Journal of Mathematical Extension
Vol. 16, No. 10, (2022) (5)1-14
URL: https://doi.org/10.30495/JME.2022.2182
ISSN: 1735-8299
Original Research Paper

# Necessary Conditions in Generalized Semi-Infinite Optimization with Nondifferentiable Convex Data

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**Abstract.** In this paper we focus on generalized semi-infinite optimization problem in which the index set of the inequality constraints depends on the decision vector and all emerging functions are assumed to be convex, not necessarily differentiable. We introduce three constraint qualifications which are based on the convex subdifferential, and derive some Kuhn-Tucker type necessary optimality conditions for the problem.

**AMS Subject Classification:** 90C34; 90C40; 49J52.
**Keywords and Phrases:** Generalized semi-infinite programming, Constraint qualification, Necessary optimality condition, Subdifferential.

## 1 Introduction

In the present paper, we consider the following "generalized semi-infinite programming problem" (GSIP in brief),

$$(P): \qquad \min f(x) \qquad s.t. \qquad x \in S,$$

where the feasible set $S$ is defined by

$$S := \{x \in \mathbb{R}^n \mid g(x, \ell) \geq 0, \quad \ell \in L(x)\}, \tag{1}$$

Received:September 2021 ; Accepted: December 2021

and the index set $L(x)$ is described by a finite number of inequalities as

$$L(x) := \big\{\ell \in \mathbb{R}^m \mid \vartheta_i(x,\ell) \leq 0,\ i \in I := \{1,\ldots,p\}\big\},$$

in which all appearing functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g, \vartheta_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are convex on their domains, and the set-valued mapping $x \mapsto L(x)$ is uniformly bounded on $S$; i.e., for each $x \in S$ there exists a neighborhood $\mathcal{U}_x$ of $x$ such that the set $\bigcup_{y\in\mathcal{U}_x} L(y)$ is bounded. The latter assumption implies that for each $\hat{x} \in S$, the index set $L(\hat{x})$ is compact and the set-valued mapping $x \mapsto L(x)$ is upper semi-continuous (u.s.c), i.e., for every sequences $\{x^r\} \to \hat{x}$ and $\{\ell^r\}$ with $\ell^r \in L(x^r)$, we can find a $\hat{\ell} \in L(\hat{x})$ such that $\hat{\ell}$ is a cluster point of $\{\ell^r\}_{r=1}^{\infty}$ (cf. [1]). These assumptions are *standing* throughout the whole paper. It should be noted that the mentioned standing assumptions are standard in the analyzing of GSIPs, see, e.g.,[5, 16, 17, 18, 20, 21, 22, 23] for smooth case, and [11] for nonsmooth case (of course, some papers, such as [6, 12], have not assumed the latter condition).

It is worth mentioning that when the index set $L^x$ does not depend to $x$ (i.e., $L^x$ is constant for all $x \in S$, named $L$), the problem $(P)$ decreases to the following standard "semi-infinite programming problem" (SIP),

$$\min f(x) \qquad s.t. \quad g_\ell(x) \geq 0,\ \ell \in L,$$

where, $g_\ell(x) := g(x,\ell)$ for all $(x,\ell) \in S \times L$. Optimality conditions of SIP problems have been studied by many authors; see for instance [9] in linear case, [13] in quasiconvex case, and [2, 7, 8, 10] locally Lipschitz case.

In almost all existing literature on GSIP theory, in order to establish optimality conditions for problem $(P)$, several kinds of lower-level constraint qualifications (CQ, briefly) are introduced. Extensive references to these CQs and optimality conditions, as well as their applications and historical notes, in the case that all appearing functions are continuously differentiable (while not necessarily convex), can be found in the book by Stein [21]. These CQs and optimality conditions have been extended to GSIPs with locally Lipschitz and DC (difference of convex functions) data by Kanzi and Nobakhtian [11] and by Kanzi [6, 12], respectively.

In the case when all appearing functions of GSIP are continuously differentiable, the optimality conditions under some upper-level CQs are presented only in [5, 21], but according to our latest information for the nonsmooth case nothing has been done so far. The aim of this paper is to fill this gap as the first task, for the convex case.

The structure of subsequent sections of this paper is as follows: In Sec. 2, we define required definitions and preliminary results which are requested in sequel. Section 3, which is devoted to the main results, contains introducing some upper-level CQs, expressing the relationships between them, and setting several necessary optimality conditions for problem $(P)$.

## 2 Preliminaries

In this section, we briefly address some notations and standard preliminaries which are used in the sequel, from [4, 14].

The symbols $\mathbb{R}_+$, $0_n$, and $a^\top b$ denote the set of non-negative real numbers, the zero vector in $\mathbb{R}^n$, and the standard inner product of two vectors $a, b \in \mathbb{R}^n$, respectively.

The function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if

$$\varphi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y), \qquad \forall x, y \in \mathbb{R}^n,\ \lambda \in [0,1].$$

In this section, we suppose that $\varphi$ is a convex function, defined on wholl of $\mathbb{R}^n$. The subdifferential of $\varphi$ at $x_0 \in \mathbb{R}^n$ is defined by

$$\begin{aligned} \partial\varphi(x_0) \;&:=\; \{\xi \in \mathbb{R}^n \mid \varphi(x) - \varphi(x_0) \geq \xi^\top (x - x_0), \;\; \forall x \in \mathbb{R}^n\} \\ &=\; \{\xi \in \mathbb{R}^n \mid \varphi'(x_0; d) \geq \xi^\top d, \;\; \forall d \in \mathbb{R}^n\}, \end{aligned}$$

where, $\varphi'(x_0; d)$ denotes directional derivative of $\varphi$ at $x_0$ in direction $d \in \mathbb{R}^n$, defined as

$$\varphi'(x_0; d) := \lim_{t \to 0} \frac{\varphi(x_0 + td) - \varphi(x_0)}{t}.$$

Also, we know from [4] that $\partial\varphi(x_0)$ is always a non-empty compact convex set in $\mathbb{R}^n$, and if $\varphi$ is differentiable at $x_0$, then $\partial\varphi(x_0) = \{\nabla\varphi(x_0)\}$, in which $\nabla\varphi(x_0)$ denotes the gradient of $\varphi$ at $x_0$. The following equality will be used in sequel:

$$\varphi'(x_0; d) = \max\{d^\top \xi \mid \xi \in \partial\varphi(x_0)\}. \tag{2}$$

For a convex function $\phi : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}$ and a point $(\bar{x}, \bar{y}) \in \mathbb{R}^r \times \mathbb{R}^s$, let $\partial_x \phi(\bar{x}, \bar{y}) \subseteq \mathbb{R}^r$ and $\partial_y \phi(\bar{x}, \bar{y}) \subseteq \mathbb{R}^s$ denote the partial subdifferentials of $\phi(\cdot, \cdot)$ at $(\bar{x}, \bar{y})$, which are defined as $\partial\phi(\cdot, \bar{y})(\bar{x})$ and $\partial\phi(\bar{x}, \cdot)(\bar{y})$.

Finally, we recall that for $D \subseteq \mathbb{R}^n$ and $x_0 \in \overline{D}$ (:=the closure of $D$), the contingent cone of $D$ at $x_0$, denoted by $T_D(x_0)$, is defined as the set of all vectors $z \in \mathbb{R}^n$ that can find two sequences $\{t^r\} \to 0^+$ and $\{z^r\} \to z$ in such a way $x_0 + t^r z^r \in D$ for all $r \in \mathbb{N}$. Notice that $T_D(x_0)$ is always a closed cone (generally non-convex) in $\mathbb{R}^n$, and it is convex when $D$ is convex.

## 3 Necessary Conditions

As beginning, for each $\hat{x} \in S$, we define the index set of active constraints and the lower-level problem at $\hat{x}$, respectively as

$$L^{\hat{x}} := \{\ell \in L(\hat{x}) \mid g(\hat{x}, \ell) = 0\},$$

$$(P^{\hat{x}}) : \qquad \min \ g(\hat{x}, \ell) \quad s.t. \quad \ell \in L(\hat{x}).$$

Also, the set (probably empty) of active inequalities of $(P^{\hat{x}})$ at each $\ell_0 \in L(\hat{x})$ is denoted by $I^{\hat{x}}(\ell_0)$,

$$I^{\hat{x}}(\ell_0) := \{i \in I \mid \vartheta_i(\hat{x}, \ell_0) = 0\}.$$

Clearly, each $\hat{\ell} \in L^{\hat{x}}$ is a global minimizer of the lower-level problem $(P^{\hat{x}})$, and by well-known Fritz-John first-order necessary condition (see, e.g., [4]), there exist non-negative scalars $\hat{\alpha} \geq 0$ and $\hat{\beta}_i \geq 0$ as $i \in I^{\hat{x}}(\hat{\ell})$, satisfying

$$\hat{\alpha} + \sum_{i \in I^{\hat{x}}(\hat{\ell})} \hat{\beta}_l = 1, \quad \text{and} \quad 0_m \in \partial_\ell \mathcal{L}^{\hat{x}}_{\hat{\ell}}(\hat{x}, \hat{\ell}, \hat{\alpha}, \hat{\beta}),$$

where, $\hat{\beta} := \big(\hat{\beta}_i\big)_{i \in I^{\hat{x}}(\hat{\ell})}$, and $\mathcal{L}^{\hat{x}}_{\hat{\ell}}$ denotes the Lagrangian function, defined as:

$$\mathcal{L}^{\hat{x}}_{\hat{\ell}}(x, \ell, \alpha, \beta) = \alpha g(x, \ell) + \sum_{i \in I^{\hat{x}}(\hat{\ell})} \beta_i \vartheta_i(x, \ell).$$

For each $\hat{x} \in S$, set

$$\Omega_{\hat{x}} := \bigcup_{\substack{\ell \in L^{\hat{x}} \\ (\alpha,\beta) \in F_{\ell}^{\hat{x}}}} \partial_x \pounds_{\ell}^{\hat{x}}(\hat{x}, \ell, \alpha, \beta),$$

where, for any $\ell \in L^{\hat{x}}$, the Fritz-John multipliers set $F_{\ell}^{\hat{x}}$ is defined as

$$F_{\ell}^{\hat{x}} := \Big\{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+^{|I^{\hat{x}}(\ell)|} \mid \alpha + \sum_{i \in I^{\hat{x}}(\ell)} \beta_i = 1,\ 0_m \in \partial_\ell \pounds_{\ell}^{\hat{x}}(\hat{x}, \ell, \alpha, \beta)\Big\}.$$

Notice, if all the appearing functions are continuously differentiable, $\Omega_{\hat{x}}$ coincides to $V(\hat{x})$, defined in [5]. In the following, we introduce two upper-level CQs for problem $(P)$.

**Definition 3.1.** We say that $(P)$ satisfies the Abadie constraint qualification (ACQ) at $\hat{x} \in S$ if the following implication holds:

$$\Big(d^\top \zeta \geq 0, \quad \forall \zeta \in \Omega_{\hat{x}}\Big) \implies d \in T_S(\hat{x}).$$

Also, wee say that $(P)$ satisfies the Guignard constraint qualification (GCQ) at $\hat{x} \in S$ if the following implication holds:

$$\Big(d^\top \zeta \geq 0, \quad \forall \zeta \in \Omega_{\hat{x}}\Big) \implies d \in \overline{cone}\big(T_S(\hat{x})\big),$$

where, $cone(A)$ denotes the smallest nonempty convex cone containing $A \subseteq \mathbb{R}^n$, and the closure of $cone(A)$ is denoted by $\overline{cone}(A)$, i.e.,

$$\overline{cone}(A) := \overline{cone(A)}. \tag{3}$$

We observe that ACQ is stronger than GCQ at any feasible point. Now, the Kahn-Tucker (KT) type necessary optimality condition for $(P)$ can be stated as follows.

**Theorem 3.2. (KT Condition under ACQ):** *Suppose that $\hat{x}$ is an optimal solution for $(P)$ and ACQ holds at $\hat{x}$. Then, we have*

$$0_n \in \partial f(\hat{x}) - \overline{cone}(\Omega_{\hat{x}}). \tag{4}$$

*In addition, if $cone(\Omega_{\hat{x}})$ is closed, there exist finite number $\ell^\nu \in L^{\hat{x}}$, $(\alpha^\nu, \beta^\nu) \in F_{\ell^\nu}^{\hat{x}}$, and $\mu^\nu \in \mathbb{R}_+$ as $\nu = 1, \ldots, q$, satisfying*

$$0_n \in \partial f(\hat{x}) - \sum_{\nu=1}^{q} \mu^\nu \partial_x \pounds_{\ell^\nu}^{\hat{x}}(\hat{x}, \ell^\nu, \alpha^\nu, \beta^\nu).$$

**Proof.** As the beginning, we note that (4) is equivalent to

$$\partial f(\hat{x}) \cap \overline{cone}(\Omega_{\hat{x}}) \neq \emptyset. \tag{5}$$

The proof of (5) is by contradiction. If

$$\partial f(\hat{x}) \cap \overline{cone}(\Omega_{\hat{x}}) = \emptyset,$$

according to well-known strong separation theorem (see, e.g., [14, Corollary 11.4.1]), and observing the compactness of $\partial f(\hat{x})$ and the closedness of $\overline{cone}(\Omega_{\hat{x}})$, we obtain a vector $d \in \mathbb{R}^n$, satisfying

$$d^\top \xi < 0 \leq d^\top \zeta, \qquad \text{for all } \xi \in \partial f(\hat{x}),\ \zeta \in \overline{cone}(\Omega_{\hat{x}}).$$

The above inequality and the fact that $\Omega_{\hat{x}} \subseteq \overline{cone}(\Omega_{\hat{x}})$, conclude that

$$\begin{cases} d^\top \xi < 0, & \text{for all } \xi \in \partial f(\hat{x}), \\ d^\top \zeta \geq 0, & \text{for all } \zeta \in \Omega_{\hat{x}}. \end{cases} \tag{6}$$

Thus, employing the ACQ assumption and (2), we deduce

$$\begin{cases} f'(\hat{x}; d) < 0, \\ \\ d \in T_S(\hat{x}), \end{cases} \Longrightarrow \begin{cases} \exists \delta > 0,\ f(\hat{x} + td) - f(\hat{x}) < 0, \quad \forall\ t \in (0, \delta), \\ \\ \exists \{(d^r, t^r)\} \to (d, 0^+),\ \hat{x} + t^r d^r \in S, \quad \forall r \in \mathbb{N}. \end{cases} \tag{7}$$

Hence, for large enough $\hat{r} \in \mathbb{N}$ we have

$$\begin{cases} f(\hat{x} + t^{\hat{r}} d^{\hat{r}}) < f(\hat{x}), \\ \hat{x} + t^{\hat{r}} d^{\hat{r}} \in S, \end{cases} \tag{8}$$

which contradicts the optimality of $\hat{x}$. Thus, the claimed (5) is proved.

Now, if $cone(\Omega_{\hat{x}})$ is closed, the inclusion (4) and definition of $\Omega_{\hat{x}}$ imply that

$$0_n \in \partial f(\hat{x}) - cone\Big( \bigcup_{\substack{\ell \in L^{\hat{x}} \\ (\alpha, \beta) \in F_\ell^{\hat{x}}}} \partial_x \mathcal{L}_\ell^{\hat{x}}(\hat{x}, \ell, \alpha, \beta) \Big). \tag{9}$$

On the other hand, it is easy to see that (see, e.g., [14]) if $\{M^\gamma \mid \gamma \in \Gamma\}$ is a collection of convex sets in $\mathbb{R}^n$, then:

$$cone\big( \bigcup_{\gamma \in \Gamma} M^\gamma \big) = \bigcup_{q \in \mathbb{N}} \ \bigcup_{(\mu^1, \ldots, \mu^q) \in \mathbb{R}^q_+} \ \sum_{\nu=1}^{q} \mu^\nu M^\nu.$$

From this, (9), and convexity of subdifferential, we get

$$0_n \in \partial f(\hat{x}) - \bigcup_{q \in \mathbb{N}} \bigcup_{(\mu^1,\ldots,\mu^q)\in\mathbb{R}^q_+} \sum_{\nu=1}^{q} \mu^\nu \partial_x \pounds^{\hat{x}}_{\ell^\nu}(\hat{x}, \ell^\nu, \alpha^\nu, \beta^\nu),$$

for some $\ell^\nu \in L^{\hat{x}}$ and $(\alpha^\nu, \beta^\nu) \in F^{\hat{x}}_{\ell^\nu}$. Therefore, we can find a natural number $q \in \mathbb{N}$, some non-negative scalars $\mu^1, \ldots, \mu^q$, as well as some $\ell^\nu \in L^{\hat{x}}$ and $(\alpha^\nu, \beta^\nu) \in F^{\hat{x}}_{\ell^\nu}$ for $\nu = 1, \ldots, q$, such that

$$0_n \in \partial f(\hat{x}) - \sum_{\nu=1}^{q} \mu^\nu \partial_x \pounds^{\hat{x}}_{\ell^\nu}(\hat{x}, \ell^\nu, \alpha^\nu, \beta^\nu).$$

The proof is complete. $\square$

As special case of problem $(P)$, we can consider the objective function $f$ is affine. For this special case, we can state the KT necessary condition under GCQ, as follows.

**Theorem 3.3. (KT Condition under GCQ):** *Suppose that $\hat{x}$ is an optimal solution of the following GSIP with affine objective function:*

$$\min(a^\top x + b), \quad s.t. \quad x \in S,$$

*where $S$ is defined as (1), and $(a, b) \in \mathbb{R}^n \times \mathbb{R}$. If GCQ holds at $\hat{x}$, then we have*

$$a \in \overline{cone}(\Omega_{\hat{x}}). \tag{10}$$

*In addition, if $cone(\Omega_{\hat{x}})$ is closed, there exist finite number $\ell^\nu \in L^{\hat{x}}$, $(\alpha^\nu, \beta^\nu) \in F^{\hat{x}}_{\ell^\nu}$, and $\mu^\nu \in \mathbb{R}_+$ as $\nu = 1, \ldots, q$, satisfying*

$$a \in \sum_{\nu=1}^{q} \mu^\nu \partial_x \pounds^{\hat{x}}_{\ell^\nu}(\hat{x}, \ell^\nu, \alpha^\nu, \beta^\nu).$$

**Proof.** As start, we observe that (10) is a special case of (4) with $f(x) = a^\top x + b$. Thus, if (10) does not hold, from (6) and GCQ we obtain a vector $d_* \in \mathbb{R}^n$ such that:

$$\begin{cases} d_*^\top a < 0, \\ d_* \in \overline{cone}\big(T_S(\hat{x})\big) \implies \exists\{d_*^r\} \subseteq cone\big(T_S(\hat{x})\big), \;\; d_* = \lim_{r\to\infty} d_*^r. \end{cases} \tag{11}$$

For each $d_*^r \in cone\big(T_S(\hat{x})\big)$, we can find some $s^r \in \mathbb{N}$, some non-negative scalars $\gamma_1^r, \ldots, \gamma_{s^r}^r$, and some vectors $d_1^r, \ldots, d_{s^r}^r \in T_S(\hat{x})$ such that

$$d_*^r = \sum_{\jmath=1}^{s^r} \gamma_\jmath^r d_\jmath^r. \tag{12}$$

If $(d_\jmath^r)^\top a < 0$ for some $\jmath \in \{1, \ldots, s^r\}$, since $d_\jmath^r \in T_S(\hat{x})$, we give a relation in state of (7), and we deduce a contradiction in state of (8). Thus, $(d_\jmath^r)^\top a \geq 0$ for all $\jmath \in \{1, \ldots, s^r\}$. From this and (12) we get

$$(d_*^r)^\top = \Big(\sum_{\jmath=1}^{s^r} \gamma_\jmath^r d_\jmath^r\Big)^\top a = \sum_{\jmath=1}^{s^r} \gamma_\jmath^r (d_\jmath^r)^\top a \geq 0.$$

The above inequality and the second relation in (11) imply that

$$d_*^\top a = \Big(\lim_{r\to\infty} d_*^r\Big)^\top a = \lim_{r\to\infty} (d_*^r)^\top a \geq 0,$$

which contradicts the first inequality in (11). This contradiction justifies (10). The second part of proof is quite similar to the proof of Theorem 3.2. $\square$

The value function of lower-level problem $(P^x)$ is defined as follows:

$$\Phi(x) := \begin{cases} \inf\big\{g(x,\ell) \mid \ell \in L(x)\big\} & \text{if} \quad L(x) \neq \emptyset, \\ +\infty & \text{if} \quad L(x) = \emptyset. \end{cases}$$

The following theorem has been proved in [19, Theorem 10.4] where the functions $g$ and $\vartheta_i$ as $i \in I$ are convex and differentiable.

**Theorem 3.4.** *The value function $\Phi(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex.*

**Proof.** Suppose that $(x, r)$ and $(y, s)$ are chosen in

$$epi(\Phi) := \big\{(z, \tau) \in \mathbb{R}^n \times \mathbb{R} \mid \Phi(z) \leq \tau\big\},$$

i.e., $\Phi(x) \leq r$ and $\Phi(y) \leq s$. The case when $L(x) = \emptyset$ and\or $L(y) = \emptyset$, the inequality

$$\Phi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y),$$

is trivially true. Thus, we assume that $L(x) \neq \emptyset$ and $L(y) \neq \emptyset$, and so, we can choose some $\ell_x \in L(x)$ and $\ell_y \in L(y)$ such that

$$g(x, \ell_x) \leq r \qquad \text{and} \qquad g(y, \ell_y) \leq s. \tag{13}$$

Since $\vartheta_i$ is convex as $i \in I$, for each $\lambda \in [0, 1]$, we have

$$\vartheta_i\Big(\lambda(x, \ell_x) + (1-\lambda)(y, \ell_y)\Big) \leq \lambda\vartheta_i(x, \ell_x) + (1-\lambda)\vartheta_i(y, \ell_y) \leq 0, \quad i \in I,$$

where, the last inequality holds by $\vartheta_i(x, \ell_x) \leq 0$ and $\vartheta_i(y, \ell_y) \leq 0$ $\big($ since $\ell_x \in L(x)$ and $\ell_y \in L(y)$ $\big)$. The above inequality and the fact that

$$\vartheta_i\Big(\lambda(x, \ell_x)+(1-\lambda)(y, \ell_y)\Big) = \vartheta_i\Big(\lambda x+(1-\lambda)y, \lambda\ell_x+(1-\lambda)\ell_y\Big), \qquad i \in I,$$

imply that $\lambda\ell_x + (1-\lambda)\ell_y \in L\big(\lambda x + (1-\lambda)y\big)$, and hence

$$\Phi\big(\lambda x + (1-\lambda)y\big) \leq g\Big(\lambda x + (1-\lambda)y, \lambda\ell_x + (1-\lambda)\ell_y\Big).$$

From this and (13), we get

$$\begin{aligned} \Phi\big(\lambda x + (1-\lambda)y\big) &\leq g\Big(\lambda(x, \ell_x) + (1-\lambda)(y, \ell_y)\Big) \\ &\leq \lambda g(x, \ell_x) + (1-\lambda)g(y, \ell_y) \\ &\leq \lambda r + (1-\lambda)s. \end{aligned}$$

This means $\lambda(x, r) + (1-\lambda)(y, s) \in epi(\Phi)$, and so $epi(\Phi)$ is a convex set. Since $epi(\Phi)$ is convex if and only if $\Phi(\cdot)$ is convex (see e.g., [4]), the proof is complete. □

It should be noted that due to the importance of function $\Phi(\cdot)$, many papers have worked on the upper estimate of its subdifferential. For example, [8, 11, 12, 21] show that the inclusion

$$\partial\Phi(\hat{x}) \subseteq conv(\Omega_{\hat{x}}), \tag{14}$$

holds for SIPs and GSIPs, under some suitable assumptions. Here, $conv(\Omega_{\hat{x}})$ denotes the smallest convex set contains $\Omega_{\hat{x}}$, named convex hull of $\Omega_{\hat{x}}$. It is easy to see ([14])

$$z_* \in conv(\Omega_{\hat{x}}) \iff \exists s \in \mathbb{N},\ \exists \lambda^1, \dots, \lambda^s \in \mathbb{R}_+,\ \exists z^1, \dots, z^s \in \Omega_{\hat{x}},$$
$$\text{such that} \quad z_* = \sum_{\jmath=1}^{s} \lambda^\jmath z^\jmath \text{ and } \sum_{\jmath=1}^{s} \lambda^\jmath = 1. \tag{15}$$

Motivated by [5], we define the Mangasarian-Fromovitz constraint qualification for problem $(P)$.

**Definition 3.5.** We say that $(P)$ satisfies the Mangasarian-Fromovitz constraint qualification (MFCQ) at $\hat{x} \in S$ if there exists a vector $d \in \mathbb{R}^n$ such that

$$d^\top \zeta > 0, \quad \text{for all } \zeta \in \Omega_{\hat{x}}. \tag{16}$$

Since the cheking of satisfying ACQ depends to calculating tangent cone of feasible set and this is difficult in general, the following theorem lets us checking it by initial data of problem.

**Theorem 3.6.** *Suppose that the inclusion (14) and MFCQ hold at a feasible point $\hat{x} \in S$. Then, ACQ is also satisfied at $\hat{x}$.*

**Proof.** Assume that $d \in \mathbb{R}^n$ is satisfied (16). If $\zeta_* \in conv(\Omega_{\hat{x}})$, by (15) we can find a natural number $s \in \mathbb{N}$, some scalars $\lambda^1, \dots, \lambda^s \in \mathbb{R}_+$, and some vectors $\zeta^1, \dots, \zeta^s \in \Omega_{\hat{x}}$, such that

$$\zeta_* = \sum_{\jmath=1}^{s} \lambda^\jmath \zeta^\jmath \quad \text{and} \quad \sum_{\jmath=1}^{s} \lambda^\jmath = 1.$$

Thus,

$$d^\top \zeta_* = d^\top \Big( \sum_{\jmath=1}^{s} \lambda^\jmath \zeta^\jmath \Big) = \sum_{\jmath=1}^{s} \lambda^\jmath d^\top \zeta^\jmath > 0,$$

where the last strict inequality holds by (16) and $\sum_{\jmath=1}^{s} \lambda^\jmath = 1$. From this and (14), we obtain $d^\top \xi > 0$ for all $\xi \in \partial\Phi(\hat{x})$. This means that

$$\Psi'(\hat{x}, d) = \max \left\{ d^\top \xi \mid \xi \in \partial\Phi(\hat{x}) \right\} > 0,$$

which together with definition of directional derivation implies that the exists a $\epsilon > 0$ such that $\Psi(\hat{x} + \delta d) - \Psi(\hat{x}) > 0$ for all $\delta \in (0, \epsilon)$. This inequality and $\Psi(\hat{x}) \geq 0$ conclude that

$$\begin{aligned} \Psi(\hat{x} + \delta d) > 0 \quad &\Rightarrow \quad \Big( g(\hat{x} + \delta d, \ell) > 0, \quad \forall \ell \in L(\hat{x} + \delta d),\ \forall \delta \in (0, \epsilon) \Big) \\ &\Rightarrow \quad \Big( \hat{x} + \delta d \in S, \quad \forall \delta \in (0, \epsilon) \Big) \ \Rightarrow \ d \in T_S(\hat{x}). \end{aligned}$$

Summarizing, we proved that

$$D := \big\{ d \in \mathbb{R}^n \mid d^\top \zeta > 0, \quad \text{for all } \zeta \in \Omega_{\hat{x}} \big\} \subseteq T_S(\hat{x}).$$

Now, if a vector $d_* \in \mathbb{R}^n$ is given such that $d_*^\top \zeta \geq 0$ for all $\zeta \in \Omega_{\hat{x}}$, we have $d_* \in \overline{D}$, and by above inclusion we obtain $d_* \in \overline{T_S(\hat{x})}$. From this and closedness of $T_S(\hat{x})$, we conclude that

$$\Big( d_*^\top \zeta \geq 0, \quad \forall \zeta \in \Omega_{\hat{x}} \Big) \implies d_* \in T_S(\hat{x}),$$

which deduces the satisfying ACQ at $\hat{x}$. □

## 4 Conclusion

As the final point of this paper, we compare the results obtained above with some known developments. In the case where all function $g$ and $\vartheta_i$ as $i \in I$ are linear, Theorem 3.3 is proved in [17]. Also, if all the appealing functions $f, g, \vartheta_i$ as $i \in I$ are continuously differentiable, Theorem 3.2 has been justified in [17, 21]. The following theorem which was proved in [21] can be deduced from Theorem 3.6. It is noteworthy that [5, Corollary 4.1] can not be deduced from Theorem 3.6, because there assumes only the continuously differentiability of the functions, not their convexity.

**Theorem 4.1.** *[21] Suppose that all the functions $f, g$, and $\vartheta_i$ as $i \in I$ are continuously differentiable and convex. Let $\hat{x}$ be an optimal solution of $(P)$ and MFCQ holds at $\hat{x}$. Then, there exist finite number $\ell^\nu \in L^{\hat{x}}$, $(\alpha^\nu, \beta^\nu) \in F^{\hat{x}}_{\ell^\nu}$, and $\mu^\nu \in \mathbb{R}_+$ as $\nu = 1, \ldots, q$, satisfying*

$$\nabla f(\hat{x}) - \sum_{\nu=1}^{q} \mu^\nu \nabla_x \pounds^{\hat{x}}_{\ell^\nu}(\hat{x}, \ell^\nu, \alpha^\nu, \beta^\nu) = 0_n.$$

**Acknowledgements**
The author expresses his gratitude to the referee of this paper, who carefully read and made a number of valuable suggestions.

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