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A Projection Method for Solving Nonlinear Volterra-Fredholm Integral Equations using Legendre Hybrid Functions

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Abstract. Since various problems in science and engineering fields can be modeled by nonlinear Volterra-Fredholm integral equations, the main focus of this study is to present an effective numerical method for solving them. This method is based on the hybrid functions of Legendre polynomials and block-pulse functions. By using this approach, a nonlinear Volterra-Fredholm integral equation reduces to a nonlinear system of mere algebraic equations. The convergence analysis and associated theorems are also considered. Test problems are provided to illustrate its accuracy and computational efficiency.

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1. Introduction

Many problems in science and engineering field such as heat transfer, diffusion process, neurosciences, etc. give rise to Volterra-Fredholm integral equations (VFIE). Usually, evaluating the exact solution of these equations may be difficult. So the numerical methods have a great appeal for

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mathematicians [1].
The aim of this work is to present a numerical method for approximating the solution of nonlinear integral equations of the form

$$f(s) = g(s) + \int_0^s k_1(s,t)G_1(t,f(t))dt + \int_0^1 k_2(s,t)G_2(t,f(t))dt, \quad 0 \leqslant s \leqslant 1, \tag{1}$$

where $f(s)$ is an unknown function defined on $[0,1]$, and $g(s)$, $k_1(s,t)$, $k_2(s,t)$, $G_1(t,f(t))$, and $G_2(t,f(t))$ are given $\mathcal{L}^2$ functions. We suppose that $G_1(t,f(t)) = [f(t)]^\alpha$ and $G_2(t,f(t)) = [f(t)]^\beta$, with arbitrary positive integers α and β.
Several numerical methods have been proposed for solving Eq. (1), using various orthogonal basis functions [9, 11-13].
In recent years, the hybrid functions of Legendre polynomials and block-pulse functions have been applied in solving various types of integral equations, control problems, time-varying descriptor systems and etc., [6-8, 10]. By these functions, the computational advantages of Legendre polynomials are combined with the simplicity of block-pulse functions, and a powerful set of basis functions is made. A numerical solution for the linear case of Eq. (1), was presented by Hsiao, using the hybrid functions [5]. Also, Maleknejad et al. used the hybrid functions to compute an approximate solution for this equation when G_1 and G_2 be the positive integer powers of f, [4].
In this paper, a novelty method based on hybrid functions in a Galerkin approach, is proposed. The method converts Eq. (1) to a system of mere algebraic equations. In this manner, only the unknown function in Eq. (1) is expanded by using hybrid functions, and the exact forms of all known functions in this equation are used. So, the proposed method is more accurate than the methods used in [4] and [5].
In the next sections, a description of the Legendre hybrid functions and some of their properties are mentioned. Some theorems regarding the convergence of function expansion with respect to the Legendre hybrid functions are also proved. Then, a method for computing numerical solutions of nonlinear Volterra-Fredholm integral equations by using Legendre hybrid functions and Galerkin conditions is proposed. Finally, The method is applied for solving several numerical examples, which follows

some conclusions.

2. Review on Legendre Hybrid Functions

Definition 2.1. *The Legendre polynomials on the interval* $[-1,1]$ *are given by the following recursive formula*

$$\begin{aligned} L_0(s) &= 1, \\ L_1(s) &= t, \\ L_{m+1}(s) &= \frac{2m+1}{m+1} s L_m(s) - \frac{m}{m+1} L_{m-1}(s), \qquad m = 1, 2, 3, \cdots . \end{aligned}$$

The set of $\{L_m(s) : m = 0, 1, \cdots\}$ *in the Hilbert space* $\mathcal{L}^2[-1,1]$ *is a complete orthogonal set. Orthogonality of Legendre polynomials on the interval* $[-1,1]$ *implies that*

$$< L_i(s), L_j(s) >= \int_{-1}^{1} L_i(s) L_j(s) ds = \begin{cases} \frac{2}{2i+1}, & i = j, \\ 0, & i \neq j, \end{cases} \tag{2}$$

for $i, j = 0, 1, \cdots$, *such that* $< \cdot, \cdot >$ *denotes the inner product [2].*

Definition 2.2. *In an* N*-set of block-pulse functions over the interval* $[0,1)$, *each component is defined as*

$$\phi_n(s) = \begin{cases} 1, & \frac{n-1}{N} \leqslant s < \frac{n}{N}, \\ 0, & otherwise, \end{cases} \tag{3}$$

where $n = 1, 2, \cdots, N$, *for arbitrary positive integer* N. *Block-pulse functions have several important properties such as disjointness, orthogonality, and completeness [3].*

Definition 2.3. *Let* $\{L_m(s)\}_{m=0}^{M-1}$ *be an* M*-set of Legendre polynomials, and* $\{\phi_n(s)\}_{n=1}^{N}$ *be an* N*-set of block-pulse functions over the interval* $[0,1)$, *too. An* MN*-set of Legendre hybrid functions (LHFs) is defined over the interval* $[0,1)$ *as*

$$h_{n,m}(s) = L_m(2Ns - 2n + 1)\phi_n(s), \qquad \begin{array}{l} n = 1, 2, \cdots, N, \\ m = 0, 1, \cdots, M-1. \end{array} \tag{4}$$

In the above definition, N and M are the order of block-pulse functions and the order of Legendre polynomials, respectively. So, the interval $[0,1)$ is divided to N-subintervals and M Legendre polynomials constructed on each of them. It is clear that $h_{n,m}(s)$ can be written as

$$h_{n,m}(s) = \begin{cases} L_m(2Ns-2n+1), & \frac{n-1}{N} \leqslant s < \frac{n}{N}, \\ 0, & otherwise, \end{cases} \tag{5}$$

for $n = 1, 2, \cdots, N$, and $m = 0, 1, \cdots, M-1$.
The set of $\{h_{n,m}(s) : n = 1, 2, \cdots, N,\ m = 0, 1, \cdots, M-1\}$ is a complete orthogonal set in the Hilbert space $\mathcal{L}^2[0,1)$, and its components can be considered in the following LHFs vector

$$\mathcal{H}(s) = [h_{1,0}(s), \cdots, h_{1,M-1}(s), \cdots, h_{N,0}(s), \cdots, h_{N,M-1}(s)]^T. \tag{6}$$

It is simple to verify that the function $h_{n,m}(s)$ is attached in k-th component of vector $\mathcal{H}$ where $k = (n-1)M + m$.

Function Expansion: The truncated LHFs expansion of any function $f(s) \in \mathcal{L}^2[0,1)$ is defined as

$$\begin{aligned} f(s) &\simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} h_{n,m}(s) \\ &= F^T \cdot \mathcal{H}(s), \end{aligned} \tag{7}$$

where $\mathcal{H}(s)$ defined in (6) and the NM-vector F contains the coefficients $c_{n,m}$ that are defined as

$$\begin{aligned} F_k = c_{n,m} &= \frac{< f(s), h_{n,m}(s) >}{< h_{n,m}(s), h_{n,m}(s) >} \\ &= N(2m+1) \cdot \int_{\frac{n-1}{N}}^{\frac{n}{N}} f(s) h_{n,m}(s) ds, \end{aligned}$$

for $k = 1, 2, \cdots, NM$, and called LHFs coefficients vector.

The uniform convergence and the expected error for (7) are showed in [14], and the results can be summarized in the following theorem.

Theorem 2.4. *Let $f(s)$ be a continuous function defined on $[0,1)$, and $\bar{f}(s)$ be its truncated LHFs expansion. If $|f''(s)| \leqslant M_2$, then we have the following error estimation*

$$\|f(s)-\bar{f}(s)\|_2^2 \leqslant \frac{3}{8}M_2^2 \sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(2m-3)^4}.$$

Proof. See [14].
The above theorem guarantees the uniform convergence of the function expansion with respect to LHFs. □

The Integration of Cross Products: In continuation of this section, we will encounter to the product of $\mathcal{H}(s)$ and $\mathcal{H}^T(s)$, which is called the product matrix of the Legendre hybrid functions. It is clear that

$$h_{n_p,m_i}(s)h_{n_q,m_j}(s) = \begin{cases} h_{n_p,m_i}(s)h_{n_p,m_j}(s), & n_p = n_q, \\ 0, & n_p \neq n_q, \end{cases}$$

for $n_p, n_q = 1, 2, \cdots, N$, and $m_i, m_j = 0, 1, \cdots, M-1$. Therefore

$$\mathcal{H}(s)\cdot\mathcal{H}^T(s) = \begin{bmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & H_N \end{bmatrix}, \tag{8}$$

in which the $M \times M$ matrices H_{n_p} are as follows

$$\left(H_{n_p}\right)_{m_i,m_j} = h_{n_p,m_i}(s)h_{n_p,m_j}(s), \qquad \begin{array}{l} n_p = 1, 2, \cdots, N, \\ m_i, m_j = 0, 1, \cdots, M-1. \end{array} \tag{9}$$

So,

$$D = \int_0^1 \mathcal{H}(s)\cdot\mathcal{H}^T(s)ds = \begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & L & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L \end{bmatrix}, \tag{10}$$

where L is an $M \times M$ diagonal matrix that is given by

$$L = \frac{1}{N}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{2M-1} \end{bmatrix}.$$

Evaluation of cross products: Let F be an arbitrary NM-vector of the form

$$F = [c_{1,0}, \cdots, c_{1,M-1}(s), \cdots, c_{N,0}(s), \cdots, c_{N,M-1}(s)]^T. \tag{11}$$

If $\mu = (n_p - 1)M + m_i$, the μth component of $\mathcal{H}(s) \cdot \mathcal{H}^T(s) \cdot F$ can be computed as

$$\left(\mathcal{H}(s) \cdot \mathcal{H}^T(s) \cdot F\right)_\mu = \sum_{k=0}^{M-1} h_{n_p,m_i}(s) h_{n_p,k}(s) c_{n_p,k}.$$

By expanding the components of $\mathcal{H}(s) \cdot \mathcal{H}^T(s) \cdot F$ in terms of LHFs, we have

$$\mathcal{H}(s) \cdot \mathcal{H}^T(s) \cdot F \simeq \tilde{F} \cdot \mathcal{H}(s). \tag{12}$$

It is remarkable that the $(NM \times NM)$-matrix $\tilde{F}$ is a block matrix too, and we have

$$\tilde{F} = \begin{bmatrix} \tilde{F}_1 & 0 & \cdots & 0 \\ 0 & \tilde{F}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{F}_N \end{bmatrix}, \tag{13}$$

in which the components of the $M \times M$ matrices $\tilde{F}_{n_p}$ can be computed as follows

$$\begin{aligned} \left(\tilde{F}_{n_p}\right)_{m_i,m_j} &= \sum_{k=0}^{M-1} \frac{< h_{n_p,m_i}(s) h_{n_p,k}(s)\ ,\ h_{n_p,m_j}(s) >}{< h_{n_p,m_j}(s), h_{n_p,m_j}(s) >} c_{n_p,k} \\ &= N(2m_j - 1) \\ &\cdot \sum_{k=0}^{M-1} \left[\int_{\frac{n_p-1}{N}}^{\frac{n_p}{N}} h_{n_p,m_i}(s) h_{n_p,k}(s) h_{n_p,m_j}(s) ds\right] c_{n_p,k}, \end{aligned} \tag{14}$$

for $n_p = 1, 2, \cdots, N$, and $m_i, m_j = 0, 1, \cdots, M-1$.
The components of vector F may be considered as the expansion coefficients in Eq. (7). In this situation $\tilde{F}$ is called the coefficients matrix. The calculation procedure of Eq. (12) for $N = 2$ and $M = 8$ can be found in [5].

Approximation for Power of Functions: Now, a truncated expansion of $[f(s)]^\alpha$, for positive integer $\alpha \geqslant 2$ and $f(s) \in \mathcal{L}^2[0,1)$ is computed. This idea comes from [8], and is indicated in the following lemma.

Lemma 2.5. *Let NM-vectors F and F_α be the LHFs coefficients of $f(s)$ and $[f(s)]^\alpha$, respectively. Then F_α can be computed from the following recursive formula*

$$\begin{cases} F_\alpha = F^T \cdot \widetilde{F^T_{\alpha-1}}, & \alpha = 3, 4, \cdots, \\ F_2 = F^T \cdot \tilde{F}. \end{cases} \tag{15}$$

where $\tilde{F}$ defined in Eq. (13).

Proof. see [8]. □

3. Solving Nonlinear Volterra-Fredholm Integral Equations

The results obtained in the previous section are applied to present an effective method for solving the nonlinear Volterra–Fredholm integral equations, numerically.
Consider the following nonlinear Volterra–Fredholm integral equation

$$f(s) = g(s) + \int_0^s k_1(s,t)[f(t)]^\alpha dt + \int_0^1 k_2(s,t)[f(t)]^\beta dt, \qquad 0 \leqslant s \leqslant 1, \tag{16}$$

where $\alpha, \beta \geqslant 1$ and $\mathcal{L}^2$ functions $k_1(s,t)$, $k_2(s,t)$ and $g(s)$ are known but $f(s)$ is not [1]. We can Approximate the functions f, $[f(s)]^\alpha$, and

$[f(s)]^\beta$ with respect to the Legendre hybrid functions as follows

$$\begin{array}{lll} f(s) & \simeq & F^T \cdot \mathcal{H}(s) = \mathcal{H}^T(s) \cdot F, \\ [f(s)]^\alpha & \simeq & F_\alpha^T \cdot \mathcal{H}(s) = \mathcal{H}^T(s) \cdot F_\alpha, \\ [f(s)]^\beta & \simeq & F_\beta^T \cdot \mathcal{H}(s) = \mathcal{H}^T(s) \cdot F_\beta, \end{array} \tag{17}$$

where $\mathcal{H}(s)$ is defined in Eq. (6), and NM-vectors F, F_α, and F_β are LHFs coefficients of f, $[f(s)]^\alpha$, and $[f(s)]^\beta$, respectively. Elements of F_α, and F_β are nonlinear combinations of the elements F.

By substituting Eqs. (17) in Eq. (16), we have

$$\begin{array}{rcl} F^T\mathcal{H}(s) & \simeq & g(s) + F_\alpha^T \int_0^s k_1(s,t)\mathcal{H}(t)dt + F_\beta^T \int_0^1 k_2(s,t)\mathcal{H}(t)dt \\ & = & g(s) + F_\alpha^T \mathcal{K}_1(s) + F_\beta^T \mathcal{K}_2(s), \end{array}$$

in which $\mathcal{K}_1(s)$ and $\mathcal{K}_2(s)$ are two NM-vectors with the following components

$$\begin{array}{rcl} (\mathcal{K}_1(s))_{n,m} & = & \int_0^s k_1(s,t)h_{n,m}(t)dt \\ & = & \begin{cases} 0, & s < \frac{n-1}{N}, \\ \int_{\frac{n-1}{N}}^s k_1(s,t)h_{n,m}(t)dt, & \frac{n-1}{N} \leqslant s < \frac{n}{N}, \\ \int_{\frac{n-1}{N}}^{\frac{n}{N}} k_1(s,t)h_{n,m}(t)dt, & s \geqslant \frac{n}{N}, \end{cases} \end{array}$$

$$(\mathcal{K}_2(s))_{n,m} = \int_0^1 k_2(s,t)h_{n,m}(t)dt = \int_{\frac{n-1}{N}}^{\frac{n}{N}} k_2(s,t)h_{n,m}(t)dt,$$

for $n = 1, 2, \cdots, N$, and $m = 0, 1, \cdots, M-1$. Now, let $r(s)$ be the residual of Eq. (16) when using the approximate solution (17), that is

$$r(s) = F^T\mathcal{H}(s) - F_\alpha^T \mathcal{K}_1(s) - F_\beta^T \mathcal{K}_2(s) g(s),$$

In order to obtain unknown vectors, we use the orthogonality conditions of $\{h_{n,m}(s) : n = 1, 2, \cdots, N,\ m = 0, 1, \cdots, M-1\}$ to $r(s)$, that means

$$\langle r(s), h_{n,m}(s) \rangle = 0, \qquad n = 1, 2, \cdots, N,\ m = 0, 1, \cdots, M-1,$$

so,

$$F^T \int_0^1 \mathcal{H}(s)\mathcal{H}^T(s)ds \; - \; F_\alpha^T \int_0^1 \mathcal{K}_1(s)\mathcal{H}^T(s)ds \\ - \; F_\beta^T \int_0^1 \mathcal{K}_2(s)\mathcal{H}^T(s)ds = \int_0^1 g(s)\mathcal{H}^T(s)ds,$$

and symbolically

$$F^T \cdot D - F_\alpha^T \cdot A - F_\beta^T \cdot B = G, \tag{18}$$

where D is defined in Eq. (10), and A is an $NM \times NM$ matrix of the form

$$A = \begin{bmatrix} AD_1 & AU_{1,2} & AU_{1,3} & \cdots & AU_{1,N} \\ \mathbf{0} & AD_2 & AU_{2,3} & \ddots & AU_{2,N} \\ \mathbf{0} & \mathbf{0} & AD_3 & \ddots & AU_{3,N} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & AD_N \end{bmatrix}, \tag{19}$$

in which AD_{n_p} and AU_{n_p,n_q} are $M \times M$ matrices and can be computed as follow

$$(AD_{n_p})_{m_i,m_j} = \int_{\frac{n_p-1}{N}}^{\frac{n_p}{N}} \int_{\frac{n_p-1}{N}}^{s} k_1(s,t)h_{n_p,m_i}(t)h_{n_p,m_j}(s)dtds,$$

$$(AU_{n_p,n_q})_{m_i,m_j} = \int_{\frac{n_q-1}{N}}^{\frac{n_q}{N}} \int_{\frac{n_p-1}{N}}^{\frac{n_p}{N}} k_1(s,t)h_{n_p,m_i}(t)h_{n_q,m_j}(s)dtds,$$

for $n_p = 1, 2, \cdots, N$, $n_q = n_p + 1, \cdots, N$ and $m_i, m_j = 0, 1, \cdots, M-1$. Similarly,

$$B_{NM\times NM} = \begin{bmatrix} BD_{1,1} & BD_{1,2} & \cdots & BD_{1,N} \\ BD_{2,1} & BD_{2,2} & \cdots & BD_{2,N} \\ \vdots & \vdots & & \vdots \\ BD_{N,1} & BD_{N,2} & \cdots & BD_{N,N} \end{bmatrix}, \tag{20}$$

and $M \times M$ matrices BD_{n_p,n_q} can be computed as follows

$$(DB_{n_p,n_q})_{m_i,m_j} = \int_{\frac{n_q-1}{N}}^{\frac{n_q}{N}} \int_{\frac{n_p-1}{N}}^{\frac{n_p}{N}} k_2(s,t)h_{n_p,m_i}(t)h_{n_q,m_j}(s)dtds,$$

for $n_p, n_q = 1, 2, \cdots, N$, and $m_i, m_j = 0, 1, \cdots, M-1$.

Eq. (18) is a nonlinear system of NM algebraic equations. The NM components of F are unknown and can be computed by solving this system using Newton method or other iterative methods. Hence, an approximate solution

$$f(s) \simeq F^T \cdot \mathcal{H}(s),$$

can be computed for Eq. (16). If an approximate value for $f(a)$, $0 \leqslant a < 1$ required, we can evaluate it as

$$f(a) \simeq \sum_{m=0}^{M-1} c_{n,m} h_{n,m}(a),$$

providing a belongs into the interval $\left[\frac{n-1}{N}, \frac{n}{N}\right)$.

4. Convergance Analysis

In the following theorem, we show that the solution obtained by our method converges to the exact solution of Eq. (16).

Theorem 4.1. *Let $f(s)$ be the exact solution of Eq. (16), and $f_{MN}(s)$ be its approximate solution obtained by the proposed method. If $K_1 = \|k_1(s,t)\| < \infty$ and $K_2 = \|k_2(s,t)\| < \infty$, then $f_{MN}(s)$ converges to $f(s)$ when $M, N \to \infty$.*

Proof. We have

$$f(s) = g(s) + \int_0^s k_1(s,t)[f(t)]^\alpha dt + \int_0^1 k_2(s,t)[f(t)]^\beta dt,$$

and

$$f_{MN}(s) \simeq g(s) + \int_0^s k_1(s,t)[f_{MN}(t)]^\alpha dt + \int_0^1 k_2(s,t)[f_{MN}(t)]^\beta dt.$$

So

$$\begin{aligned} e_{MN} &= \|f(s) - f_{MN}(s)\|_2 \\ &\leqslant \| \int_0^s k_1(s,t)([f(t)]^\alpha - [f_{MN}(t)]^\alpha) dt \|_2 \\ &+ \| \int_0^1 k_2(s,t)([f(t)]^\beta - [f_{MN}(t)]^\beta) dt \|_2 \\ &\leqslant K_1 \int_0^s \|[f(t)]^\alpha - [f_{MN}(t)]^\alpha\|_2 dt \\ &+ K_2 \int_0^1 \|[f(t)]^\beta - [f_{MN}(t)]^\beta\|_2 dt. \end{aligned}$$

Suppose that

$$\|[f(t)]^\alpha - [f_{MN}(t)]^\alpha\|_2 \leqslant C_\alpha \|f(t) - f_{MN}(t)\|_2,$$

$$\|[f(t)]^\beta - [f_{MN}(t)]^\beta\|_2 \leqslant C_\beta \|f(t) - f_{MN}(t)\|_2.$$

Then, using theorem 2., we have

$$\begin{aligned} e_{MN} &\leqslant K_1 C_\alpha \int_0^s \|f(t) - f_{MN}(t)\|_2 dt + K_2 C_\beta \int_0^1 \|f(t) - f_{MN}(t)\|_2 dt \\ &\leqslant (sK_1 C_\alpha + K_2 C_\beta) \cdot \sqrt{\frac{3}{8} M_2^2 \sum_{n=N+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^5 (2m-3)^4}}. \end{aligned}$$

Hence $e_{MN} \to 0$ when $M, N \to \infty$, and the proof is completed. □

5. Numerical Examples

In this section, we implement the proposed method on some examples. In example 1, a simple integral equation is considered to illustrate the vector G and the matrices A, B, and $\tilde{F}$ in details.

Examples 2 and 3 are selected from [5, 13]. So, we can compare our results with the results obtained by another method based on Legendre hybrid functions [5], and rationalized Haar functions method [13]. The results of these methods and the exact solution of integral equations are

compared for 17 terms and reported in Tables 1 and 2.
Furthermore, the accuracy of the method is studied by computing

$$e_2 = \|\bar{f}(s) - f(s)\|_2 = \left[\sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left(\bar{f}(s) - f(s)\right)^2 ds \right]^{\frac{1}{2}}, \qquad (21)$$

where $f(s)$ and $\bar{f}(s)$ are the exact and approximate solutions of the integral equation, respectively. The results are tabulated for $N = 2$ and different values of M in Table 3.
The computations associated with the examples were performed using Matlab 7.0 software on a personal computer.

Example 5.1. Consider the following nonlinear Volterra–Fredholm integral equation

$$f(s) = \frac{1}{2}s^3 - \frac{11}{12}s^2 + s + 1 + \int_0^s (s-3t)f(t)dt + \int_0^1 s^2 t[f(t)]^2 dt, \qquad 0 \leqslant s < 1, \qquad (22)$$

with the exact solution $f(s) = s + 1$. Choosing $N = 2$ and $M = 4$, we have

$$A = \frac{1}{2}\begin{bmatrix} \frac{-1}{24} & \frac{-1}{48} & \frac{-1}{240} & 0 & 0 & \frac{1}{24} & 0 & 0 \\ \frac{-1}{48} & \frac{-1}{60} & \frac{-1}{120} & \frac{-1}{560} & \frac{-1}{8} & 0 & 0 & 0 \\ \frac{1}{48} & \frac{1}{120} & \frac{-1}{420} & \frac{-1}{280} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{240} & \frac{1}{280} & \frac{-1}{1260} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-7}{24} & \frac{-5}{48} & \frac{-1}{240} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{-1}{60} & \frac{-1}{40} & \frac{-1}{560} \\ 0 & 0 & 0 & 0 & \frac{1}{48} & \frac{1}{40} & \frac{-1}{420} & \frac{-3}{280} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{240} & \frac{3}{280} & \frac{-1}{1260} \end{bmatrix},$$

$$B = \frac{1}{2}\begin{bmatrix} \frac{1}{96} & \frac{1}{192} & \frac{1}{960} & 0 & \frac{7}{96} & \frac{1}{64} & \frac{1}{960} & 0 \\ \frac{1}{288} & \frac{1}{576} & \frac{1}{2880} & 0 & \frac{7}{288} & \frac{1}{192} & \frac{1}{2880} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{32} & \frac{1}{64} & \frac{1}{320} & 0 & \frac{7}{32} & \frac{3}{64} & \frac{1}{320} & 0 \\ \frac{1}{288} & \frac{1}{576} & \frac{1}{2880} & 0 & \frac{7}{288} & \frac{1}{192} & \frac{1}{2880} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{F} = \begin{bmatrix} \widetilde{F_1} & \mathbf{0} \\ \mathbf{0} & \widetilde{F_2} \end{bmatrix},$$

$$\widetilde{F_i} = \begin{bmatrix} c_{i,0} & c_{i,1} & c_{i,2} & c_{i,3} \\ \frac{1}{3}c_{i,1} & c_{i,0} + \frac{2}{5}c_{i,2} & \frac{2}{3}c_{i,1} + \frac{3}{7}c_{i,3} & \frac{3}{5}c_{i,2} \\ \frac{1}{5}c_{i,2} & \frac{2}{5}c_{i,1} + \frac{9}{35}c_{i,3} & c_{i,0} + \frac{2}{7}c_{i,2} & \frac{3}{5}c_{i,1} + \frac{4}{15}c_{i,3} \\ \frac{1}{7}c_{i,3} & \frac{9}{35}c_{i,2} & \frac{3}{7}c_{i,1} + \frac{4}{21}c_{i,3} & c_{i,0} + \frac{4}{15}c_{i,2} \end{bmatrix}, \qquad i = 1, 2$$

and

$$G = \begin{bmatrix} \frac{685}{1152} & \frac{157}{5760} & \frac{-13}{5760} & \frac{1}{4480} & \frac{835}{1152} & \frac{13}{640} & \frac{1}{1152} & \frac{1}{4480} \end{bmatrix}^T.$$

Solving the nonlinear system (18), the unknown vector F is obtained as

$$F = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} & 0 & 0 & \frac{7}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}^T,$$

which confirms that the proposed method gives the analytical solution of Eq. (22).

Example 5.2. [5, 13] Consider the following Volterra–Fredholm integral equation

$$f(s) = e^{2s+\frac{1}{3}} - \int_0^1 \frac{1}{3} e^{2s-\frac{1}{3}t} f(t)dt, \qquad 0 \leqslant s < 1, \qquad (23)$$

with the exact solution $f(s) = e^{2s}$. The comparison between the results of presented method, and two other methods are shown in Table 1, which confirms that our method is more efficient than Haar method [2] and another hybrid method [10].

Table 1: The results for Example 5.2.

s	Hybrid method [5] $N=2,\ M=11$	Haar method [13] $m=64$	Proposed method $N=2,\ M=11$	Analytic solution
0.0000	1.0000000000	1.0000000000	1.00000000000000	1.000000000000000
0.0625	1.1331484528	1.1311579430	1.13314845306683	1.133148453066826
0.1250	1.2840254167	1.2837603834	1.28402541668774	1.284025416687741
0.1875	1.4549914143	1.4527005824	1.45499141461820	1.454991414618201
0.2500	1.6487212706	1.6481159278	1.64872127070013	1.648721270700128
0.3125	1.8682459571	1.8655695040	1.86824595743222	1.868245957432222
0.3750	2.1170000165	2.1159577076	2.11700001661268	2.117000016612675
0.4375	2.3988752936	2.3957036931	2.39887529396710	2.398875293967098
0.5000	2.7182818283	2.7166784439	2.71828182845904	2.718281828459046
0.5625	3.0802168485	3.0764094661	3.08021684891803	3.080216848918031
0.6250	3.4903429572	3.4880191376	3.49034295746184	3.490342957461841
0.6875	3.9550767224	3.9504529800	3.95507672292058	3.955076722920577
0.7500	4.4816890699	4.4784401932	4.48168907033806	4.481689070338065
0.8125	5.0784190365	5.0727470670	5.07841903718008	5.078419037180082
0.8750	5.7546026755	5.7501660019	5.75460267600573	5.754602676005730
0.9375	6.5208191195	6.5138011998	6.52081912033011	6.520819120330113
1.0000	7.3890560982	7.3830942633	7.38905609893065	7.389056098930650

Table 2: The results for Example 5.3.

s	Hybrid method [5] $N=2,\ M=11$	Haar method [13] $m=64$	Proposed method $N=2,\ M=11$	Analytic solution
0.0000	1.00000000000	1.00000000000	1.000000000000000	1.0000000000000000
0.0625	0.99902225265	0.99499092661	0.996097563207461	0.9960975632074611
0.1250	0.98440552375	0.97582086257	0.984435939868604	0.9844359398686043
0.1875	0.96801596822	0.95627379531	0.965151656227364	0.9651516562273642
0.2500	0.93835060930	0.92441288933	0.938470479377739	0.9384704793777390
0.3125	0.90745186009	0.88989178359	0.904704774138347	0.9047047741383467
0.3750	0.86398669789	0.84418099810	0.864249846100545	0.8642498461005446
0.4375	0.82015918128	0.80152307757	0.817579313663175	0.8175793136631750
0.5000	0.76478771780	0.74919674323	0.765239563234971	0.7652395632349713
0.5625	0.71021580511	0.68888550827	0.707843352519283	0.7078433525192829
0.6250	0.64538775011	0.62093763904	0.646062636769458	0.6460626367694575
0.6875	0.58275778520	0.56004085150	0.580620702000130	0.5806207020001298
0.7500	0.51136461677	0.49332917634	0.512283697253356	0.5122836972533560
0.8125	0.44373937747	0.43081228158	0.441851665053983	0.4418516650539831
0.8750	0.36897929214	0.36370557901	0.370149175063496	0.3701491750634954
0.9375	0.29965494014	0.30573454224	0.298015670587060	0.2980156705870596
1.0000	0.22488345576	0.24264748419	0.226295640950206	0.2262956409502063

Table 3:The values of e_2 for Examples 5.2. and 5.3.

M	N	Example 5.2.	Example 5.3.
4	2	$6.7848e-04$	$4.9436e-05$
6	2	$1.4204e-06$	$7.8036e-08$
8	2	$1.5892e-09$	$6.5687e-11$
10	2	$1.1053e-12$	$3.4333e-14$
12	2	$7.5409e-16$	$3.4333e-14$

Example 5.3. [5, 13] For the following Volterra–Fredholm integral equation

$$f(s)=\cos(s)-\int_0^s (s-t)\cos(s-t)f(t)dt, \qquad 0\leqslant t\leqslant s<1, \quad (24)$$

with the exact solution $f(s)=\frac{1}{3}(2\cos\sqrt{3}s+1)$, the results for various methods are reported in table 2.

6. Comments on the Results

In this approach, applying the hybrid Legendre polynomials and block-pulse functions, a nonlinear Volterra–Fredholm integral equation can be reduced to a system of algebraic equations. Since Eq. (18) is set up in a simple manner, the suggested method can be used easily in practical cases.

The accuracy and applicability of method is checked on some examples. Example 1 shows that the exact solution of the integral equation can be computed by the method with suitable choice of M and N, when the kernel and the known term is selected by polynomials. The function approximation with respect to LHFs is uniformly converges to the function. Furthermore, Since in this method, only the unknown function is expanded by LHFs, it provides more accurate solutions than some of other existing methods.

References

[1] K. E. Atkinson, *The Numerical Solutions of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, 1997.

[2] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, Science Publisher Inc., New York, 1978.

[3] P. R. Ganti, *Piecewise Constant Orthogonal Functions and Their Application to Systems and Control*, Springer-Verlag, 1983.

[4] E. Hashemizadeh, K. Maleknejad, and B. Basirat, Hybrid functions approach for the nonlinear Volterra-Fredholm integral equations, *Procedia Computer Science*, 3 (2011), 1189-1194.

[5] C. H. Hsiao, Hybrid function method for solving Fredholm and Volterra integral euations of the second kind, *J. Comput. Appl. Math.,* 230 (2009), 59-68.

[6] C. H. Hsiao, Numerical solutions of linear time–varying descriptor systems via hybrid functions, *Appl. Math. Comput.,* 216 (2010), 1363-1374.

[7] K. Maleknejad and M. Tavassoli Kajani, Solving second kind integral equations by Galerkin methods with hybrid Legendre and Block–Pulse functions, *Appl. Math. Comput.,* 145 (2003), 623-629.

[8] K. Maleknejad, B. Basirat, and E. Hashemizadeh, Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations, *Appl. Math. Comput.,* 61 (2011), 2821-2828.

[9] K. Maleknejad, E. Hashemizadeh, and B. Basirat, Computational method based on Bernstein operational matrices for nonlinear Volterra-Fredholm-Hammerstein integral equations, *Commun. Nonlinear Sci. Numer. Simulat.,* 17 (2012), 52-61.

[10] H. R. Marzban and M. Razzaghi, Hybrid functions approach for linearly constrained quadratic optimal control problems, *Appl. Math. Model.,* 27 (2003), 471-485.

[11] Y. Ordokhani, Solution of nonlinear Volterra–Fredholm–Hammerstein integral equations via rationalized Haar functions, *Appl. Math. Comput.,* 180 (2006), 436-443.

[12] K. Parand and J. A. Rad, Numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via collocation method based on radial basis functions, *Appl. Math. Comput.,* 218 (2012), 5292-5309.

[13] M. H. Reihani and Z. Abadi, Rationalized Haar functions method for solving Fredholm and Volterra integral equations, *J. Comput. Appl. Math.,* 200 (2007), 12-20.

[14] S. Sohrabi, Study on convergence of hybrid functions method for solution of nonlinear integral equations, *Appl. Anal.,* 0 (2011), 1-13.

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