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Original Research Paper

Solving a Class of Variable-Order Differential Equations by the Ritz Method and Genocchi Functions

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Abstract. In this article, a class of variable-order differential equations is solved by the Ritz-approximation. Firstly, the unknown function is estimated using the Ritz-approximation and Genocchi polynomials as the basis functions. Then, by collocation method and preference of Genocchi roots as the node points, a set of algebraic equations is obtained. This system of nonlinear equations is solved by *Mathematica 10* software. Finally, by solving some numerical examples and comparing the achieved results with other methods, the validity and efficiency of the presented method are shown.

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1 Introduction

Fractional calculus indicates the differential and integral with fractional order. In recent years, many phenomena in the real world, such as finance [10], medical [15], transfer of heat [18], Geo-Hydrology [3], are modelled by fractional calculus. In [26], an iterative scheme to solve the fractional-order model for brushless DC motor is suggested. Fractal calculus is a field of fractional for the functions that are not integrable or differentiable in their domain. Fractal calculus is used in electromagnetic fields and various branches of physics. The stability of a class of fractal-order equation in [23] is studied. Fractional equations can be classified into fractional differential equations, optimal control problems, boundary and initial value problems, fractional partial differential, and partial integro-differential equations [5].

Recently, it has been observed that many phenomena in dynamic processes can not be characterized by constant fractional operators [7, 14]. So, the order of differential operator possesses an active matter that can vary a function of time, space, or parameters of the system. Ross and Samko, in 1993, introduced the concept of variable-order(VO) fractional operators and studied their properties [16]. After that, the outlook of VO fractional calculus is generalized by many scholars, and its applications are investigated in many fields such as viscoelasticity oscillators [19], petroleum engineering [13], signal processing [21], and engineering [11]. The VO operators have the variable exponent kernel so, getting the analytical solutions is complex and in, many cases impossible. So, many researchers investigate numerical methods. In [17, 6] the VO differential equations are solved by the finite difference method. Bhrawey and Zaky [4] suggested the shifted Legendre polynomials to solve the cable VO differential equation. A numerical approximation is applied by Tavares et al. [22] to solve VO partial differential equations. The authors in [12], with the kernel method, solved the VO problems by boundary conditions.

In this study, a class of VO differential equations is brought up as:

$$\begin{cases} D^{\alpha(t)}\mathcal{X}(t) + a(t)\mathcal{X}'(t) + b(t)\mathcal{X}(t) + c(t)\mathcal{X}(\tau(t)) = f(t), \quad t \in [0,1], \\ \mathcal{X}(0) = \lambda_0, \quad \mathcal{X}(1) = \lambda_1, \end{cases} \tag{1}$$

where $a(t), b(t), c(t) \in C^2[0,1], \alpha(t), \tau(t), f(t) \in C[0,1], 0 \leq \tau(t) \leq 1$ and $D^{\alpha}(t)$ represents the VO Caputo derivative that, $n-1 \leq \alpha(t) < n$.

This article is arranged as follows: In Sect. 2, certain preliminary of VO calculus are introduced. In Sect. 3, the Genocchi polynomials and function approximation are expressed. In Sect. 4, the numerical method is explained. In Sect. 5, the error bound is shown. In Sect. 6, the numerical conclusions achieved by this method are announced. Results display the presented method be efficient for detecting the numerical yields of VO differential equations.

2 Preliminaries

Some primary definitions of VO operators, in this section, will be mentioned.

Definition 2.1. The VO integral operator of function $\mathcal{X}(t)$ in Riemann-Liouville sense with the $\alpha(t) \geq 0$ order is defined as [20]

$$I^{\alpha(t)}\mathcal{X}(t) = \frac{1}{\Gamma(\alpha(t))}\int_0^t \frac{u(x)}{(t-x)^{1-\alpha(t)}}\mathrm{d}x \qquad \alpha(t) > 0,$$

the Gamma function is shown as $\Gamma()$.

Definition 2.2. The VO Caputo derivative operator for a differentiable and continuous function $u(x)$ is defined as [8]

$$^{C}D^{\alpha(t)}\mathcal{X}(t) = \frac{1}{\Gamma(n-\alpha(t))}\int_0^t (t-x)^{n-\alpha(t)-1}\frac{d^n\mathcal{X}(t)}{dx^n}\mathrm{d}x, \tag{2}$$

where $n-1 < \alpha(t) < n,\ n \in \mathbb{N}$.

If the values of $\alpha(t)$ are an integer, then the VO Caputo derivative becomes identical to the classical derivative.

Corollary 2.3. *As a direct conclusion from (2), the VO Caputo derivative is a linear operator such as the fractional and integer-order derivative.*

$$D^{\alpha(t)}(\mu g(t) + \lambda f(t)) = \mu D^{\alpha(t)}g(t) + \lambda D^{\alpha(t)}f(t),$$

where μ and λ are constants.

Corollary 2.4. *Let* $\mathcal{X}(t) = t^k, k > 1$*, then* [1]

$$I^{\alpha(t)}\mathcal{X}(t) = \frac{\Gamma(1+k)}{\Gamma(\alpha(t)+1+k)}t^{k+\alpha(t)}, \quad k > 1,$$

and

$$D^{\alpha(t)}\mathcal{X}(t) = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha(t))}t^{k-\alpha(t)}, & k \in \mathbb{N}, k \geq n, \\ 0, & \textit{otherwise}, \end{cases}$$

that $n = [\alpha(t)]$*. Furthermore*

$$D^{\alpha(t)}C = 0 \quad (C \textit{ is a constant}).$$

3 The Properties of the Genocchi Polynomials

In this section, first, the Genocchi function is introduced then the function approximation is described.

3.1 Genocchi polynomials

The Genocchi polynomials of degree n can be defined with generating function [2]

$$\frac{2xe^{xt}}{1+e^x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.G_n(t), \qquad (|t| < \pi). \tag{3}$$

Several first values of the Genocchi polynomials are

$$G_1(t) = 1, \quad G_2(t) = 2t-1, \quad G_3(t) = 3t^2 - 3t,$$
$$G_4(t) = 4t^3 - 6t^2 + 1, \quad G_5(t) = 5t^4 - 10t^3 + 5t.$$

Also, $g_n := G_n(0)$ is named the Genocchi numbers, be defined with the generating function

$$\frac{2x}{1+e^x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.g_n, \qquad (|x| < \pi).$$

The Genocchi polynomial in (3) can be presented by the Genocchi number as follows

$$G_n(t) = \sum_{l=0}^{n} \binom{n}{l} x^l.g_{n-l}.$$

In the following, some properties of the Genocchi polynomials are recalled

$$\begin{aligned}
&g_n + G_n(1) = 0, \quad n > 1,\\
&\frac{dG_n(t)}{dt} = nG_{n-1}(t), \quad n \geq 1,\\
&\int_0^1 G_m(t)G_n(t)\mathrm{d}t = \frac{2(-1)^n n!m!}{(n+m)!} g_{n+m}(x), \quad m, n \geq 1.
\end{aligned}$$

3.2 Function approximation

Assume $H = span\{G_1(t), \ldots, G_N(t)\} \subset L^2[0,1]$ be space with the finite-dimensional. Any arbitrary $\mathcal{X}(t) \in L^2[0,1]$ has a best unique approximation $\tilde{\mathcal{X}}(t)$ in H such as [9]

$$\exists \tilde{\mathcal{X}}(t) \in H; \quad \forall y(t) \in H \qquad \|\mathcal{X}(t) - \tilde{\mathcal{X}}(t)\| \leq \|\mathcal{X}(t) - y(t)\|.$$

We can decompose the L^2 as $L^2 = H \oplus H^\perp$ since $H \subset L^2$ is close subspace. Also, we have $\mathcal{X}(t) = y^\perp(t) + y(t)$, therefore $\mathcal{X}(t) - y(t) \in H^\perp$ then

$$\forall y(t) \in H, \quad \left\langle \mathcal{X}(t) - \tilde{\mathcal{X}}(t), y(t) \right\rangle = 0,$$

where $\langle .,. \rangle$ shows the inner product. On the other hand, as $\tilde{\mathcal{X}} \in H$ then $\{c_i\}_{i=0,\ldots,n}$ are unique coefficients that

$$\mathcal{X}(t) \cong \tilde{\mathcal{X}}(t) = \sum_{n=0}^{N} c_n G_n(t).$$

4 Description Method

Ritz, in 1908, introduced an approximation method to solve the initial and boundary differential equations. Now we apply this method for estimating VO differential equations.
Suppose $\mathcal{X}(t) \in L^2[0,1]$. we approximate $\mathcal{X}(t)$ as

$$\mathcal{X}(t) \cong \tilde{\mathcal{X}}(t) = \sum_{n=0}^{N} c_n \phi(t) G_n(t) + w(t) \tag{4}$$

where $\phi(t)$ is a non-unique function that must be chosen in some way that $\phi(t_n)$ is not zero. Meanwhile, the (4) satisfy the homogenous initial problem (1). Also, $w(t)$ that is called "*satisfier function*" must be satisfied in boundary conditions as

$$w(0) = \mathcal{X}(0), \quad w(1) = \mathcal{X}(1).$$

Several ways are to choose the satisfier function, but regularly, interpolation is applied. Experience perceived that when selected satisfier function is closer to the exact solution, obtained numerical results are cost-effective computational [25].
Concerning boundary conditions (1), the $\phi(t)$ is considered as

$$\phi(t) = (t-1)t.$$

Now the unknown function $\mathcal{X}(t)$ is estimated as

$$\mathcal{X}(t) \cong \mathcal{X}_N(t) = \sum_{n=0}^{N} c_n t(t-1)G_n(t) + w(t). \tag{5}$$

This approximation (5) is substituted in (1) and yields

$$D^{\alpha(t)}\mathcal{X}_k(t) + a(t)\mathcal{X}_k'(t) + b(t)\mathcal{X}_k(t) + c(t)\mathcal{X}_k(\tau(t)) = f(t), \tag{6}$$

and using the collocation method for (6) in $t_i = \frac{i}{1+N}, i = 1, 2, \cdots, N$ points, an algebraic system of linear or nonlinear equations is achieved. This resulting system is solved with *Mathematica 10* software and unknown coefficients c_n are obtained.

5 Error Analysis

In the last section, H is considered the collection of Geocchi functions on $[0, 1]$. Suppose $\mathcal{X}(t) \in C^{N+1}[0, 1]$ and $\tilde{\mathcal{X}}(t)$ be the best approximation for $\mathcal{X}(t)$. Also, $\mathcal{X}_N(x) \in H$ in $[0, 1]$ is defined as

$$\mathcal{X}_N(t) = \sum_{n=1}^{N} \frac{1}{n!}\mathcal{X}^{(n)}(\xi)(t-\xi)^n, \quad \xi \in [0, 1].$$

Utilizing Taylor polynomials is inferred that

$$|\mathcal{X}(t) - \mathcal{X}_N(t)| \leq \frac{(t-\xi)^{N+1}}{(N+1)!} \sup_{t\in[0,1]} |\mathcal{X}^{N+1}(t)|. \tag{7}$$

Consider $M = \sup_{t\in[0,1]} |\mathcal{X}^{N+1}(t)|$ and using (7), we have

$$\begin{aligned}
\|\mathcal{X} - \tilde{\mathcal{X}}\|^2_{L^2[0,1]} &\leq \|\mathcal{X} - \mathcal{X}_N\|^2_{L^2[0,1]} \\
&= \int_0^1 |\mathcal{X}(t) - \mathcal{X}_N(t)|^2 \mathrm{d}t \\
&\leq \int_0^1 \left| \mathcal{X}^{1+n}(\xi) \frac{(t-\xi)^{(1+n)}}{(1+n)!} \right|^2 \mathrm{d}t \\
&= \frac{M^2}{[(1+N)!]^2} \int_0^1 |(t-\xi)|^{1+2N},
\end{aligned}$$

therefore,

$$\int_0^1 |(t-\xi)|^{2+2N} \mathrm{d}x \leq \frac{2}{3+2N}.$$

Finally, we conclude that

$$\| \mathcal{X} - \mathcal{X}_N \|_{L^2[0,1]} \leq \frac{M}{1+N} \sqrt{\frac{2}{3+2N}}, \tag{8}$$

when N (the number of Genocchi functions) is increasing, the right value of (8) inclines to zero, which means the approximation converges to $u(x)$.

6 Illustrative Examples

The proposed numerical method is used for solving two instances. The simulation is performed for $N = 5$ (the number of truncated terms in the Genocchi series is N). The collocation points are $x = \frac{i}{N+1}$, $i = 1, \cdots, k$. The fractional-order is $1 \leq \alpha(x) < 2$. The absolute error is calculated with $E_N(x) = |u(x) - u_N(x)|$. The result is compared with the previous work and, tables; graphs, indicate that with a low volume of conclusions, high accuracy is obtained.

Example 6.1. Assume the boundary value VO differential equation as

$$\begin{cases} D^{\alpha(x)}u(x) + 4u(x) + cos(x)u'(x) + 5u(x^2) = f(x), & x \in [0,1], \\ u(0) = 0, \quad u(1) = 1, \end{cases} \tag{9}$$

where $f(x) = \frac{2x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 5x^4 + 4x^2 + 2xcos(x)$ and $\alpha(x) = \frac{5+sinx}{4}$. $u(x) = x^2$ be the exact solution.

According to the last section, firstly, $u(x)$ is estimated by the Ritz-approximation. The *satisfier function* that applies in boundary conditions (9) be as follows:

$$w(x) = x.$$

Then, the unknown c_k coefficients are calculated with the collocation method and x points:

$$\begin{aligned} c_0 &= 1, & c_1 &= -5.0445 \times 10^{-15}, \\ c_2 &= -1.8983 \times 10^{-14}, & c_3 &= -3.7484 \times 10^{-15}, \\ c_4 &= -7.3741 \times 10^{-15}. \end{aligned}$$

These Coefficients specify the $\tilde{u}(x)$. The error of the method is drawn in Fig 1. The error of this method is compared with RKSM (reproducing kernel splines method) [24] in Table 1.

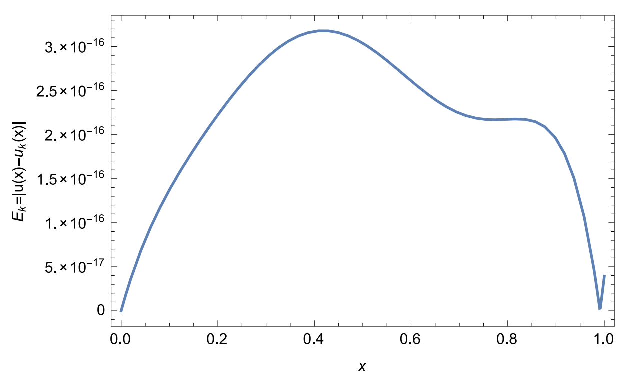

Figure 1: The absolute error of the present method in Example 6.1.

The above table detected the advantages of the proposed method, since selecting much fewer points than last work, and high accuracy be obtained.

Table 1: Comparison of absolute errors in Example 1.

x	E_{20}[24]	E_5 present method
0.1	$2.79655E-14$	$1.38778E-16$
0.2	$3.12597E-14$	$2.22045E-16$
0.3	$3.50414E-14$	$3.33067E-16$
0.4	$3.91076E-14$	$3.33067E-16$
0.5	$4.37705E-14$	$3.33067E-16$
0.6	$4.86278E-14$	$3.05311E-16$
0.7	$5.37903E-14$	$2.498E-16$
0.8	$5.92859E-14$	$2.22045E-17$
0.9	$6.51701E-14$	$2.35922E-16$

Example 6.2. Assume the boundary value VO fractional problem of the form [24]

$$\begin{cases} D^{\alpha(x)}u(x) + 2u(x) + e^x u'(x) + 8u(e^{x-1}) = f(x), \quad x \in [0,1], \\ u(0) = 4, \quad u(1) = 9, \end{cases} \tag{10}$$

where $f(x) = \frac{2x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 2(x^2+4x+4) + 8(4e^{x-1} + e^{2x-2} + 4) + e^x(2x+4)$ and $\alpha(x) = \frac{1}{4}(6 + \cos x)$. The exact solution is $u(x) = 4 + 4x + x^2$. The satisfier function, with respect to equation (10) is considered as $w(x) = (x+2)^2$. Using the discussed method the c_k are as:

$$\begin{aligned} c_0 &= 5.4656 \times 10^{-14}, \quad c_1 = 3.1455 \times 10^{-14}, \quad c_2 = 1.9810 \times 10^{-13}, \\ c_3 &= 2.8209 \times 10^{-14}, \quad c_4 = 7.762 \times 10^{-14}. \end{aligned}$$

The absolute error be drawn in Fig 2, on $[0,1]$.

7 Conclusion

In the current paper, a class of VO fractional differential equations is estimated via Genocchi polynomials and Ritz-approximation. Then the new approximated equation is solved by the collocation method. The error bound of the scheme is explained. Solved examples demonstrated the reliability and accuracy of the mentioned method.

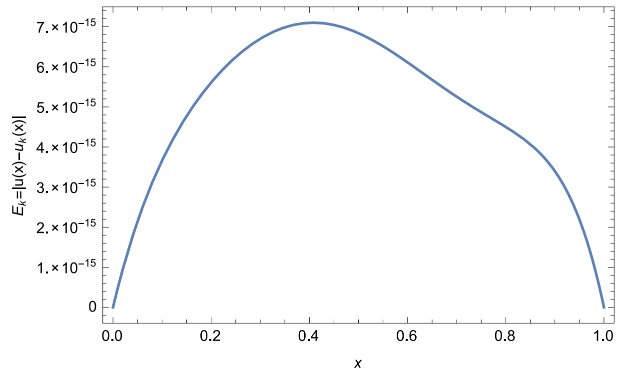

Figure 2: The absolute error of the proposed method in Example 2.

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