a positive increasing function on $(a,b]$. The left-sided and right-sided fractional integrals of a function f with respect to another function Ψ in $[a,b]$ are defined by:*

$$I^{\varrho}_{a^+;\Psi} f(\varrho) = \frac{1}{\Gamma(\varrho)} \int_a^{\varsigma} \frac{\Psi^{'}(\zeta) f(\zeta) d\zeta}{[\Psi(\varsigma) - \Psi(\zeta)]^{1-\varrho}}, \quad (\varsigma > a),$$

and

$$I^{\varrho}_{b^-;\Psi} f(\varrho) = \frac{1}{\Gamma(\varrho)} \int_{\varsigma}^{b} \frac{\Psi^{'}(\zeta) f(\zeta) d\zeta}{[\Psi(\zeta) - \Psi(\varsigma)]^{1-\varrho}}, \quad (\varsigma < b).$$

Definition 1.4. ([1]) *Let $\beta \in \mathbb{C}, \mathbb{R}(\beta) > 0$, the left-sided and right-sided conformable fractional integrals are defined by:*

$$^{\rho}_{\beta}\Im^{\varrho} f(\varsigma) = \frac{1}{\Gamma(\beta)} \int_a^{\varsigma} \left(\frac{(\varsigma - a)^{\varrho} - (\zeta - a)^{\varrho}}{\varrho} \right)^{\beta - 1} f(\zeta) \frac{d\zeta}{(\zeta - a)^{1-\varrho}},$$

and

$$^{\rho}_{\beta}\Im^{\varrho} f(\varsigma) = \frac{1}{\Gamma(\beta)} \int_{\varsigma}^{b} \left(\frac{(b - \varsigma)^{\varrho} - (b - \zeta)^{\varrho}}{\varrho} \right)^{\beta - 1} f(\zeta) \frac{d\zeta}{(\zeta - a)^{1-\varrho}}.$$

Katugampola in [31], proposed the following definition of generalized conformable fractional integral.

Definition 1.5. *Let Φ be conformable fractional integrable on the interval $[p,q] \subseteq (0,\infty)$. The left-sided and right-sided generalized conformable fractional integrals of $^{\tau}_{\varrho}K^{\beta}_{p^+}$ and $^{\tau}_{\varrho}K^{\beta}_{q^-}$ of order $\beta > 0, \tau \in \mathbb{R}, \varrho + \tau \neq 0$, are defined by:*

$$^{\tau}_{\varrho}K^{\beta}_{p^+} \Phi(\zeta) = \frac{1}{\Gamma(\beta)} \int_p^{\zeta} \left(\frac{\zeta^{\varrho+\tau} - \varsigma^{\varrho+\tau}}{\varrho + \tau} \right)^{\beta - 1} \Phi(\varsigma) \varsigma^{\tau} d_{\varrho}\varsigma, \tag{1}$$

and

$$^{\tau}_{\varrho}K^{\beta}_{q^-} \Phi(\zeta) = \frac{1}{\Gamma(\beta)} \int_{\zeta}^{q} \left(\frac{\varsigma^{\varrho+\tau} - \zeta^{\varrho+\tau}}{\varrho + \tau} \right)^{\beta - 1} \Phi(\varsigma) \varsigma^{\tau} d_{\varrho}\varsigma, \tag{2}$$

respectively, ${}^{0}_{\varrho}K^{\beta}_{p^+}\Phi(\zeta) = {}^{0}_{\varrho}K^{\beta}_{q^-}\Phi(\zeta) = \Phi(\zeta)$.

Here the integral $\int_p^\zeta d_\varrho\varsigma$ is the conformable fractional integral and it is defined as:

$$\int_p^\zeta \Phi(\varsigma)d_\varrho\varsigma = \int_p^\zeta \Phi(\varsigma)\varsigma^{\varrho-1}d\varsigma.$$

Remark 1.6. ([33]) For $\tau = 0$ in definition (1.5), new Riemann Liouville type conformable fractional integrals are obtained as:

$${}^{0}_{\varrho}K^{\beta}_{p^+}\Phi(\zeta) = \frac{1}{\Gamma(\beta)}\int_p^\zeta \left(\frac{\zeta^\varrho - \varsigma^\varrho}{\varrho}\right)^{\beta-1}\Phi(\varsigma)d_\varrho\varsigma, \tag{3}$$

and

$${}^{0}_{\varrho}K^{\beta}_{q^-}\Phi(\zeta) = \frac{1}{\Gamma(\beta)}\int_\zeta^q \left(\frac{\varsigma^\varrho - \zeta^\varrho}{\varrho}\right)^{\beta-1}\Phi(\varsigma)d_\varrho\varsigma. \tag{4}$$

Remark 1.7. ([33])For $\varrho = 1$ in (3), the well known Riemann Liouville fractional integral operator is obtained as:

$${}^{0}_{1}K^{\beta}_{p^+}\Phi(\zeta) = \frac{1}{\Gamma(\beta)}\int_p^\zeta (\zeta - \varsigma)^{\beta-1}\Phi(\varsigma)d_\varrho\varsigma, \tag{5}$$

Remark 1.8. For $\beta = 1, \tau = 0$ in (1), we obtain the conformable fractional integral and For $\varrho = \beta = 1, \tau = 0$ the classical Riemann Liouville integral is obtained .

Remark 1.9. Under the above conditions, all these definitions can also be obtained for (2).

The structure of this work is as follows: next we have a Preliminary Section, where we present some ideas and references about Integral Inequalities, mainly the well-known Hermite-Hadamard Inequality, in the Main Results Section, we obtain various generalizations of known integral inequalities and We close with a Conclusions Section, where some future work directions are presented.

2 Preliminaries

One of the most developed mathematical areas in recent decades are the integral inequalities using different class of convex functions, the concept itself has undergone innumerable extensions and ramifications in the last 30 years, we recommend [49] to have a fairly complete picture of this development. The convex function is defined as:

Definition 2.1. *A function $\psi : I \to \mathbb{R}$, $I := [a, b]$ is said to be convex if*

$$\psi\left(\tau\xi + (1-\tau)\varsigma\right) \leq \tau\psi(\xi) + (1-\tau)\psi(\varsigma),$$

holds $\forall\ \xi, \varsigma \in I, \tau \in [0, 1]$.

In 2014, with the develpoment of the conformable derivative ([32]), a new direction of work has been opened, which also favorably influenced the aforementioned topic. For different applications and results ([2, 11, 13, 28, 29, 38, 39, 59, 61, 62]).

Among the well-known integral inequalities, the classic Hermite-Hadamard inequality for convex functions occupies a prominent place.

Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$, then

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \psi(\xi)d\xi \leq \frac{\psi(a)+\psi(b)}{2}. \tag{6}$$

The inequality is known as Hermite-Hadamard inequality for convex functions.

It gives an estimation of the mean value of a convex function, interpolates the image of the average of the interval and the average of the images of the extremes of the interval and it is important to note that it also provides a refinement to the Jensen inequality (various extensions and applications of this inequality can be consulted in [3, 4, 5, 6, 9, 12, 14, 15, 17, 18, 19, 20, 21, 33, 40, 45, 46, 47, 50, 60, 64, 66, 67]).

The main purpose of this paper is, using the generalized conformable integral operators of Definition 1.5, to establish several integral inequalities of Hermite-Hadamard type (6), which contain as particular cases, several of those reported in the literature.

3 Main Results

In this section, we establish new integral inequalities within the framework of the generalized conformable fractional operators order β

Theorem 3.1. *Let Φ and Ψ be continuous functions defined on the interval $[1,+\infty)$ where that $\Phi \leq \Psi$. If $\frac{\Phi}{\Psi}$ is decreasing and Φ is increasing over $[0,+\infty)$, then for any convex function Ω satisfying $\Omega(0)=0$, the following inequality holds:*

$$\frac{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Phi(u)\right]}{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Psi(u)\right]} \geq \frac{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\right]}{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Psi(u))\right]}. \tag{7}$$

Proof. Using the convexity of Ω and using the assumption $\Omega(0)=0$, the function $\frac{\Omega(\Phi(x))}{x}$ is increasing. As the function Φ is increasing then the function $\frac{\Omega(\Phi(x))}{\Phi(x)}$ is also increasing. It is obvious that the function $\frac{\Phi}{\Psi}$ is decreasing, then for all $s,t\in[0,+\infty)$, we have

$$\left(\frac{\Omega(\Phi(s))}{\Phi(s)}-\frac{\Omega(\Phi(t))}{\Phi(t)}\right)\left(\frac{\Phi(t)}{\Psi(t)}-\frac{\Phi(s)}{\Psi(s)}\right)\geq 0. \tag{8}$$

From (8), we have

$$\frac{\Omega(\Phi(s))}{\Phi(s)}\frac{\Phi(t)}{\Psi(t)}+\frac{\Omega(\Phi(t))}{\Phi(t)}\frac{\Phi(s)}{\Psi(s)}\geq\frac{\Omega(\Phi(t))}{\Phi(t)}\frac{\Phi(t)}{\Psi(t)}+\frac{\Omega(\Phi(s))}{\Phi(s)}\frac{\Phi(s)}{\Psi(s)}. \tag{9}$$

Multiplying inequality (9) by $\Psi(s)\Psi(t)$, we obtain

$$\frac{\Omega(\Phi(s))}{\Phi(s)}\Psi(s)\Phi(t)+\frac{\Omega(\Phi(t))}{\Phi(t)}\Psi(t)\Phi(s)\geq\Omega(\Phi(t))\Psi(s)+\Omega(\Phi(s))\Psi(t). \tag{10}$$

Multiply (10) by $\frac{1}{\Gamma(\beta)}\left(\frac{u^{\varrho+\tau}-t^{\varrho+\tau}}{\varrho+\tau}\right)^{\beta-1}t^{\tau}d_{\varrho}t$ and integrating over $[p,u]$ with respect to t, we obtain

$$\begin{aligned}&\frac{\Omega(\Phi(s))}{\Phi(s)}\Psi(s){}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Phi(u)\right]+\Phi(s)\,{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right]\\ \geq\;&\Psi(s)\,{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\right]+\Omega(\Phi(s))\,{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Psi(u)\right].\end{aligned} \tag{11}$$

Similarly, multiplying the inequality (11) by $\frac{1}{\Gamma(\beta)}\left(\frac{u^{\varrho+\tau}-s^{\varrho+\tau}}{\varrho+\tau}\right)^{\beta-1} s^{\tau} d_{\varrho}s$, integrating the resulting inequality over $[p,u]$ with respect to s, we obtain

$$\begin{aligned} &{}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right]\ {}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Phi(u)\right] + \ {}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Phi(u)\right]\ {}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right] \\ &\geq \ {}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Omega(\Phi(u))\right]\ {}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Psi(u)\right] + \ {}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Omega(\Phi(u))\right]\ {}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Psi(u)\right]. \quad (12) \end{aligned}$$

From (12), we have

$$\frac{{}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Phi(u)\right]}{{}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Psi(u)\right]} \geq \frac{{}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Omega(\Phi(u))\right]}{{}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right]}. \quad (13)$$

Since $\Phi \leq \Psi$ and from the properties of Ω, it is easy to obtain for $t \in [0,+\infty)$

$$\frac{\Omega(\Phi(t))}{\Phi(t)} \leq \frac{\Omega(\Psi(t))}{\Psi(t)},$$

and

$${}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right] \leq \ {}_{\varrho}^{\tau}K_{p^+}^{\beta}\left[\Omega(\Psi(u))\right]. \quad (14)$$

On utilizing (14) in (13), we obtain required result. □

Remark 3.2. Since ${}_{\varrho}^{\tau}K_{p^+}^{\beta}\Phi(u) = {}_{\varrho}^{\tau}K_{q^-}^{\beta}\Phi(u)$ so above inequality is also proved for ${}_{\varrho}^{\tau}K_{q^-}^{\beta}\Phi(u)$.

Remark 3.3. If $\varrho = 1$ then this result is proved for Riemann Liouville Fractional Integral.

Remark 3.4. On taking $\beta = \varrho = 1$and $\tau = 0$, the integral becomes Classical Riemann Liouville integral and we obtain theorem 3.1 of [16].

Remark 3.5. Let's look at some particular cases of this Theorem reported in the literature. For example, if we work with the classic Riemann Integral, we obtain Theorem 9 of [36] taking the function Ω as convex. In addition, if we consider the Riemann Liouville integral, then we obtain Theorem 3 of [55].

Next, we present a more general variation from the previous result, in which two fractional orders are considered.

Theorem 3.6. *Let Φ and Ψ be continuous functions defined on the interval $[p,u]\subseteq[1,+\infty)$ where $\Phi\leq\Psi$. If $\frac{\Phi}{\Psi}$ is decreasing and Φ is increasing over $[0,+\infty)$, then for any convex function Ω satisfying $\Omega(0)=0$, the following inequality holds:*

$$\frac{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Phi(u)\right]\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Omega(\Psi(u))\right]+\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[(\Phi(u))\right]^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Psi(u))\right]}{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Psi(u)\right]\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Omega(\Phi(u))\right]+^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[(\Psi(u))\right]^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\right]}\geq 1. \tag{15}$$

Proof. Using the convexity of Ω and using the assumption $\Omega(0)=0$, the function $\frac{\Omega(\Phi(x))}{x}$ is increasing. As the function Φ is increasing then the function $\frac{\Omega(\Phi(x))}{\Phi(x)}$ is also increasing. It is obvious that the function $\frac{\Phi}{\Psi}$ is decreasing, then for all $s,t\in[0,+\infty)$, we have

$$\left(\frac{\Omega(\Phi(s))}{\Phi(s)}-\frac{\Omega(\Phi(t))}{\Phi(t)}\right)\left(\frac{\Phi(t)}{\Psi(t)}-\frac{\Phi(s)}{\Psi(s)}\right)\geq 0. \tag{16}$$

From (16), we have

$$\frac{\Omega(\Phi(s))}{\Phi(s)}\frac{\Phi(t)}{\Psi(t)}+\frac{\Omega(\Phi(t))}{\Phi(t)}\frac{\Phi(s)}{\Psi(s)}\geq\frac{\Omega(\Phi(t))}{\Phi(t)}\frac{\Phi(t)}{\Psi(t)}+\frac{\Omega(\Phi(s))}{\Phi(s)}\frac{\Phi(s)}{\Psi(s)}. \tag{17}$$

Multiplying inequality (9) by $\Psi(s)\Psi(t)$, we obtain

$$\frac{\Omega(\Phi(s))}{\Phi(s)}\Psi(s)\Phi(t)+\frac{\Omega(\Phi(t))}{\Phi(t)}\Psi(t)\Phi(s)\geq\Omega(\Phi(t))\Psi(s)+\Omega(\Phi(s))\Psi(t). \tag{18}$$

Multiply (18) by $\frac{1}{\Gamma(\beta)}\left(\frac{u^{\varrho+\tau}-t^{\varrho+\tau}}{\varrho+\tau}\right)^{\beta-1}t^{\tau}d_{\varrho}t$, and integrating the resulting inequality over $[p,u]$ with respect to t, we obtain

$$\begin{aligned}&\frac{\Omega(\Phi(s))}{\Phi(s)}\Psi(s)^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Phi(u)\right]+\Phi(s)\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right] \qquad (19)\\ \geq\ &\Psi(s)\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\right]+\Omega(\Phi(s))\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Psi(u)\right].\end{aligned}$$

Similarly, on multiplying the inequality (19) by $\frac{1}{\Gamma(\lambda)}\left(\frac{u^{\varrho+\tau}-s^{\varrho+\tau}}{\varrho+\tau}\right)^{\lambda-1} s^{\tau} d_{\varrho}s$ and integrating the resulting inequality over $[p,u]$ with respect to s, we obtain

$$
\begin{aligned}
&{}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right]\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Phi(u)\right] + \ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Phi(u)\right]\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right] \qquad (20)\\
&\geq \ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\right]\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Psi(u)\right] + \ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Omega(\Phi(u))\right]\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Psi(u)\right].
\end{aligned}
$$

Since $\Phi \leq \Psi$ and from the properties of Ω, it is easy to obtain for $t \in [0,+\infty)$

$$
\frac{\Omega(\Phi(t))}{\Phi(t)} \leq \frac{\Omega(\Psi(t))}{\Psi(t)}, t \in [0,+\infty). \tag{21}
$$

From the inequality (21), we can obtain

$$
{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\right] \leq \ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Psi(u))}{\Psi(u)}\Psi(u)\right], \tag{22}
$$

and

$$
{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\varpi(u)\right] \leq \ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Psi(u))\varpi(u)\right]. \tag{23}
$$

Utilizing (22) and (23) in (20) , obtain the required result. □

Remark 3.7. If $\lambda = \Omega$ this result becomes Theorem 3.1. On the other hand, if we consider the classic Riemann integral, our theorem is reduced to Theorem 3.3 of [16].

We can to obtain a more general conclusion to Theorem 3.1, if we consider a positive, continuous and increasing function in addition.

Theorem 3.8. *Let Φ, ϖ and Ψ be continuous functions defined on the interval $[p,u] \subseteq [1,+\infty)$ where $\Phi \leq \Psi$. If $\frac{\Phi}{\Psi}$ is decreasing and Φ is increasing over $[0,+\infty)$, then for any convex function Ω satisfying $\Omega(0) = 0$, the following inequality holds:*

$$
\frac{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Phi(u)\right]}{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Psi(u)\right]} \geq \frac{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right]}{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Psi(u))\varpi(u)\right]}. \tag{24}
$$

Proof. Using the convexity of Ω and using the assumption $\Omega(0)=0$, the function $\frac{\Omega(\Phi(x))}{x}$ is increasing. As the function Φ is increasing then the function $\frac{\Omega(\Phi(x))}{\Phi(x)}$ is also increasing. It is obvious that the function $\frac{\Phi}{\Psi}$ is decreasing, then for all $t\in[0,+\infty)$, we have

$$\frac{\Phi(t))}{\Phi(t)} \leq \frac{\Omega(\Psi(t))}{\Psi(t)}. \tag{25}$$

Multiplying inequality (25) by $\Psi(t)\varpi(t)\frac{1}{\Gamma(\beta)}\left(\frac{u^{\varrho+\tau}-t^{\varrho+\tau}}{\varrho+\tau}\right)^{\beta-1} t^{\tau}d_{\varrho}t$ and integrating over $[p,u]$ with respect to t, we obtain

$${}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\varpi(u)\right] \leq {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Psi(u))\varpi(u)\right]. \tag{26}$$

Now from the assumptions, we consider the inequality

$$\left(\frac{\Omega(\Phi(t))}{\Phi(t)}\varpi(t)-\frac{\Omega(\Phi(s))}{\Phi(s)}\varpi(s)\right)\left(\Phi(s)\Psi(t)-\Phi(t)\Psi(s)\right)\geq 0. \tag{27}$$

The above inequality can be written as

$$\begin{aligned} &\frac{\Omega(\Phi(t))}{\Phi(t)}\varpi(t)\Psi(t)\Phi(s)-\Omega(\Phi(s))\varpi(s)\Psi(t)-\Omega(\Phi(t))\varpi(t)\Psi(s)\\ +\;&\frac{\Omega(\Phi(s))}{\Phi(s)}\varpi(s)\Phi(t)\Psi(s)\geq 0. \end{aligned} \tag{28}$$

Multiplying this inequality (28) by $\frac{1}{\Gamma(\beta)}\left(\frac{u^{\varrho+\tau}-t^{\varrho+\tau}}{\varrho+\tau}\right)^{\beta-1} t^{\tau}d_{\varrho}t$ and integrating over $[p,u]$ with respect to t, we obtain

$$\begin{aligned} &\Phi(s){}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\varpi(u)\Psi(u)\right]-\Omega(\Phi(s))\varpi(s){}^{\tau}_{\varrho}K^{\beta}_{p^+}\Psi(t) \\ &-\Psi(s)\,{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right]+\frac{\Omega(\Phi(s))}{\Phi(s)}\varpi(s)\Psi(s)\,{}^{\tau}_{\varrho}K^{\beta}_{p^+}\Phi(u)\geq 0. \end{aligned} \tag{29}$$

Again multiplying (29)by $\frac{1}{\Gamma(\beta)}\left(\frac{u^{\varrho+\tau}-s^{\varrho+\tau}}{\varrho+\tau}\right)^{\beta-1} s^{\tau}d_{\varrho}s$ and integrating over

$[p, u]$ with respect to s, we obtain

$$
\begin{aligned}
&{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\varpi(u)\Psi(u)\right]\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\Phi(u) - {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right]\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\Psi(u) \\
&-{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right]\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\Psi(u) + {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\varpi(u)\Psi(u))\right]\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\Phi(u) \geq 0.
\end{aligned}
\tag{30}
$$

It follows from (30)

$$
\frac{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left(\Phi(u)\right)}{{}^{\tau}_{\varrho}K^{\beta}_{p^+}(\Psi(u))} \geq \frac{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega\Phi(u)\varpi(u)\right]}{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u)}{\Phi(u)}\Psi(u)\varpi(u)\right]}. \tag{31}
$$

Utilizing (31) and (26) , we obtain required result. □

Remark 3.9. If $\varrho = 1$ then this result is proved for Riemann Liouville fractional integral.

Remark 3.10. If we put $\beta = \varrho = 1$ and $\tau = 0$, it follows Classical Riemann Liouville Fractional Integral and we obtain theorem 3.5 of [16].

Remark 3.11. It is not difficult to obtain the Theorem 10 of [36].

The following consequence is the generalization to Theorem 3.6.

Theorem 3.12. *Let Φ, Ψ and ϖ be continuous and positive functions defined on the interval $[1, +\infty)$ where $\Phi \leq \varpi$. Under the condition that $\frac{\Phi}{\varpi}$ is decreasing and Φ is increasing over $[1, +\infty)$ then, for any convex function Ω that satisfies $\Omega(0) = 0$, the following inequality holds:*

$$
\frac{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Phi(u)\right]\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Omega(\Psi(u))\varpi(u)\right] + {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[(\Phi(u))\right]{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Psi(u))\varpi(u)\right]}{{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Psi(u)\right]\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right] + {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[(\Psi(u))\right]{}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right]} \geq 1. \tag{32}
$$

Proof. Using the convexity of Ω and using the assumption $\Omega(0) = 0$, the function $\frac{\Omega(\Phi(x))}{x}$ is increasing. As the function Φ is increasing then the function $\frac{\Omega(\Phi(x))}{\Phi(x)}$ is also increasing. It is obvious that the function $\frac{\Phi}{\Psi}$ is decreasing, then for all $t \in [0, +\infty)$, we have

$$
\left(\frac{\Omega(\Phi(t))}{\Phi(t)}\varpi(t) - \frac{\Omega(\Phi(s))}{\Phi(s)}\varpi(s)\right)\left(\Phi(s)\Psi(t) - \Phi(t)\Psi(s)\right) \geq 0. \tag{33}
$$

The inequality (33) can be written as

$$\frac{\Omega(\Phi(t))}{\Phi(t)}\varpi(t)\Phi(s)\Psi(t)-\Omega(\Phi(s))\varpi(s))\Psi(t)-\Omega(\Phi(t))\varpi(t)\Psi(s)$$
$$+\quad \frac{\Omega(\Phi(s))}{\Phi(s)}\varpi(s)\Phi(t)\Psi(s)\geq 0, \tag{34}$$

where $s,t\in[0,+\infty)$

On multiplying (34) by $\frac{1}{\Gamma(\beta)}\left(\frac{u^{\varrho+\tau}-t^{\varrho+\tau}}{\varrho+\tau}\right)^{\beta-1}t^{\tau}d_{\varrho}t$ and integrating over $[p,u]$ with respect to t, we obtain

$$\Phi(s){}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\varpi(u)\Psi(u)\right]-\Omega(\Phi(s))\varpi(s)){}^{\tau}_{\varrho}K^{\beta}_{p^+}\Phi(u)$$
$$-\Psi(s)\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right]+\frac{\Omega(\Phi(s))}{\Phi(s)}\varpi(s)\Psi(s){}^{\tau}_{\varrho}K^{\beta}_{p^+}\Phi(u)\geq 0. \tag{35}$$

Now multiplying (35) by $\frac{1}{\Gamma(\lambda)}\left(\frac{u^{\varrho+\tau}-s^{\varrho+\tau}}{\varrho+\tau}\right)^{\lambda-1}s^{\tau}d_{\varrho}s$ and integrating over $[p,u]$ with respect to s, we obtain

$${}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\varpi(u)\Psi(u){}^{\tau}_{\varrho}\right]K^{\lambda}_{p^+}\Phi(u)-\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right]^{\tau}_{\varrho}K^{\beta}_{p^+}\Psi(u) \tag{36}$$
$$-\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Phi(u))\varpi(u)\right]^{\tau}_{\varrho}K^{\lambda}_{p^+}\Psi(u)+\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\varpi(u)\Psi(u)\right]\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\Phi(u)\geq 0,$$

Again considering

$$\frac{\Phi(t))}{\Phi(t)}\leq\frac{\Omega(\Psi(t))}{\Psi(t)}. \tag{37}$$

for all $s,t\in[0,+\infty)$.

From (37), we can easily obtain

$${}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\varpi(u)\right]\leq\ {}^{\tau}_{\varrho}K^{\beta}_{p^+}\left[\Omega(\Psi(u))\varpi(u)\right], \tag{38}$$

and

$${}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\frac{\Omega(\Phi(u))}{\Phi(u)}\Psi(u)\varpi(u)\right]\leq\ {}^{\tau}_{\varrho}K^{\lambda}_{p^+}\left[\Omega(\Psi(u))\varpi(u)\right]. \tag{39}$$

Utilizing(36), (38) and (39) , we obtain the required result. □

Remark 3.13. If we put $\beta = \lambda$ then this result becomes the Theorem 3.8.

Remark 3.14. If $\varrho = 1$ then this result is proved for Riemann Liouville Fractional Integral.

Remark 3.15. If we put $\lambda = \beta = 1$, $\varrho = 1$ and $\tau = 0$ then we obtain the Theorem 3.7 of [16].

4 Conclusions

In this paper, we established certain inequalities by employing the generalized proportional Hadamard conformable fractional integral operator. The inequalities obtained in this present paper will lead to the classical inequalities which are established earlier by Dahmni and Liu [16, 36]. The results established in this paper give some contribution in the field of fractional calculus and Hadamard fractional integral inequalities. One can establish various integral inequalities by employing the newly defined Hadamard fractional integral operators.

From these results, some work directions remain open, for example:

i) Extending these results to other types of integral fractional operators, let's say, the one defined in [25, 35, 44, 65], which contains as particular cases many of those reported in the literature.

ii) Obtain new results for other well-known inequalities such as Chevishev, Grüss, among others.

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References

[1] T. Abdeljawad, S. Iqbal, M. Samraiz, K. S.Nisar,G. Rehman & M. Adil, Some generalized pólya szegö and čebyšev type inequalities with general kernel and measure, *Advances in Difference Equations*, (2020).

[2] M. Abu-Shady and M. K. A. Kaabar, A generalized definition of the fractional derivative with applications, *Mathematical Problems in Engineering*, (2021).

[3] B. Ahmad, A. Alsaedi, M. Kirane & B. T. Torebek, Hermite-hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals, *Journal of Computational and Applied Mathematics,* 353, 120-129, (2019). https://doi.org/10.1016/j.cam.2018.12.030

[4] M. Aleem, M. Ur Rehman, J. Alzabut, S. Etemad & S. Rezapour, On solutions of nonlinear BVPs with general boundary conditions by using a generalized Riesz-Caputo operator, Adv Differ Equ 2021, 303 (2021). https://doi.org/10.1186/s13662-021-03459-w

[5] M. A. Ali, J. E. Nápoles V., A. Kashuri and Z. Zhang, Fractional non conformable hermite-hadamard inequalities for generalized-convex functions, *Fasciculi Mathematici*, Nr 64 2020, 5-16 DOI: 10.21008/j.0044-4413.2020.0007

[6] J. Alzabut, B. Ahmad, S. Etemad, S. Rezapour & A. Zada, Novel existence techniques on the generalized ϕ-Caputo fractional inclusion boundary problem, Adv Differ Equ 2021, 135 (2021). https://doi.org/10.1186/s13662-021-03301-3

[7] A. Atangana, Derivative with a new parameter theory, *Methods and Applications*, Academic Press, (2016).

[8] D. Baleanu, COMMENTS ON: Ortigueira M., Martynyuk V., Fedula M. & J.A.T. Machado, The failure of certain fractional calculus operators in two physical models, *Fract. Calc. Appl. Anal.*, Volume 23, Issue 1, (2019). https://doi.org/10.1515/fca-2020-0012

[9] D. Baleanu, S. Etemad & S. Rezapour, On a Caputo conformable inclusion problem with mixed Riemann-Liouville conformable integro-derivative conditions, Adv Differ Equ 2020, 473 (2020). https://doi.org/10.1186/s13662-020-02938-w

[10] D. Baleanu & A. Fernandez, On fractional operators and their classifications, *Mathematics*, 7(830), (2019).

[11] D. Baleanu, A. Jajarmi, H. Mohammadi & S. Rezapour, A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative, *Chaos, Solitons and Fractals*, 109705, (2020). https://doi.org/10.1016/j.chaos.2020.109705

[12] B. Bayraktar, S. I. Butt, Sh. Shaokat, J. E. Nápoles V., New Hadamard-type inequalities via (s,m1,m2)-convex functions, Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika, Komp'yuternye Nauki, 2021, vol. 31, issue 4, pp. 597-612.

[13] S. A. Bhanotar & M. K. A. Kaabar, Analytical solutions for the nonlinear partial differential equations using the conformable triple Laplace transform decomposition method, *International Journal of Differential Equations*, (2021).

[14] M. Bohner, A. Kashuri, P. O. Mohammed & Juan E. Nápoles V., Hermite-Hadamard-type inequalities for integrals arising in conformable fractional calculus, submited.

[15] H. Budak, On refinements of Hermite-Hadamard type inequalities for riemann-liouville fractional integral operators, *IJOCTA*, Vol.9, No.1, pp.41-48, (2019). http://doi.org/10.11121/ijocta.01.2019.00585

[16] Z. Dahmani, A note on some new fractional results involving convex functions, *Acta Math. Univ. Comen.*, LXXXI, 241-246, (2012).

[17] S. Etemad, S. Rezapour & M. E. Samei, On a fractional Caputo-Hadamard inclusion problem with sum boundary value conditions by using approximate endpoint property, Math Meth Appl Sci. 2020;1-16. https://doi.org/10.1002/mma.6644

[18] S. Etemad, S. Rezapour & M. E. Samei, $\alpha - \phi$-contractions and solutions of a q-fractional differential inclusion with three-point boundary value conditions via computational results, Adv Differ Equ 2020, 218 (2020). https://doi.org/10.1186/s13662-020-02679-w

[19] S. Etemad, M. S. Souid, B. Telli, M. K. A. Kaabar & S. Rezapour, Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments via Kuratowski MNC technique, Adv Differ Equ 2021, 214 (2021). https://doi.org/10.1186/s13662-021-03377-x

[20] J. Galeano, J. Lloreda, J. E. Nápoles V. & E. Pérez, CERTAIN INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR H-CONVEX FUNCTIONS, Journal of Mathematical Control Science and Applications Vol. 7 No. 2 (July-December, 2021), 129-140

[21] J. Galeano, J. E. Nápoles V. & E. Pérez, New Hermite-Hadamard inequalities in the framework of generalized fractional integrals, Annals of the University of Craiova, Mathematics and Computer Science Series, Volume 48(2), 2021, Pages 319-327.

[22] R. Gorenflo, F. Mainardi, FRACTIONAL CALCULUS :Integral and Differential Equations of Fractional Order, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien and New York 1997, pp.223-276

[23] P. M. Guzmán, P. Kórus & Juan E. Nápoles Valdés, Generalized integral inequalities of Chebyshev type, *Fractal Fract.*, (2020). http//doi:10.3390/fractalfract4020010

[24] P. M. Guzmán & Juan E. Nápoles Valdés, Generalized fractional Grüss-type inequalities, *Contrib. Math.* 2, 16-21, (2020). http//doi:10.47443/cm.2020.0029

[25] P. M. Guzmán, J. E. Nápoles V. & Y. S. Gasimov, Integral inequalities within the framework of generlized fractional integrals, *Fractional Differential Calculus*, Volume 11, Number 1, 69-84, (2021). http//doi:10.7153/fdc-2021-11-05

[26] C. Hermite, Sur deux limites d'une intégrale définie, *Mathesis*, 3, 82, (1883).

[27] M. K. A. Kaabar, F. Martínez, J. F. Gómez-Aguilar, B. Ghanbari, M. Kaplan & H. Günerhan, New approximate analytical solutions for nonlinear fractional Schrödinger equation with second order spatiotemporal dispersion via double Laplace transform method, *Mathematical Methods in the Applied Sciences*, 44(14), 11138-11156, (2021).

[28] M. K. A. Kaabar, F. Martínez, J. F. Gómez-Aguilar, B. Ghanbari, M. Kaplan & H. Günerhan, New approximate analytical solutions forthe nonlinear fractional Schrödinger equation with second-order spatio-temporal dispersion via double Laplace transform method, *Mathematical Methods in the Applied Sciences*, 44(14), 11138-11156, (2021)

[29] M. K. A. Kaabar, M. Shabibi, J. Alzabut, S. Etemad, W. Sudsutad, F. Martínez & S. Rezapour, Investigation of the Fractional Strongly Singular Thermostat Model via Fixed Point Techniques. Mathematics, 9(18), 2298 .

[30] U. N. Katugampola, New approach generalized fractional integral, *Applied Math and Computation*, 218, 860-865, (2010).

[31] U. N. Katugampola, New fractional integral unifying six existing fractional integrals. arXiv:1612.08596 [math.CA]

[32] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65-70, (2014).

[33] T. U. Khan & M. A. Khan, On hermite hadamard inequaltiy for new generalized conformable fractional operator, *AIMS Mathematics*, Volume 6, Issue 1: 23-38, (2020). http//doi:10.3934/math.2021002

[34] A. A. Kilbas, H. M. Srivastava & J. J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, New York (2006).

[35] P. Kórus, L. M. Lugo & J. E. Nápoles V., Integral inequalities in a generalized context, *Studia Scientiarum Mathematicarum Hungarica* 57(3), 312-320, (2020).

[36] W. J. Liu, Q. A. Ngo, & V. N. Huy, Several interesting integral inequalities, *J. Mathe. Inequalities*, 3, 201-212, (2009).

[37] F. Mainardi, Fractional calculus and waves in linear viscoelsticity, *Ed. Imperial College Press*, (2010).

[38] F. Martínez, I. Martínez, M. K. A. Kaabar & S. Paredes, New results on complex conformable integral, *AIMS Mathematics*, 5(6), (2020), 7695-7710. doi: 10.3934/math.2020492

[39] M. M. Matar, M.I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad & S. Rezapour, Investigation of the p-laplacian nonperiodic nonlinear boundary value problem via generalized caputo fractional derivatives, *Advances in Difference Equations*, Article no. 68, (2021). https://doi:10.1186/s13662-021-03228-9

[40] H. Mohammadi, D. Baleanu, S. Etemad & S. Rezapour, Criteria for existence of solutions for a Liouville-Caputo boundary value problem via generalized Gronwall's inequality, J Inequal Appl 2021, 36 (2021). https://doi.org/10.1186/s13660-021-02562-6

[41] H. Mohammadi, S. Kumar, Sh. Rezapour & S. Etemad, A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to mumps virus with optimal control, *Chaos, Solitons & Fractals*, 144, (2021). https://doi.org/10.1016/j.chaos.2021.110668

[42] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue & V. Feliu-Batle, Fractional order systems and controls, *Fundamentals and Applications*, Londres: Springer-Verlag London Limited, (2010).

[43] S. Mubeen & G. M. Habibullah, K-fractional integrals and application, *Int. J. Contemp. Math. Sciences, International Journal of Mathematics and Mathematical Sciences*, Vol. 7(2): 89-94, (2012).

[44] J. E. Nápoles V., A generalized k-proportional fractional integral operators with general kernel, submited.

[45] Juan E. Nápoles Valdés, SOME INTEGRAL INEQUALITIES IN THE FRAMEWORK OF GENERALIZED K-PROPORTIONAL FRACTIONAL INTEGRAL OPERATORS WITH GENERAL KERNEL, Honam Mathematical J. 43 (2021), No. 4, pp. 587-596 https://doi.org/10.5831/HMJ.2021.43.4.587

[46] J. E. Nápoles V. & B. Bayraktar, On The Generalized Inequalities Of The Hermite-Hadamard Type, Filomat 35:14 (2021), 4917-4924 https://doi.org/10.2298/FIL2114917N

[47] J. E. Nápoles V., B. Bayraktar & S. I. Butt, New integral inequalities of Hermite-Hadamard type in a generalized context, Punjab University Journal of Mathematics (2021),53(11),765-777 https://doi.org/10.52280/pujm.2021.531101

[48] J. E. Nápoles V. & F. Rabossi, A note on Chebyshev inequality via k-generalized fractional integrals, *Electron. J. Math.* 1, 41-51, (2021). http//doi:10.47443/ejm.2021.0004

[49] J. E. Nápoles V., F. Rabossi & A. Samaniego, Convex functions: Ariadne's thread or Charlotte's spiderweb?, *Advanced Mathematical Models & Applications*, Vol.5, No.2 (2020), pp.176-191.

[50] J. E. Nápoles V., J. M. Rodríguez & J. M. Sigarreta, New Hermite-Hadamard type inequalities involving nonconformable integral operators, *Symmetry*, 11-1108, (2019). https://doi.org/10.3390/sym11091108

[51] T. F. Nonnenmacher & R. Metzler, Applications of fractional calculus ideas to biology, *World Scientific*, (1998).

[52] K. Oldham & J. Spanier, Applications of differentiation and integration to arbitrary order, *Elsevier Science*, Volume 111, *(*1974).

[53] I. Podlubny, Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, *Mathematics in science and engineering*, v. 198, San Diego: Academic Press, (1999).

[54] F. Qi & B. N. Guo, Integral representations and complete monotonicity of remainders of the binet and stirling formulas for the gamma function, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 111(2), 425-434, (2017). https://doi:10.1007/s13398-016-0302-6

[55] G. Rahman, T. Abdeljawad, F. Jarad, A. Khan & K. S. Nisar, Certain inequalities via generalized proportional Hadamard fractional integral operators, *Adv. Diff. Eqs.*, 2019, 454, (2019).

[56] E. D. Rainville, Special functions, *Macmillan Co.*, New York, (1960).

[57] S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom & Y. M. Chu, Inequalities by means of generalized proportional fractional integral operators with respect to another function, *Mathematics*, 7, 1225, (2020). https://doi:10.3390/math7121225

[58] H. U. Rehman, M. Darus & J. Salah, A note on caputo's derivative operator interpretation in economy, *Journal of Applied Mathematics,* Article ID 1260240, (2018). https://doi.org/10.1155/2018/1260240

[59] S. Rezapour, A. Imran, A. Hussain, F. Martínez, S. Etemad & M. K. A. Kaabar, Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs, *Symmetry*, 13(3), 469, (2021).

[60] S. Rezapour, S. K. Ntouyas, A. Amara, S. Etemad & J. Tariboon, Some Existence and Dependence Criteria of Solutions to a Fractional Integro-Differential Boundary Value Problem via the Generalized Gronwall Inequality, Mathematics 2021, 9(11), 1165; https://doi.org/10.3390/math9111165

[61] M. Shabibi, M. E. Samei, M. Ghaderi & Sh. Rezapour, Some analytical and numerical results for a fractional q-differential inclusion problem with double integral boundary conditions, *Advances in Difference Equations*, (2021). https://doi.org/10.1186/s13662-021-03623-2

[62] A. K. Sethi, M. Ghaderi, Sh. Rezapour, M. K. A. Kaabar, M. Inc & H. P. Masiha, Suffcient conditions for the existence of oscillatory solutions to nonlinear second order differential equations, *Journal of Applied Mathematics and Computing*, (2021), 1-18. https://doi.org/10.1007/s12190-021-01629-3

[63] F. S. Silva, On conformable fractional integral of second kind, *Mathematica Aeterna*, Vol. 8,no. 4, 199-205 (2018).

[64] S. T. M. Thabet, S. Etemad & S. Rezapour, On a new structure of the pantograph inclusion problem in the Caputo conformable setting, Bound Value Probl 2020, 171 (2020). https://doi.org/10.1186/s13661-020-01468-4

[65] M. Vivas-Cortez, P. Kórus & Juan E. Nápoles V., Some generalized hermite-hadamard-fejér inequality for convex functions, *Advances in Difference Equations* (2021), 2021:199. https://doi.org/10.1186/s13662-021-03351-7

[66] X. You, M. A. Ali, H. Budak, J. Reunsumrit & T. Sitthiwirattham, Hermite-Hadamard-Mercer-Type Inequalities for Harmonically Convex Mappings, Mathematics 2021, 9(20), 2556; https://doi.org/10.3390/math9202556

[67] X. You, H. Kara, H. Budak & H. Kalsoom, Quantum Inequalities of Hermite-Hadamard Type for r-Convex Functions, Journal of Mathematics, vol. 2021, Article ID 6634614, 14 pages, 2021. https://doi.org/10.1155/2021/6634614

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