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Original Research Paper

Domination Number Based on Fuzzy Bridge And Application

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Abstract. This paper considers the notion of α-strong dominating set based on fuzzy bridges in fuzzy graphs and computes the domination number of wheel fuzzy graphs and complete (multi) partite fuzzy graphs. In this regard, we consider the fuzzy cycles and compute the domination number of wheel fuzzy graphs, supremum center-based wheel fuzzy graph, infimum center-based wheel fuzzy graph, and complete (multi) partite fuzzy graphs with any given fuzzy vertices. It investigated the relationship between the domination number of wheel fuzzy graphs and complete (multi) partite fuzzy graphs via some critical fuzzy vertices in these classes of fuzzy graphs. The new conception of domination number of wheel fuzzy graphs and complete (multi)partite fuzzy graphs based on fuzzy bridges, was given for the first time in this paper and we found an Algorithm in this regard. In the final, we apply the domination number of fuzzy graphs in modeling of real problems of complex networks.

AMS Subject Classification: 05C72, 05C69, 03E72
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1 Introduction

Graph theory as an important branch of mathematics was introduced as an algebraic structure as a graph that is a convenient tool to model real-life problems and has many applications in other sciences [6]. A graph on any given set considers a relationship between elements set (as objects or vertices) and describes this relationship if it is a weighted graph. It is an ideal condition if proper weights are known, but in most situations, the weights may not be known, and the relationships are hesitant in a natural sense. With the advent of the fuzzy graph, the importance of this theory increased and the fuzzy graph as a generalization of a graph provides more information on real-life problems. Based on Zadeh's fuzzy relations [21], Kaufman [12], gave the first definition of a fuzzy graph, and later Rosenfeld, Yeh and Bang introduced and extended the basic algebraic structures of fuzzy graph theory. Due to the importance of fuzzy graphs, many researchers have worked in this field and a lot of research has been produced such as m-polar fuzzy graph structures and fuzzy hypergraph structures [5, 12]. Today, Fuzzy graphs have important applications and are used in complex networks such as computer science, wireless sensor networks, machine learning, and other complex hyper networks. In addition, dominating set in graph and domination number was studied by Ore in 1962, [18], E. J. Cockayne and S. T. Hedetnieme, [9], which were motivated by solving some problems, such as queens problem, locating radar stations, communications in a network, nuclear power plants, modeling biological networks, modeling social networks, facility location, coding theory, etc. So fuzzy dominating set in fuzzy graphs (as one of the most important substructures of fuzzy graphs) as a generalization of dominating set in graphs in similar applications but uncertainty data were introduced by A. Somasundaram and S. Somasundaram in 1998, [19]. Recently, diverse and practical works on domination in fuzzy graphs were also done by researchers such as domination in fuzzy incidence graphs based on valid edges [1], the energy of double dominating bipolar fuzzy graphs [2], novel decision-making method based on domination in m-polar fuzzy graphs [3], certain types of domination in m-polar fuzzy graphs [4], a novel concept of domination in m-polar interval-valued fuzzy graph and its application [7], on computing domination set in intuitionistic fuzzy graph[8], domination of bipolar fuzzy graphs in various settings [10], double domination on intuitionistic fuzzy graphs [15], double vertex-edge domination in graphs: complexity and algorithms [16], dominating broadcasts in fuzzy graphs [17] and domination of vertex-edges in bipolar fuzzy graphs [20]. Hamidi et al. introduced a novel concept of fuzzy dominating sets as α-strong dominating set and fuzzy domination number in fuzzy graphs based on fuzzy bridges. They computed fuzzy domination number of special fuzzy graphs as strong cycle fuzzy graphs and complete fuzzy graphs based on an application Algorithm [11]. Regarding these points, we consider some notations of fuzzy dominating sets in fuzzy graphs, such as effective dominating set, strong (weak) dominating set, and (α, β)-strong dominating sets in fuzzy graphs and apply the novel notion of fuzzy dominating sets as α-strong dominating set and fuzzy domination number in fuzzy graphs based on fuzzy bridges. The concept of α-strong dominating sets in fuzzy graphs is stronger than (α, β)-strong dominating sets, such that in α-strong dominating sets equality of fuzzy values does not occur, and strict inequality of fuzzy values, play an important role. This study investigates some main results regarding fuzzy bridge-based domination number of fuzzy graphs and provides some theorems for simplifying calculations. Our motivation for his work is computations of fuzzy domination numbers of complete fuzzy graphs based on some applications in real-world problems. We try to compute the fuzzy domination numbers of a wheel fuzzy graph and complete (multi)partite fuzzy graph with distinct vertices and indistinct vertices and so to compute the fuzzy domination numbers of any wheel fuzzy graph and complete (multi)partite fuzzy graph. In the final, we introduce some examples of applications of fuzzy domination numbers of fuzzy graphs in the real world.

Motivation, main advantages, contribution, and application: The proposed method in this study is to introduce the smallest domination number in fuzzy graphs with fuzzy cyclic in comparison to previous definitions that theorems, examples, and applications confirm. The main motivation of this work is the extraction of the minimum domination number in fuzzy graphs and so introduce a novel method for computation of domination number in this paper. Another advantage is the different answers to real-world applications, so, there's an open way to compare the definitions. As it's mentioned in upcoming sections, optimal numbers and optimal sets provide proper answers to real-world applications. So there's a big gap in research concerning these notions and their applications. These issues imply the main advantages of new definitions, hence new definition highlights these problems and their answers. These are the statement of problems that highlight motivation for us to fill the gap in previous research and previous study. Another objective is to get some mathematical results that conclude general results to have specific mathematical results and specific real-world problems. So another novelty of the conducted research is to introduce a new definition to get some chains of mathematical results which are applied to solve some specific real-world applications. Most previous methods in the computation of the domination number of fuzzy graphs, were not optimal, since these methods considered either the fuzzy value in edges or fuzzy value in vertices to compute of domination number of fuzzy graphs, which is a defect. Indeed previous methods, can't be the appropriate modeling in application in the real world. Therefore, this motivated us to present this method to solve this problem by combining the fuzzy value in edges and fuzzy value in vertices to compute of domination number of fuzzy graphs. Thus the novelty of conducted research is to introduce the new definition, comparative usages of results in mathematical viewpoint and applications, specific mathematical results, addressing some issues in applications alongside solutions to them. In section 2, the preliminaries are represented. In section 3, a new notion of domination number in wheel fuzzy graphs via a fuzzy bridge and with a combination of vertex and edge is introduced. Also, basic properties and clarifications are mentioned in this section. In subsection 3.1, the concept of infimum center-based wheel strong fuzzy graph, in section 3.2, the concept of supremum center-based wheel strong fuzzy graph is described. In section 4, the notations of the fuzzy bridge in complete bipartite fuzzy graphs, and complete multipartite fuzzy graphs are introduced. In section 5, applications of fuzzy domination number ((supremum, infimum) center-based wheel strong, complete (bipartite, multipartite)) fuzzy graphs and the study, literature review, and presentation are described. The main application of this work is in complex networks that add some examples in this paper.

2 Preliminaries

In what follows, we recall some results from [14], that are needed in our work. Let $G^* = (V, E)$ be a simple graph. Then an algebraic structure $G = (V, \sigma, \mu)$ is called a fuzzy graph on G^*, if $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ are fuzzy subsets and for all $xy \in E, \mu(xy) = \mu(x,y) \leq \sigma(x) \wedge \sigma(y)$. It is called σ as fuzzy vertex set and μ as fuzzy edge set of G. A fuzzy graph $G = (\sigma, \mu)$ is called a *complete fuzzy graph* if for all $x, y \in V$, we have $\mu(xy) = \sigma(x) \wedge \sigma(y)$ and is called a *strong fuzzy graph* if for all $xy \in E$, we have $\mu(xy) = \sigma(x) \wedge \sigma(y)$. If $G = (\sigma, \mu)$ and $H = (\tau, \nu)$ are fuzzy graphs, then H is called a *partial fuzzy subgraph* of G, if $\sigma \subseteq \tau, \mu \subseteq \nu$ and partial fuzzy subgraph H is called a *spanning fuzzy subgraph* of G, if $\tau = \sigma$. A *path* $P(x,y)$ between x, y of length n(will denote by $l(P(x,y)) = n$) is a sequence of distinct vertices $P : x = x_0, x_1, \cdots, x_n = y$ such that for all $i \in \{1, 2, \cdots, n\}, \mu(x_{i-1}x_i) > 0$ and $s(P) = \bigwedge_{i=1}^{n} \mu(x_{i-1}x_i)$ is defined as its strength. The *strength of connectedness* between two vertices x and y is defined by $\mu^\infty(xy) = \max\{s(P) \mid P \text{ is a path between } x, y\}$(a strongest path joining any two vertices x, y has strength $\mu^\infty(xy)$). A path $P : x_0, x_1, \cdots, x_n$ is called a *cycle* of length n, if $x_0 = x_n$ and $n \geq 3$. A fuzzy graph $G = (V, \sigma, \mu)$ is called a *wheel fuzzy graph*(it's denoted by W_n), if $V = V_1 \cup V_2$, that $V_1 = \{c\}$(it's called the center of wheel fuzzy graph), $V_2 = \{x_1, x_2, \cdots, x_n\}$, where $P : x_1, x_2, \cdots, x_n$ is a cycle, $n \geq 3$, and for all $y \in V_2$, we have $\mu(cy) > 0$. A fuzzy graph $G = (V, \sigma, \mu)$ is called a *multipartite fuzzy graph*, if $V = \bigcup_{i=1}^{m} V_i$, such that for all $1 \leq i \neq j \leq m$, $V_i \cap V_j = \emptyset$, for all $x, y \in V_i, \mu(xy) = 0$ and $\mu(xy) \neq 0$ implies that $(x,y) \in V_i \times V_j$ or $(x,y) \in V_j \times V_i$ and it is called a *complete multipartite fuzzy graph* (it's denoted by $K_{n_1, n_2 \cdots, n_m}$), if $\mu(xy) = 0$, implies that there exists $1 \leq i \leq m$ such that $x, y \in V_i$, where $|V_i| = n_i$. A (complete)multipartite fuzzy graph is called a (*complete*)*bipartite fuzzy graph*, if $m = 2$.

In the following we introduce some types of dominating sets in fuzzy graphs.

Definition 2.1. Let $G = (\sigma, \mu)$ be a fuzzy graph and $D \subseteq V$. Then

(i) (**A. Somasundaram 1998**)[19] $D \subseteq V$ is said to be an *effective dominating set* in G, if for every $v_1 \in V \setminus D$, there exists $v \in D$ such that $\mu(vv_1) = \sigma(v) \wedge \sigma(v_1)$(it called that v dominates v_1). If $\mathcal{D}$ is a set of all effective dominating sets in G, define $\gamma(G) = \bigwedge_{D\in\mathcal{D}} \sum_{v\in D} \sigma(v)$ as an *effective domination number* of G;

(ii) (**O. T. Manjusha 2014**)[13] $D \subseteq V$ is said to be *dominating set*, if for every $v_1 \in V \setminus D$, there exists $v \in D$ such that $\mu(vv_1) \geq \mu'^{\infty}(v, v_1)$(it called that v dominates v_1 as (α, β)-strong). If $\mathcal{D}$ is the set of all dominating sets in G, define $\gamma(G) = \bigwedge_{D\in\mathcal{D}} \sum_{v\in D} \sigma(v)$ as *domination number* of G.

3 Fuzzy Bridge In Wheel Fuzzy Graphs

In this section, we computed the fuzzy domination number of wheel fuzzy graphs, based on center of wheel fuzzy graphs.

Let $G = (\sigma, \mu)$ be a fuzzy graph and $xy \in E$. Then an edge $xy \in E$ is called an *α-strong* edge(fuzzy *bridge*), if $\mu(xy) > \mu'^{\infty}(xy)$, where G' is the partial fuzzy subgraph of G obtained by deleting the edge xy, that is, $G' = (\sigma, \mu')$, where $\mu'(xy) = 0$ and $\mu' = \mu$ for all other pairs.

Definition 3.1. Let $G = (\sigma, \mu)$ be a fuzzy graph, $D \subseteq V$ and $x, y \in V$. Then

(i) if $\mu(xy) > \mu'^{\infty}(xy)$, then will say a vertex x *dominates* a vertex y, α-strongly in G and will denote it by $x \overset{\alpha}{\hookrightarrow} y$,

(ii) a set D^α is called an *α-strong dominating set* in G, if for every $y \in V \setminus D$, there exists $x \in D$ such that $x \overset{\alpha}{\hookrightarrow} y$;

(iii) the weight of D^α is defined by $\omega(D^\alpha) = \sum_{\substack{x\in D^\alpha \\ xy\in\mathcal{E}(D^\alpha,x)}} \sigma(x)\mu(xy)$, where $\mathcal{E}(D^\alpha, x) = \{xy \in E \mid x \overset{\alpha}{\hookrightarrow} y\}$, $\mathcal{E}(D^\alpha) = \bigcup_{x\in D^\alpha} \mathcal{E}(D^\alpha, x)$ and $x \in D$;

(*iv*) the fuzzy *domination number* of G is defined by $\gamma_t(G) = \bigwedge_{D^\alpha \in \mathcal{D}^\alpha} \omega(D^\alpha)$, where $\mathcal{D}^\alpha(G) = \{D^\alpha \mid D^\alpha$ is α-*strong* dominating set in $G\}$ and the α-strong dominating set with minimum weight is called by *vertex fuzzy dominating set.*

Example 3.2. Let $G = (\sigma, \mu)$ be a fuzzy graph in Figures 1 and 2.

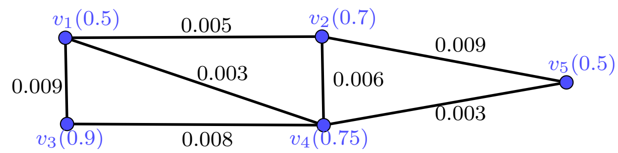


Figure 1: Fuzzy graph G

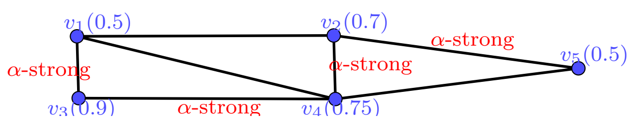


Figure 2: Fuzzy graph G with its α-strong edges

Computations show that $\mathcal{D}^\alpha(G) = \big\{D_1^\alpha = \{v_2, v_3\}, D_2^\alpha = \{v_2, v_3, v_1\}, D_3^\alpha = \{v_2, v_3, v_1, v_4\}, D_4^\alpha = \{v_2, v_3, v_1, v_4, v_5\}, D_5^\alpha = \{v_1, v_5, v_4\}, D_6^\alpha = \{v_1, v_5, v_4, v_2\}\big\}$ so $Min(\mathcal{D}) = \{D_1^\alpha, D_5^\alpha\}$. Then for all $D^\alpha \in \mathcal{D}^\alpha(G)$,

$$\mathcal{E}(v_1) = \{v_1v_3\}, \mathcal{E}(v_2) = \{v_2v_4, v_2v_5\}, \mathcal{E}(v_3) = \{v_1v_3, v_3v_4\},$$
$$\mathcal{E}(v_4) = \{v_4v_3, v_4v_2\}, \mathcal{E}(v_5) = \{v_5v_2\},$$

and so $\mathcal{E} = \{v_2v_5, v_2v_4, v_3v_4, v_1v_3\}$. Now,

$$\gamma_t(G) = \bigwedge_{D \in Min(\mathcal{D})} \omega(D) = T_{min}(\omega(D_1^\alpha), \omega(D_5^\alpha)) = T_{min}(0.0258, 0.0195) = 0.0195.$$

So $\gamma_t(G) = 0.0195$ and $D_5^\alpha = \{v_1, v_5, v_4\}$ is the vertex dominating set.

In Table 1, we present the computations to compare the dominating number based on our definition and other definitions, which determine

Table 1: Mathematical Discussion on Definitions Based on Fuzzy graph Figure 1.

Definitions	Dominating Set	Domination Number
New Definition	$\{v_1, v_5, v_4\}$	0.0195
(**A. Somasundaram 1998**)[19]	$\{v_1, v_2, v_3, v_4, v_5\}$	3.35
(**O. T. Manjusha 2014**) [13]	$\{v_1, v_2\}$	1.2

a comparison in our results.

In [11], Hamidi et al. computed fuzzy domination number of complete fuzzy graphs and cyclic strong fuzzy graphs as follows.

Theorem 3.3. *Let $K_n = (\sigma, \mu)$ be a complete fuzzy graph, $V = \{v_1, v_2, \ldots, v_n\}$ and $k = \max\{\mu(xy) \mid x, y \in V\}$. If $\sigma(v_1) > \sigma(v_2) > \sigma(v_3) > \ldots > \sigma(v_{n-1}) > \sigma(v_n)$ is an ascending sequence, then*

(i) $\gamma_t(K_n) = \sigma^2(v_{n-1})$.

(ii) $\{v_2, v_3, \ldots, v_{n-1}, v_n\}$ *is the vertex fuzzy dominating set.*

(i) $\gamma_t(K_n) = 0$ *if and only if complete fuzzy graph (K_3, k) is a fuzzy subgraph of K_n.*

Theorem 3.4. *Let $C_n = (\sigma, \mu)$ be a cyclic strong fuzzy graph, $V = \{y_1, y_2, \ldots, y_n\}$ and $n \geq 5$. If there exists $1 \leq i \leq k$ such that $\bigwedge\limits_{x \in V} \sigma(x) = \sigma(y_i)$ and $\sigma(v_1) > \sigma(v_2) > \sigma(v_3) > \ldots > \sigma(v_{n-1}) > \sigma(v_n)$ is an ascending sequence, then*

(i) *if* $3 \mid n-k-2$, *then* $\gamma_t(C_n) = \sigma(y_n)((\mu(y_n y_{n-1}) + \sum\limits_{i \in \{n-3, n-6, \cdots, k+2\}} \sigma(y_i)((\mu(y_i y_{i-1}) + \mu(y_i y_{i+1})),$

(ii) *if* $3 \mid n-k-3$, *then*

$$\gamma_t(C_n) = \sigma(y_{k+1})\mu(y_{k+1} y_{k+2}) + \sigma(y_n)\mu(y_n y_{n-1}) + \sum_{i \in \{n-3, n-6, \cdots, k+3\}} \sigma(y_i)((\mu(y_i y_{i-1}) + \mu(y_i y_{i+1})),$$

(*iii*) *if* $3 \mid n-k-1$, *then*

$$\gamma_t(C_n) = \sigma(y_{k+1})\mu(y_{k+1}y_{k+2}) + \sigma(y_n)\mu(y_n y_{n-1}) + \sum_{i\in\{n-3,n-6,\cdots,k+4\}} \sigma(y_i)((\mu(y_i y_{i-1}) + \mu(y_i y_{i+1})).$$

In the following, we consider the wheel fuzzy graph $W_n = (\sigma,\mu)$, in Figure 3, where for all $1 \le i \le n, \sigma(y_i) < \sigma(y_{i+1})$. In what follows,

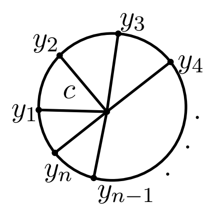


Figure 3: Strong Wheel W_n

consider a special wheel fuzzy graph $W_n = (V,\sigma,\mu)$, so we assume that $c = \bigwedge_{v_i \in V} \sigma(v_i)$, and showed that $\gamma_t(W_n) = \gamma_t(C_n)$, in this regard.

Definition 3.5. A wheel fuzzy graph $W_n = (V,\sigma,\mu)$, is called an infimum center-based wheel fuzzy graph, if $c = \bigwedge_{v_i\in V} \sigma(v_i)$.

Example 3.6. Consider Figure 4(a), and for all $x_i \in V$, $\sigma(x_i) = \dfrac{i}{10}$, $c = \bigwedge_{v_i\in V} \sigma(v_i)$. Then, $W_7 = (V,\sigma,\mu)$ is an infimum center-based wheel strong fuzzy graph.

Theorem 3.7. *Let* $W_n = (V,\sigma,\mu)$ *be an infimum center-based wheel strong fuzzy graph and* $V = \{y_1, y_2, \ldots, y_n\}$. *Then*

(*i*) *if* $3 \mid n-k-2$, *then* $\gamma_t(W_n) = \sigma(y_n)((\mu(y_n y_{n-1}) + \sum_{i\in\{n-3,n-6,\cdots,k+2\}} \sigma(y_i)((\mu(y_i y_{i-1}) + \mu(y_i y_{i+1})),$

(*ii*) *if* $3 \mid n-k-3$, *then*

$$\gamma_t(W_n) = \sigma(y_{k+1})\mu(y_{k+1}y_{k+2}) + \sigma(y_n)\mu(y_n y_{n-1}) + \sum_{i\in\{n-3,n-6,\cdots,k+3\}} \sigma(y_i)((\mu(y_i y_{i-1}) + \mu(y_i y_{i+1})),$$

(*iii*) *if* $3 \mid n-k-1$, *then*

$$\gamma_t(W_n) = \sigma(y_{k+1})\mu(y_{k+1}y_{k+2}) + \sigma(y_n)\mu(y_n y_{n-1}) + \sum_{i\in\{n-3,n-6,\cdots,k+4\}} \sigma(y_i)((\mu(y_i y_{i-1}) + \mu(y_i y_{i+1})).$$

Proof. Since for any $y_i \in V, \mu(cy_i) = \sigma(c)$, there exists the cycle $P : y_i, c, y_{i+1}, y_i$. So for any $y_i \in V$, $\mu'^{\infty}(cy_i) = \sigma(c) = \mu(cy_i)$. Hence for any $y_i \in V$, $\mu'^{\infty}(y_i y_{i+1}) = \sigma(c) < \sigma(y_i) = \mu(y_i y_{i+1})$. Using Theorem 3.4, the result is straightforward. □

In the following example, we present the computations of fuzzy domination number of infimum center-based wheel fuzzy graph of orders $7, 8, 9$.

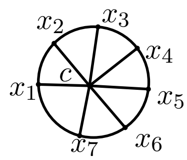
Example 3.8. (*i*) Let $W_7 = (V_1, \sigma_1, \mu_1)$ be an infimum center-based wheel strong fuzzy graph as Figure 4(a) and for all $x_i \in V_1$, $\sigma(x_i) = \dfrac{i}{10}$. Then, by Theorem 3.7 (*ii*), $\gamma_t(W_7) = 0.74$. If $\sigma(x_1) = \sigma(x_2)$, then by Theorem 3.7 (*i*), $\gamma_t(W_7) = 0.70$.

(*ii*) Let $W_8 = (V_2, \sigma_2, \mu_2)$ be an infimum center-based wheel strong fuzzy graph as Figure 4(b) and for all $x_i \in V_2$, $\sigma(x_i) = \dfrac{i}{10}$. Then by Theorem 3.7(*iii*), $\gamma_t(W_8) = 1.05$. If $\sigma(x_1) = \sigma(x_2) = \sigma(x_3)$, then by Theorem 3.7 (*i*), $\gamma_t(W_8) = 1.01$.

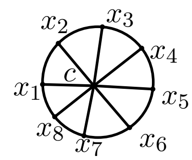
(*iii*) Let $W_9 = (V_3, \sigma_3, \mu_3)$ be an infimum center-based wheel strong fuzzy graph as Figure 4(c) and for all $x_i \in V_3$, $\sigma(x_i) = \dfrac{i}{10}$. Then by Theorem 3.7 (*i*), $\gamma_t(W_9) = 1.53$. If $\sigma(x_1) = \sigma(x_2) = \sigma(x_3) = \sigma(x_4) = \sigma(x_5)$, then by Theorem 3.7 (*iii*), $\gamma_t(W_9) = 1.08$.

Let $W_n = (\sigma, \mu)$ be a cyclic strong fuzzy graph and $e, e' \in E$. Define a relation R on E, by $(e, e') \in R$ if and only if $\mu(e) = \mu(e')$. Clearly R is an equivalence relation on E and we have the following corollary.

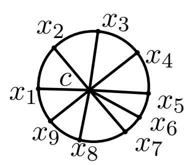
Theorem 3.9. *$W_n = (V, \sigma, \mu)$ be an infimum center-based wheel strong fuzzy graph. Then $|E/R| \leq 1$ if and only if $\gamma_t(W_n) = 0$.*



(a) Wheel fuzzy graphs W_7



(b) Wheel fuzzy graphs W_8



(c) Wheel fuzzy graphs W_9

Figure 4: Infimum center-based wheel strong fuzzy graphs

Proof. Since $|E/R| \leq 1$, for all $e, e' \in E(W_n)$, $\mu(e) = \mu(e')$. Then for all $e \in E(W_n)$, ${\mu'}^{\infty}(e) = \mu(e)$. Thus for all $e \in E(W_n)$, e is not an $\alpha-$strong edge. It implies $\gamma_t(W_n) = 0$. The converse is clear. $\square$

In what follows, consider a special wheel fuzzy graph $W_n = (V, \sigma, \mu)$, so we assume $c = \bigvee\limits_{v_i \in V} \sigma(v_i)$, then show that $\gamma_t(W_n) = \gamma_t(K_n)$.

Definition 3.10. A wheel fuzzy graph $G = (V, \sigma, \mu)$, is called a suprimum center-based wheel fuzzy graph, if $c = \bigvee\limits_{v_i \in V} \sigma(v_i)$.

Example 3.11. Consider Figure 5(a), and for all $x_i \in V$, $\sigma(x_i) = 0.i$, $c = \bigvee_{v_i \in V} \sigma(v_i)$. Then, $W_5 = (V, \sigma, \mu)$ is a suprimum center-based wheel strong fuzzy graph.

Theorem 3.12. *Let $W_n = (V, \sigma, \mu)$ be a suprimum center-based wheel strong fuzzy graph. Then $\gamma_t(W_n) = \gamma_t(K_n)$.*

Proof. Since for any $y_n \neq y_i \in V, \mu(cy_i) = \sigma(y_i)$, there exists a cycle $P : y_i, c, y_{i+1}, y_i$. So for any $1 \leq i \leq n-1$, the edge $y_i y_{i+1}$ is not an $\alpha-$strong, because of $\mu'^{\infty}(y_i y_{i+1}) = \sigma(y_i) = \mu(y_i y_{i+1})$. Applying Theorem 3.3, the result is straightforward. $\square$

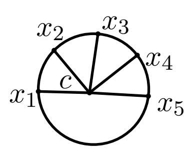
Corollary 3.13. *Let $W_n = (V, \sigma, \mu)$ be a suprimum center-based wheel strong fuzzy graph. Then $\gamma_t(W_n) = \sigma^2(v_n)$.*

Proof. by Theorems 3.12 and Theorem 3.3, the proof is obtained. □

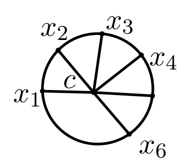
Corollary 3.14. *$W_n = (V, \sigma, \mu)$ be a suprimum center-based wheel strong fuzzy graph. Then $|E/R| \leq 1$ if and only if $\gamma_t(W_n) = 0$.*

In the following example, we present the computations of fuzzy domination number of suprimum center-based wheel strong fuzzy graph of orders $5, 6, 10$.

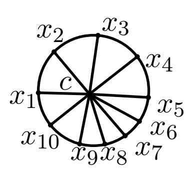
Example 3.15. Let $W_5 = (V_1, \sigma_1, \mu_1), W_6 = (V_2, \sigma_2, \mu_2), W_{10} = (V_3, \sigma_3, \mu_3)$ be suprimum center-based wheel strong fuzzy graphs as shown in Figures 5(a),5(b),5(c), where for all $j = 1, 2, 3 x_i \in V_j$, we define $\sigma(x_i) = 0.i$. Then, by Theorem 3.12 and Corollary 3.13, $\gamma_t(W_5) = \gamma_t(K_5) = 0.25.$, $\gamma_t(W_6) = \gamma_t(K_6) = 0.36.$, and $\gamma_t(W_{10}) = \gamma_t(K_{10}) = 1$.



(a) Wheel fuzzy graph W_5



(b) Wheel fuzzy graph W_6



(c) Wheel fuzzy graph W_{10}

Figure 5: Suprimum center-based wheel strong fuzzy graph

4 Fuzzy Bridge In Complete Bipartite Fuzzy Graphs

In this section, we compute the fuzzy domination number of complete bipartite fuzzy graphs and prove some important properties of them.

From now on, for complete bipartite fuzzy graph $K_{n,m} = (V, \sigma, \mu)$, we consider $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ and for $x, y \in V$, if $\mu(xy) = 0$, then either $x, y \in V_1$ or $x, y \in V_2$.

Theorem 4.1. *Let $K_{n,m} = (V, \sigma, \mu)$ be a complete bipartite fuzzy graph and $D^\alpha \in \mathcal{D}^\alpha(K_{n,m})$. If there exists $y_1, y_2, \dots, y_m \in V$ such that $\bigwedge_{a \in V} \sigma(a) = \sigma(y_1) = \sigma(y_2) = \ldots = \sigma(y_m)$. Then*

(i) *if $x \in V$, then $x \overset{\alpha}{\not\hookrightarrow} y_i$ or $y_i \overset{\alpha}{\not\hookrightarrow} x$, where $1 \leq i \leq m$,*

(ii) *if $x \in \{y_1, y_2, \dots, y_m\}$, then $\mathcal{E}(D^\alpha, x) = \emptyset$.*

Proof. (i) Let $x \in V$. If either $x, y \in V_1$ or $x, y \in V_2$, then $x \overset{\alpha}{\not\hookrightarrow} y_i$. Otherwise, let $x \in V_1$ and $y \in V_2$. Hence, for any $z \in V_1$, $\mu(xy_i) = \mu(y_i z)$. It follows that for any $x \in V, x \overset{\alpha}{\not\hookrightarrow} y_i$. Thus in any cases, if $x \in V$, then $x \overset{\alpha}{\not\hookrightarrow} y_i$ or $y_i \overset{\alpha}{\not\hookrightarrow} x$, where $1 \leq i \leq m$.

(ii) By item (i), it is obtained. $\square$

Theorem 4.2. *Let $K_{n,m} = (V, \sigma, \mu)$ be a complete bipartite fuzzy graph and for all $w, w' \in V, \sigma(w) \neq \sigma(w')$. If there exist $y' \in V_1$ and $y \in V_2$ such that $\bigvee_{x \in V_1} \sigma(x) = \sigma(y)$, $\bigvee_{x \in V_2} \sigma(x) = \sigma(y')$, $v \in V_1$ and $\sigma(y) \leq \sigma(y')$, then*

(i) *if $\sigma(v) < \sigma(y)$ and $\sigma(v') < \sigma(y')$, then $v \overset{\alpha}{\not\hookrightarrow} v'$, where $\sigma(v) \leq \sigma(v')$ and $v' \in V_2$,*

(ii) *$\sigma(v) = \sigma(y)$, if and only if $v \overset{\alpha}{\hookrightarrow} v'$, where $\sigma(v) \leq \sigma(v')$ and $v' \in V_2$,*

(iii) *if $\sigma(v) \neq \sigma(y)$, then $\mathcal{E}(D^\alpha, v) = \emptyset$, where $D^\alpha \in \mathcal{D}^\alpha(K_{n,m})$,*

(iv) *$\mathcal{E}(D^\alpha, y) = \{yz \mid \sigma(z) \geq \sigma(y), z \in V_2\}$, where $D^\alpha \in \mathcal{D}^\alpha(K_{n,m})$.*

Proof. (i) Let $v \overset{\alpha}{\hookrightarrow} v'$. Then $\mu'^\infty(vv') < \mu(vv')$ and so there's a path $P : v, y', y, v'$ such that $\sigma(v) = \sigma(v) \wedge \sigma(y') \wedge \sigma(y) \wedge \sigma(v') < \mu'^\infty(vv') < \mu(vv') = \sigma(v)$. Thus $\sigma(v) < \sigma(v)$ which is a contradiction.

(ii) $(\Rightarrow)$ Assume $v \overset{\alpha}{\not\hookrightarrow} v'$. Then there exists a path $P: v, x, t, v'$ such that $\sigma(v) \wedge \sigma(x) \wedge \sigma(t) \wedge \sigma(v') = \sigma(v)$. It implies $\sigma(y) \geq \sigma(t) \geq \sigma(v)$. Since $v \neq t$, $\sigma(y) \geq \sigma(t) > \sigma(v)$, we get that $\sigma(y) \neq \sigma(v)$.

$(\Leftarrow)$ Suppose that $v \overset{\alpha}{\hookrightarrow} v'$, where $\sigma(v) \leq \sigma(v')$ and $v' \in V_2$. Thus $\mu'^{\infty}(vv') < \mu(vv')$ and so there's a path $P: v, y', y, v'$ such that $\sigma(v) = \sigma(v) \wedge \sigma(y') \wedge \sigma(y) \wedge \sigma(v') < \mu'^{\infty}(vv') < \mu(vv') = \sigma(v)$. So $\sigma(v) = \sigma(y)$.

$(iii), (iv)$ are obtained by item (ii). $\square$

Theorem 4.3. *Let $K_{n,m} = (V = V_1 \cup V_2, \sigma, \mu)$ be a complete bipartite fuzzy graph and for all $w, w' \in V, \sigma(w) \neq \sigma(w')$. If there exists $y' \in V_1$ and $y \in V_2$ such that $\bigvee_{x \in V_1} \sigma(x) = \sigma(y)$, $\bigvee_{x \in V_2} \sigma(x) = \sigma(y')$, $v \in V_1$, and $\sigma(y) \leq \sigma(y')$, then*

(i) $Op(\mathcal{D}(K_{n,m})) = \{V \setminus \{y\}, V \setminus \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}\}$,

(ii) $\gamma_t(K_{n,m}) = T_{min}\{\sigma(y)\mu(yz), \sigma(z)\mu(yz) \mid \sigma(z) \geq \sigma(y), z \in V_2\}\}$,

(iii) $\gamma_t(K_{n,m}) = t\sigma^2(y)$, *where* $t = |\{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}|$,

(iv) $V \setminus \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}$ *is the vertex fuzzy dominating set.*

Proof. (i) By Theorem 4.2(ii), $v \overset{\alpha}{\hookrightarrow} v'$ implies $\sigma(v) = \sigma(y)$. Hence $V \setminus \{y\} \in Op(\mathcal{D}(K_{n,m}))$. Using Theorem 4.2$(iv)$, $\mathcal{E}(D^\alpha, y) = \{yz \mid \sigma(z) \geq \sigma(y), z \in V_2\}$, where $D^\alpha \in \mathcal{D}^\alpha(K_{n,m})$. So by $y \overset{\alpha}{\hookrightarrow} z, V \setminus \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\} \in Op(\mathcal{D}(K_{n,m}))$.

(ii) By item (i), $Op(\mathcal{D}(K_{n,m})) = \{V \setminus \{y\}, V \setminus \{z\} \mid \sigma(z) \geq \sigma(y), z \in V_2\}$. So $w(V \setminus \{y\}) = \sigma(z)\mu(yz)$ and $w(V \setminus \{z\}) = \sigma(y)\mu(yz)$ imply that $\gamma_t(K_{n,m}) = T_{min}\{\sigma(y)\mu(yz), \sigma(z)\mu(yz) \mid \sigma(z) \geq \sigma(y), z \in V_2\}$.

(iii) Since $\sigma(y) < \sigma(z)$, we get that $\sigma(z)\mu(yz) > \sigma(y)\mu(yz) = \sigma(y)(\sigma(z) \wedge \sigma(y)) = \sigma^2(y)$. Hence $\gamma_t(G) = \sigma^2(y)$. Since $t = |\{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}|$, we get that

$$\begin{aligned} \sigma(z)\mu(yz) + \cdots + \sigma(z)\mu(yz) &> \sigma(y)\mu(yz) + \cdots + \sigma(y)\mu(yz) \\ &= \sigma(y)(\sigma(z) \wedge \sigma(y) + \cdots + \sigma(z) \wedge \sigma(y)) \\ &= \sigma(y)(\sigma(y) + \cdots + \sigma(y)). \end{aligned}$$

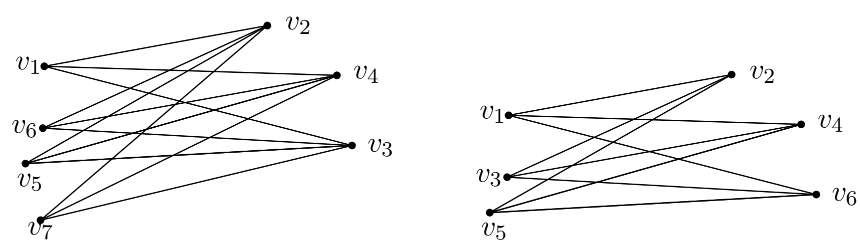
Thus $\gamma_t(K_{n,m}) = t\sigma^2(y)$.

(iv) For any $z \in V_2$ such that $\sigma(z) \geq \sigma(y)$, we have $y \overset{\alpha}{\hookrightarrow} z$. Hence $V \setminus \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\} \in Op(\mathcal{D}(K_{n,m}))$.

□ In the following example, we compute the fuzzy domination number of some complete bipartite fuzzy graph.

Example 4.4. (i) Consider the complete bipartite fuzzy graph $K_{4,3} = (V_1, \sigma_1, \mu_1)$ as shown in Figure 6(a), where and for all $v_i \in V_1$, $\sigma_1(v_i) = \dfrac{i}{10}$. By Theorem 4.2, we get that $v_4 \overset{\alpha}{\hookrightarrow} v_5$, $v_4 \overset{\alpha}{\hookrightarrow} v_6$ and $v_4 \overset{\alpha}{\hookrightarrow} v_7$. Computations show that $\mathcal{D} = \{D_1^\alpha = \{v_1, v_2, v_3, v_4\}, D_2^\alpha = \{v_1, v_2, v_3, v_5, v_6, v_7\}\}$ and so $Op(\mathcal{D}) = \{D_1^\alpha, D_2^\alpha\}$. Using Theorem 4.3, $\gamma_t(G) = \sigma^4(y) = 0.0256$ and so a set $\{v_1, v_2, v_3, v_4\}$ is the vertex fuzzy dominating set.

(ii) Consider the complete bipartite fuzzy graph $K_{3,3} = (V_2, \sigma_2, \mu_2)$ as shown in Figure 6(b), where for all $v_i \in V_2$, we have $\sigma_2(v_i) = \dfrac{i}{10}$. Applying Theorem 4.2, $v_5 \overset{\alpha}{\hookrightarrow} v_6$, so routine computations show that $\mathcal{D} = \{D_1^\alpha = \{v_1, v_2, v_3, v_4, v_5\}, D_2^\alpha = \{v_1, v_2, v_3, v_4, v_6\}\}$ and so $Op(\mathcal{D}) = \{D_1^\alpha, D_2^\alpha\}$. Using Theorem 4.3, $\gamma_t(G) = \sigma^2(y) = 0.25$ and $\{v_1, v_2, v_3, v_4, v_5\}$ is the vertex fuzzy dominating set.



(a) $K_{4,3}$ (b) $K_{3,3}$

Figure 6: Complete Bipartite Fuzzy Graphs $K_{3,3}$ and $K_{4,3}$.

Theorem 4.5. *Let $K_{n,m} = (V = V_1 \cup V_2, \sigma, \mu)$ be a complete bipartite fuzzy graph. If there exists $y' \in V_1$ and $y \in V_2$ such that $\{a \mid a \in V_1,\ \sigma(a) = \bigvee_{x \in V_1} \sigma(x)\} = \{y\}$, $\bigvee_{x \in V_2} \sigma(x) = \sigma(y')$, $v \in V_1$, and $\sigma(y) \leq \sigma(y')$, then*

(i) $Op(\mathcal{D}(K_{n,m})) = \{V \setminus \{y\}, V \setminus \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}\}$,

(ii) $\gamma_t(K_{n,m}) = T_{min}\{\sigma(y)\mu(yz), \sigma(z)\mu(yz) \mid \sigma(z) \geq \sigma(y), z \in V_2\}$,

(iii) $\gamma_t(K_{n,m}) = t\sigma^2(y)$, *where* $t = |\{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}|$,

(iv) $V \setminus \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}$ *is the vertex fuzzy dominating set.*

Proof. $(i), (ii), (iii), (iv)$ are obtained by Theorem 4.3, □

Example 4.6. (i) Let $K_{4,5} = (V_1, \sigma_1, \mu_1)$ be a complete bipartite fuzzy graph as shown in Figure 7(a), where for all $v_i \in V_1$, $\sigma_1(v_i) = \dfrac{i}{10}$. By Theorem 4.2, $v_7 \overset{\alpha}{\hookrightarrow} v_8$ and $v_7 \overset{\alpha}{\hookrightarrow} v_9$, so computations show that

$$\mathcal{D} = \{D_1^\alpha = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, D_2^\alpha = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9\}\}$$

and so $Op(\mathcal{D}) = \{D_1^\alpha, D_2^\alpha\}$. Using Theorem 4.3, $\gamma_t(G) = \sigma^2(y) = 0.49$ and so $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is the vertex fuzzy dominating set.

(ii) Consider the complete bipartite fuzzy graph $K_{5,6} = (V_2, \sigma_2, \mu_2)$ as shown in Figure 7(b), where $v_i \in \{v_1, v_2, v_3, v_4, v_5, v_6\}$, implies that $\sigma_2(v_i) = 0.2$ and otherwise, $\sigma_2(v_i) = \dfrac{i}{10}$. By Theorem 4.5, $v_{10} \overset{\alpha}{\hookrightarrow} v_{11}$, so computations show that $\mathcal{D} = \{D_1^\alpha = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, $D_2^\alpha = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{11}\}\}$ and so $Op(\mathcal{D}) = \{D_1^\alpha, D_2^\alpha\}$. Using Theorem 4.5, $\gamma_t(G) = \sigma^2(y) = 1$ and $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ is the vertex fuzzy dominating set.

Theorem 4.7. *Let* $K_{n,m} = (V = V_1 \cup V_2, \sigma, \mu)$ *be a complete bipartite fuzzy graph. Suppose there exists* $y' \in V_1$ *and* $y \in V_2$ *such that* $\bigvee\limits_{x \in V_1} \sigma(x) = \sigma(y)$, $\bigvee\limits_{x \in V_2} \sigma(x) = \sigma(y')$, $v \in V_1$ *and* $\sigma(y) \leq \sigma(y')$. *For* $v_i, v_j \in V$,

(i) *if* $\sigma(v_j) \neq \sigma(y)$ *and* $\sigma(v_i) \neq \sigma(y)$, *then* $v_i \overset{\alpha}{\not\hookrightarrow} v_j$,

(ii) *if* $\bigwedge\limits_{x \in V_1} \sigma(x) = \sigma(v_i)$ *and* $\sigma(v_i) \neq \sigma(y)$, *then* $v_i \overset{\alpha}{\not\hookrightarrow} v_j$,

(iii) *if* $v_i \in \{a \mid a \in V_1,\ \sigma(a) = \bigvee\limits_{x \in V_1} \sigma(x)\}$ *and* $v_i \neq y$, *then* $v_i \overset{\alpha}{\not\hookrightarrow} v_j$,

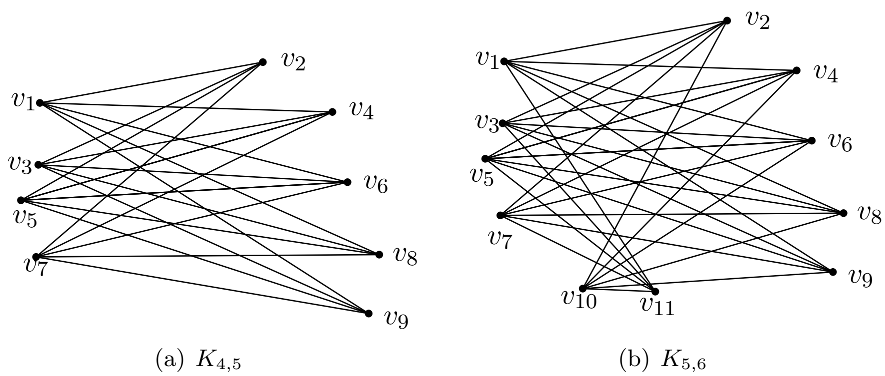


Figure 7: Complete Bipartite Fuzzy Graphs

(*iv*) *if* $v_j \notin \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}$, *then* $v_i \overset{\alpha}{\not\to} v_j$.

Proof. (*i*) Let $v_i, v_j \in V$. Because $\sigma(v_i) \neq \sigma(y)$ and $\sigma(v_j) \neq \sigma(y)$, there's a cycle $P : v_i, v_j, y, y', v_i$. It implies $v_iv_j \notin \mathcal{E}(D^\alpha, v_i)$ and so $v_i \overset{\alpha}{\not\to} v_j$.

(*ii*) Because $\sigma(v_i) \neq \sigma(y)$, we get $\sigma(v_i) < \sigma(y)$. If $\sigma(v_i) < \sigma(v_j)$, then there's a cycle $P : v_i, v_j, y, y', v_i$. It implies that $v_iv_j \notin \mathcal{E}(D^\alpha, v_i)$. Now if $\sigma(v_i) > \sigma(v_j)$, then there's a cycle $P : v_j, v_i, y', y, v_j$. It implies $v_iv_j \notin \mathcal{E}(D^\alpha, v_j)$ and so $v_i \overset{\alpha}{\not\to} v_j$.

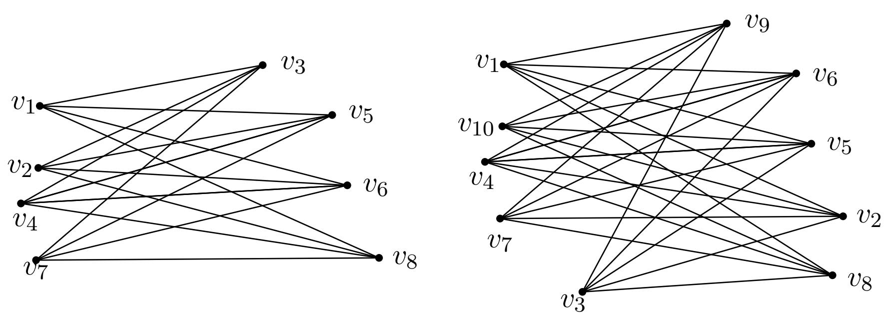
(*iii*) Since $v_i \in \{a \mid a \in V_1,\ \sigma(a) = \bigvee_{x \in V_1} \sigma(x)\}$, we obtain that $\sigma(v_i) = \sigma(y)$. So by $v_i \neq y$, there's a cycle $P : v_i, v_j, y, y', v_i$. It implies that $v_iv_j \notin \mathcal{E}(D^\alpha, v_i)$ and then $v_i \overset{\alpha}{\not\to} v_j$.

(*iv*) Since $v_j \notin \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}$, we get that $\sigma(v_j) < \sigma(y)$. Then there's a cycle $P : v_i, v_j, y, y', v_i$. It implies $v_iv_j \notin \mathcal{E}(D^\alpha, v_i)$, and hence $v_i \overset{\alpha}{\not\to} v_j$. $\square$

Example 4.8. (*i*) Let $K_{4,4} = (V_1, \sigma_1, \mu_1)$ be a complete bipartite fuzzy graph as shown in Figure 8(a). Define $\sigma_2(v_i) = 0.1.$, where $v_i \in \{v_1, v_2, v_4\}$ and for all $v_i \in V_1$, define $\sigma_1(v_i) = \dfrac{i}{10}$. Using Theorem 4.7, $v_7 \overset{\alpha}{\not\to}$

v_3, $v_7 \overset{\alpha}{\nrightarrow} v_5$ and $v_7 \overset{\alpha}{\nrightarrow} v_6$, because of $v_j \notin \{z \mid \sigma(z) \geq \sigma(y), z \in V_2\}$. Also using Theorem 4.7, $v_1 \overset{\alpha}{\nrightarrow} v_8$, $v_2 \overset{\alpha}{\nrightarrow} v_8$ and $v_4 \overset{\alpha}{\nrightarrow} v_8$, because of $\bigwedge\limits_{x\in V_1} \sigma(x) = \sigma(v_i)$ and $\sigma(v_7) \neq \sigma(v_i)$. In addition, Theorem 4.7, shows that $v_1 \overset{\alpha}{\nrightarrow} v_3$, $v_2 \overset{\alpha}{\nrightarrow} v_5$, $v_4 \overset{\alpha}{\nrightarrow} v_6$, since $\sigma(v_j) \neq \sigma(v_8)$ and $\sigma(v_i) \neq \sigma(v_7)$.

(ii) Let $K_{5,5} = (V_2, \sigma_2, \mu_2)$ be a complete bipartite fuzzy graph as shown in Figure 8(b). Define $\sigma_2(v_i) = 0.9$, where $v_i \in \{v_9, v_6, v_5\}$, and for $v_i \in V \setminus \{v_9, v_6, v_5\}$, $\sigma_2(v_i) = \dfrac{i}{10}$. Thus by Theorem 4.7, for all $v_i, v_j \in V$, we get that $v_i \overset{\alpha}{\nrightarrow} v_j$.



(a) $K_{4,4}$ (b) $K_{5,5}$

Figure 8: Complete Bipartite Fuzzy Graphs

Corollary 4.9. *Let $K_{n,m} = (V = V_1 \cup V_2, \sigma, \mu)$ be a complete bipartite fuzzy graph and there exist $y' \in V_1$ and $y \in V_2$ such that $\bigvee\limits_{x\in V_1} \sigma(x) = \sigma(y)$, $\bigvee\limits_{x\in V_2} \sigma(x) = \sigma(y')$. If $|\{a \mid a \in V_1,\ \sigma(a) = \bigvee\limits_{x\in V_1} \sigma(x)\}| \geq 2$, then $\gamma_t(K_{n,m}) = 0$.*

Proof. Since there's $v_i \in \{a \mid a \in V_1,\ \sigma(a) = \bigvee\limits_{x\in V_1} \sigma(x)\}$ and $v_i \neq y$, by Theorem 4.7, for all $v_i, v_j \in V$, $v_i \overset{\alpha}{\nrightarrow} v_j$. It implies $\gamma_t(K_{n,m}) = 0$. $\square$

4.1 Complete Multipartite Fuzzy Graphs

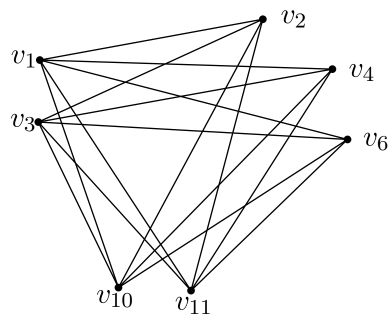
In this subsection, we compute the fuzzy domination number of complete multipartite fuzzy graph and prove that it is equal to zero.

Theorem 4.10. *Let $K_{n_1,n_2,\cdots,n_r} = (V, \sigma, \mu)$ be a complete multipartite fuzzy graph such that $V = \bigcup_{i=1}^{r} V_i$, $V_i = \{x_{i1}, x_{i2}, \cdots, x_{in_i}\}$ and $n_i \geq 2$. If $r \geq 3$, and for all $w, w' \in V, \sigma(w) \neq \sigma(w')$, then $\gamma_t(K_{n_1,n_2,\cdots,n_r}) = \gamma_t(K_{n_{r-1},n_r})$, where $y_i = \bigvee_{x\in V_i} \sigma(x)$ and $y_1 < y_2 < \cdots < y_r$.*

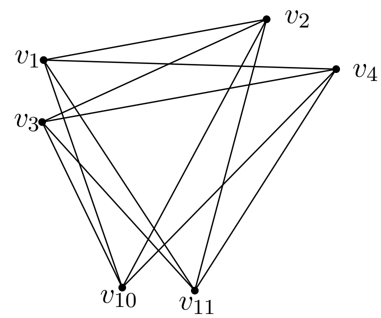
Proof. Assume $y = \bigvee_{x\in V_i} \sigma(x) < y' = \bigvee_{x\in V_j} \sigma(x) < y'' = \bigvee_{x\in V_m} \sigma(x)$. By $n \geq 3$ and $y < y' < y^{''}$, there's $y'' \in V_m$ such that $P : y, y', y'', y$ is a cycle. It implies $yy' \notin \mathcal{E}(D^\alpha, y)$. Hence $y \overset{\alpha}{\not\hookrightarrow} y'$. By $n \geq 3$ and $y < y' < y^{''}$, there's $y' \in V_j$ such that $P : y, y'', y', y$ is a cycle. It implies $yy'' \notin \mathcal{E}(D^\alpha, y)$. Hence $y \overset{\alpha}{\not\hookrightarrow} y''$. By Theorem 4.5, $y^{'} \overset{\alpha}{\hookrightarrow} y''$ and $V \setminus \{z \mid \sigma(z) \geq \sigma(y), z \in V_m\}$ is the vertex fuzzy dominating set. Thus $\gamma_t(K_{n_1,n_2,\cdots,n_r}) = \gamma_t(K_{n_{r-1},n_r}) = t\sigma^2(y)$, where $y_i = \bigvee_{x\in V_i} \sigma(x)$, $y_1 < y_2 < \cdots < y_r$ and $t = |\{z \mid \sigma(z) \geq \sigma(y), z \in V_m\}|$. By induction on r the result in this case is obvious.
Otherwise, the result is straightforward. □

Example 4.11. (i) Let $K_{2,2,3} = (V_1, \sigma_1, \mu_1)$ be a complete multipartite fuzzy graph as shown in Figure 9(a). Define $\sigma_2(v_i) = 0.6$, where $v_i \in \{v_6, v_3\}$, and for all $v_i \in V_1$, $\sigma_1(v_i) = \frac{i}{10}$. It is clear that $v_3 \overset{\alpha}{\not\hookrightarrow} v_6$, because of the cycle $P : v_3, v_{11}, v_{10}, v_3$ and $v_6 \overset{\alpha}{\not\hookrightarrow} v_{10}$, because of the cycle $P : v_6, v_{10}, v_3, v_6$. $v_6 \overset{\alpha}{\not\hookrightarrow} v_{11}$ because of the cycle $P : v_6, v_{11}, v_3, v_6$. Then $\gamma_t(K_{2,2,3}) = 0$.

(ii) Let $K_{2,2,2} = (V_2, \sigma_2, \mu_2)$ be a complete multipartite fuzzy graph as shown in Figure 9(b). Define $\sigma_2(v_i) = \frac{i}{10}$, where $v_i \in V$. One can see that $v_3 \overset{\alpha}{\not\hookrightarrow} v_{10}$, because of the cycle $P : v_3, v_{10}, v_4, v_3$. By Theorem 4.7, we get that $v_4 \overset{\alpha}{\hookrightarrow} v_{10}$ and $v_4 \overset{\alpha}{\hookrightarrow} v_{11}$. Applying Theorem 4.10, $\gamma_t(K_{2,2,2}) = \gamma_t(K_{2,2}) = 2 \times \sigma^2(v_4) = 0.32$.



(a) $K_{2,2,3}$



(b) $K_{2,2,2}$

Figure 9: Complete Multipartite Fuzzy Graphs

5 Fuzzy Bridges And Applications

In this section, we consider the concepts of fuzzy domination number ((supremum, infimum) center-based wheel strong, complete (bipartite, multipartite)) fuzzy graph and introduce some applications in real-life related to these concepts. The results of this paper are applied in four applications. The applications are simulated. In every case, by using the results of this paper, the issue in the model is solved and the solutions are addressed by the introduced results. In some cases, one solution is advised. So this could be considered a limitation and drawback of this article. Using the model which is simulated by the complete fuzzy graph, allows us to simulate a situation that has full connections between its part. So the advantages of this paper are highlighted by proposing some solutions to the model when full connections are simulated by complete bipartite fuzzy graphs and complete multipartite fuzzy graphs. These models are studied and they address some solutions to the situation. In other models, using a specific wheel gives us a special case when one part has a different position in the situation. So it's stimulated by the model and it gets some attention. It's considered an advantage of this study. This study uses two different models. First model, the position of all parts is the same but in the second model, one part has a different position and this part has full connections to which other parts.

We present some applications with the following Algorithm (with two cases):
Step 1. Let $\{v_1, v_2, \cdots, v_n\}$ be a set of vertices and $\{v_i v_{i+1}\}_{i=1}^{n-1}$ be a set of edges.
Step 2. Consider a model with Table.
Step 3. Define the values of vertices and edges in the model.
Step 4. Consider a model with a Figure, as a strong cyclic fuzzy graph and descriptions of the model to clarify.
Step 5. Using Theorems for an initial model is done.
Step 6. Using Theorems for modified models is done as two cases with clarifications about issues and solutions.

Common Server Problem

The server is the decider about what's going on on the world wide web. To visit a website, it's expected to face some problems. The common server problems are slow page loads, hardware failure, viruses, and overload. These problems of servers are common when the subject is about the relationship between servers and clients. Consider the problem concerning slow page loads. In this case, either server takes too long to respond or the website's contents are postponed. The model is built based on the obsession with both the server and the website's contents. In this case, we represent an application of fuzzy domination numbers and the common server problem. Let $\{v_1, v_2, \cdots, v_8\}$ be a set of websites, $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_7v_8, v_8v_1\}$ be a set of connections between websites, $\{cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_8\}$ be a set of connections between websites and the server. Consider the current obsession with websites and the current obsession between websites in Table 2. The current obsession between website and server is the obsession of the website. The current obsession of the server is fixed and it equals the maximum obsession of websites.
Define $\sigma(v_i) = \dfrac{\text{Current Obsession of } v_i}{1000}$ and $\mu(v_i v_{i+1}) = \sigma(v_i)$, be fuzzy

Table 2: Current Obsession.

Websites	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Obsession	100	200	300	400	500	600	700	800
Connections	v_1v_2	v_2v_3	v_3v_4	v_4v_5	v_5v_6	v_6v_7	v_7v_8	v_8v_1
Obsession	100	200	300	400	500	600	700	100

value of current obsession of website v_i and fuzzy value of current obsession of connections v_iv_{i+1}, respectively, where $i \in \mathbb{N}$ and $1 \leq i \leq 8$. We modelify it in Figure 10, as a supremum center-based wheel strong fuzzy graph such that starting a website is v_1 and a website only connects to another website in the clockwise motion on the connections as Table 2, whereby Theorem 3.12, the optimal vertex domination of this model is optimal vertex domination of complete fuzzy graph from the order 8, K_8, $\gamma_t(W_8) = \gamma_t(K_8)$. By Corollary 3.13, $\gamma_t(W_8) = \sigma(v_8)^2 = 0.64$. This number is a warning about the time when the obsession of the model takes the maximum and it has the structure of the model, a complete fuzzy graph. When the obsession of the server c is the maximum, there's only one website, v_8, to connect to the server c. The obsession of the system is 0.64. Thus in the case that, the server has the maximum obsession, website v_8 has only a connection with the server, and in this case, the model has the situation in that, two given websites have connections and obsessions with each other and with the server as the model complete fuzzy graph.

Civil engineer's problem

A civil engineer engages in many problems usually concerning design. The design of highways is the solution to the problem of traffic, where the reduction is the concern. Commuters are involved with this problem which the iteration facing is daily. Commuters and the traffic on highways are two subjects which need to make balanced to achieve the best situation possible. Consider the problem concerning reducing traffic through highway design. In this case, the center is the junction, where all other junctions in this model are connected to it. The model is built based on the traffic inside both the center and other places. In this case,

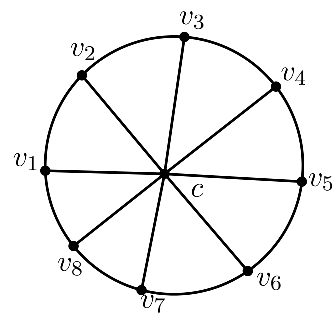


Figure 10: Suprimum Center-based Wheel Strong Fuzzy Graph W_8

we represent an application of fuzzy domination numbers and the highway design problem concerning reducing traffic. Let $\{v_1, v_2, \cdots, v_8\}$ be a set of places, $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_7v_8, v_8v_1\}$ be a set of connections between places, $\{cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_8\}$ be a set of connections between places and the center. Current traffic between places and the center is the traffic of the places. The current traffic of the center is fixed and it equals the minimum traffic of all places.

Table 3: Current Traffic.

Places	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Traffic	100	200	300	400	500	600	700	800
Roads	v_1v_2	v_2v_3	v_3v_4	v_4v_5	v_5v_6	v_6v_7	v_7v_8	v_8v_1
Traffic	100	200	300	400	500	600	700	100

The traffics is twofold: The traffic, inside of the place, and the traffic, outside of the place. So the traffic of the place means the traffic inside of the place and the traffic of connection means the traffic of the outside of the place. Define $\sigma(v_i) = \dfrac{\text{Current Traffic of } v_i}{1000}$ and $\mu(v_iv_{i+1}) = \sigma(v_i)$, be fuzzy value of current traffic of place v_i and fuzzy value of current traffic of connections v_iv_{i+1}, respectively, where $i \in \mathbb{N}$ and $1 \leq i \leq 8$. We modelify it in Figure 11, as an infimum center-based wheel strong fuzzy graph such that starting place is v_1 and a place only connects to another place in the clockwise motion on the connections as Table 3, whereby Theorem 3.7, the optimal vertex domination of this model is to redesign to the time when the traffic of the place has the minimum values. Consider the place v_2, v_3 has decreased to take the same traffic as the place v_1. Two models are redesigned: the first design is as the Table 3, and the second design is obtained from Table 3. By Theorem(iii) 3.7, $\gamma_t(W_8) = 1.05$. for the design as Table 3. By Theorem(iii) 3.7, $\gamma_t(W_8) = 1.01$. for the design as Table 3, where $\sigma(v_2) = \sigma(v_3) = \sigma(v_1) = 0.1$. If the same traffic spreads to all vertices, by Theorem 3.9, $\gamma_t(W_8) = 0$ which is the ideal case. The center has minimum traffic inside so other place connects to the center when one road outside the place is occupied. The design proposes roads to avoid traffic outside. In this case, the design suggests the places and precisely, the navigation $v_8cv_5cv_2cv_1$. So there's time to decrease the traffic of other ways which are occupied. The beginning is about the time when the number is zero and after that, the roads take the maximum capacity of their traffic until they take the situation as Table 3. In this case, the number is changed from 0 and 1.01 to 1.05. The number 1.05 is the critical number that is aware of the full capacity of the traffic in this design.

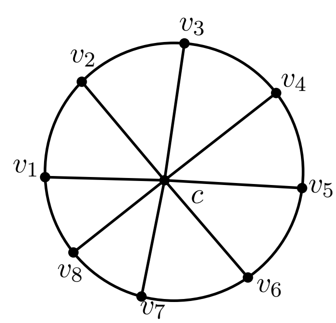


Figure 11: Infimum Center-based Wheel Strong Fuzzy Graph W_8

Common challenge in the manufacturing industry

The manufacturing industry is a big part of the industry. It's one of the biggest employers. Challenge in this industry is about the skills and the adaptation of the skills with the roles. There are challenges in this industry about selling directly, being trends, managing the projects, and lack of skills. The skills related to a role are the common challenges among manufacturers in the topic of the manufacturing industry. One solution is to be in partnership with institutions to fill the gap between skills and roles. Other solutions are to make a flexible environment for working and upgrading skills. Hence initial decision about the employees is prominent.

Consider the problem concerning employee's skills. In this case, the skills of the nominated and the roles of employees are a matter of mind. The model is built based on the skills of both the nominate and the roles. In this case, we represent an application of fuzzy domination numbers and the common problem in the manufacturing industry. Let $\{v_1, v_3, v_4\}$ be a set of nominates, $\{v_1v_2, v_1v_5, v_1v_6, v_3v_2, v_3v_5, v_3v_6,$ $v_4v_2, v_4v_5, v_4v_6\}$ be a set of connections between nominates' skills and employees' roles. Consider rates of skills and rates of roles in Table 4.

Table 4: Current Rates of Skills and Roles

Nominates	v_1	v_2	v_3	v_4	v_5	v_6
Rates	100	200	300	400	500	600
Connections	v_1v_2	v_2v_3	v_3v_4	v_4v_5	v_5v_6	v_iv_j
Rates	100	200	300	400	500	j00

Define $\sigma(v_i) = \dfrac{\text{Current Rate of } v_i}{1000}$ and $\mu(v_iv_{i+1}) = \sigma(v_i)$, be fuzzy value of current rate of skills of v_i and fuzzy value of current rate of roles of connections v_iv_{i+1}, respectively, where $i \in \mathbb{N}$ and $1 \leq i \leq 6$. We modelify it in Figure 12, as a complete bipartite fuzzy graph $K_{3,3}$ such that starting nominate is v_1 and a nominate only connects to all roles as Table 2. By Theorems 4.3 and 4.5, $\gamma_t(K_{3,3}) = 0.44$. This number is interpreted to the situation, where the adaptions of nominates' skills and employees' roles have the least costs. This case suggests the optimal situation to decide about the positions and people. If some nominate and some people have the same rates but the maximum rate is uniquely assigned, by Theorem 4.5, $\gamma_t(K_{3,3}) = 0.44$. Therefore the number isn't changed. The optimal set and the optimal number advise some choices about the employees and their roles to occupy the positions.

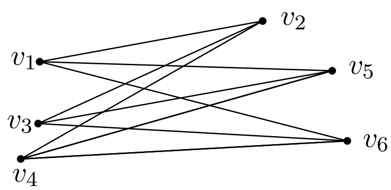


Figure 12: Complete Bipartite Fuzzy Graph $K_{3,3}$

Common challenges in the medical laboratory

A medical laboratory is an environment to research diseases. It's the heart of medical science. The research and treatment take away to the laboratory. So the problems of this place are important. This place is stressful. There are a lot of challenges concerning the limited time to do the work, pressure for obtaining results, pressure on doing fast, and so on. The challenge to decide fast and obtain results is of them.

Consider the problem concerning deciding fast about adding some dozes of material, where the same material are in a category and they don't count to combine. In this case, the amounts of the elements in different categories are a matter of mind. The model is built based on amounts of elements in different categories. In this case, we represent an application of fuzzy domination numbers and the common problem in a medical laboratory. Let V_1, V_2 and V_3 be three different categories and elements of same categories have no connection inside of categories but they connect to all elements in different categories, where $V_1 = \{v_1, v_3, v_5, v_7\}, V_2 = \{v_{10}, v_{11}\}, V_3 = \{v_2, v_4, v_6, v_8, v_9\}$. Consider amounts of dozes, and rates of combinations in Table 5.

Define $\sigma(v_i) = \dfrac{\text{Doze of } v_i}{1000}$ and $\mu(v_i v_{i+1}) = \sigma(v_i)$, be fuzzy value of doze of element of v_i and fuzzy value of rate of combination $v_i v_{i+1}$, respectively, where $i \in \mathbb{N}$ and $1 \leq i \leq 11$. We modelify it in Figure 13, as a

Table 5: Amounts of Dozes, Rates of Combinations

Elements	v_1	v_2	v_3	v_4	v_5	v_i
Dozes	100	200	300	400	500	i00
Combination	v_1v_2	v_2v_3	v_3v_4	v_4v_5	v_5v_6	v_iv_j
Rates	100	200	300	400	500	j00

complete multipartite fuzzy graph $K_{4,5,2}$ such that starting element is v_1 and an element only connects to all other elements in different categories as Table 5. By Theorem 4.1, $\gamma_t(K_{4,5,2}) = 0$. This number is interpreted to the situation, where the maximum dozes of elements and maximum rates of combinations devise the model. In this situation, the number is zero so the decision about finding the best match between three different categories has, by Theorem 4.1, no result. This decision is immediately obtained by modeling the situation, where the fast decision and right decision are parameters reflecting the skill in such a situation, where fast decision leading to result is critical.

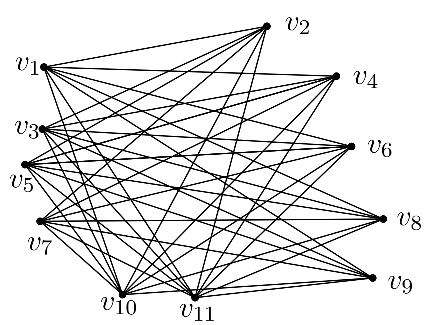


Figure 13: Complete Multipartite Fuzzy Graph $K_{4,5,2}$

6 Conclusion

The current paper has applied a novel concept of fuzzy dominating sets as α-strong dominating set and fuzzy domination number in fuzzy graphs based on fuzzy bridges. We modelify some real problems in fuzzy graphs and apply the properties of fuzzy vertex domination numbers in this regard. Based on the application of fuzzy vertex domination numbers, we proved some results that make the optimal fuzzy vertex domination. In more detail, the notion of dominating set and domination number is applied in some classes of fuzzy graphs. Specific classes of wheel strong fuzzy graphs are defined and the notions of dominating set and domination number are applied to them. Also, complete bipartite fuzzy graphs and complete multipartite fuzzy graphs are some classes of fuzzy graphs which are studied as some cases for domination numbers and dominating sets. Some applications from the real world are simulated as concluding sections and the results of the paper address some solutions to some issues which are arising from the real world. We hope that these results are helpful for further studies in the theory of graphs. In our future studies, we hope to obtain more results regarding fuzzy dominating sets as α-strong dominating sets and fuzzy domination numbers in fuzzy hypergraph and their applications.

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