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Original Research Paper

On Infinite Systems of Caputo Fractional Differential Inclusions for Convex-Compact Multivalued Maps

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Abstract. In this work, we discuss the existence of solutions of infinite systems for a boundary value problem of Caputo fractional differential inclusions involving convex-compact multivalued maps. Our results are based on an appropriate fixed point theorem for condensing maps in the multivalued case. Eventually, we demonstrate an example to show the effectiveness and usefulness of the obtained result.

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1 Introduction

In this paper, we study the existence of solutions of the infinite system of Caputo fractional differential inclusions with initial conditions involving

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convex-compact multivalued maps (IBVP for short)

$$\begin{cases} {}^cD^{\kappa}u_i(t) \in G_i(t, (\sum_{j=i}^{i+k} |u_j(t)|^2)^{\frac{1}{2}}), \qquad t \in J = [0, \rho] \\ u_i(0) = 0, \quad u_i(\rho) = au_i(\zeta), \quad i = 1, 2, \dots \\ 1 < \kappa < 2, \quad a\zeta^{\kappa-1} < \rho^{\kappa-1}, \end{cases} \tag{1}$$

where $G_i : J \times C(J, \mathbb{R}) \to P(\mathbb{R})$, $i = 1, 2, \dots$, are convex-compact L^1-Carath*é*odory multivalued maps, $a > 0$, $0 < \zeta < 1$ and $k \in \mathbb{N}$ is an arbitrary fixed number.

The fractional calculus, an essential part of mathematics analysis, is as old as the classical calculus. The main ideas of fractional calculus can be traced back to the seventeenth century when the integral calculus and classical differential theories were introduced and developed by Newton and Leibniz [12]. For more works on this topic; see [3, 10, 20, 21, 25, 27, 33]. Fractional differential equations (FDEs) appear naturally in extensive volume of scientific problems in the fields of physics, engineering, chemistry, control, porous media, and so on. To concentrate on several applications, we refer to the recent results; see, for example works of Caponetto et al. [11], Metzler et al. [26], and Shaw et al. [35]. Boundary value problems (BVPs) of fractional differential inclusions equipped with some types of conditions like integral boundary conditions, classical, nonlocal, multipoint, and fractional have been studied by a number of scholars. For example, in [4] solvability of a class of the fractional inclusion problems have been derived via the generalized $\phi-$Riemann-Liouville and $\phi-$Caputo maps with the help of analytical techniques on $\alpha - \psi-$contractive maps and multifunctions. In [6] it has been discussed several existence results for a Caputo conformable differential inclusion with initial conditions by employing some analytical procedures on $\alpha - \psi-$contractive maps possessing the approximate endpoint property. In [7] the solvability of fractional hybrid multi-term integro-differential inclusions with initial conditions has been investigated with the help of Dhage's fixed point theorem. In [16] non-hybrid types of these inclusions are studied by using the approximate endpoint property. In [13] by applying a new class of specific functions on the spaces with properties (C_α) and (B), it has been derived the existence results for some fractional multi-term BVPs possessing integral conditions. In [14] with the help of some kinds of Dhage's fixed point theorem, the existence of solutions for a category of FDEs of hybrid type has been discussed. In [15] some existence theorems for three-point q-FDEs and their corresponding inclusions have been investigated with the help of $\alpha - \psi-$contractions and multifunctions. In [17] by applying the set-valued version of M$\ddot{o}$nch fixed point theorem together with the method of measure of noncompactness, the solvability of the neutral fractional differential inclusions of Katugampola fractional derivative has been discussed. In [28] a sequential hybrid inclusion BVP with three-point integro-derivative boundary conditions has been studied using analytical methods based on the fixed points of the product operators, $\alpha - \psi-$contractive mappings, and endpoints.

Fixed point theory for multivalued mappings is a substantial subject of multivalued analysis and finds many applications to control theory, differential and integral inclusions, and optimization. Several common fixed point theorems for single-valued maps such as those of Schauder and Banach have been extended to multivalued maps in Banach spaces; see [5, 18, 19].

There are many infinite systems of differential equations arising from the real world problems, for instance in the field of artificial intelligence, stochastic process, decomposition of polymers and so many other things. In the recent times, fixed point theory has been applied to obtain some existence results; see [30, 31], and so on. Motivated by the above papers, in this work, we discuss the solvability of infinite systems for BVPs of Caputo fractional differential inclusions involving convex-compact multivalued maps. Our results are based on an appropriate fixed point theorem for condensing maps attributed to Martelli [24].

Now, we organize this paper as follows. In Section 2, we provide some auxiliary facts that will be needed further on. In Section 3, we study the existence of solutions of IBVP (1). Eventually, an example is given to indicate effectiveness of the obtained results.

2 Preliminaries

Here, we preliminarily collect some auxiliary facts which will be needed later.

Definition 2.1. *[22, 32] Assume that* $\mathbb{R}_+ = [0, \infty)$*. The Caputo frac-*

tional derivative of order κ for a continuously differentiable function $g : \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$^cD^\kappa g(t) = \frac{1}{\Gamma(n-\kappa)} \int_0^t \frac{g^{(n)}(\upsilon)}{(t-\upsilon)^{\kappa-n+1}} d\upsilon, \quad \kappa > 0,$$

when Γ is the gamma function and $n = [\kappa] + 1$.

Definition 2.2. *[34] The Riemann-Liouville fractional integral of order α for a continuous function f on $[a,b]$ is defined by*

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \ \alpha \geq 0, \ a < x \leq b.$$

Proposition 2.3. *[29] Let $g \in C([0,\rho], \mathbb{R})$ be a given function, $1 < \kappa < 2$, $a > 0$ and $0 < \zeta < 1$. Then the unique solution of*

$$^cD^\kappa u(t) = g(t), \ u(0) = 0, \ u(\rho) = a\ u(\zeta), \ a\zeta^{\kappa-1} < \rho^{\kappa-1},$$

is

$$\begin{aligned} u(t) \quad &= \quad \int_0^t \big[(t-\upsilon)^{\kappa-1} - \frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1} - a\zeta^{\kappa-1}}\big] \frac{g(\upsilon)}{\Gamma(\kappa)} d\upsilon \\ &\quad - \frac{1}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)} \int_t^\rho (t(\rho-\upsilon))^{\kappa-1} g(\upsilon) d\upsilon \\ &\quad + \frac{a}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)} \int_0^\zeta (t(\zeta-\upsilon))^{\kappa-1} g(\upsilon) d\upsilon. \end{aligned}$$

Let $\mathcal{A}$ be a Banach space and $P(\mathcal{A}) = 2^{\mathcal{A}}$. We denote

$$P_l(\mathcal{A}) = \{C \subset \mathcal{A}, C \text{ is nonempty and possessing a property } l\}.$$

Further, we write

$$\begin{aligned} P_{cl}(\mathcal{A}) &= \{C \subset \mathcal{A}, C \text{ is nonempty and closed}\}, \\ P_{cp}(\mathcal{A}) &= \{C \subset \mathcal{A}, C \text{ is nonempty and compact}\}, \\ P_{cv}(\mathcal{A}) &= \{C \subset \mathcal{A}, C \text{ is nonempty and convex}\}, \\ P_{bd}(\mathcal{A}) &= \{C \subset \mathcal{A}, C \text{ is nonempty and bounded}\}, \\ P_{cp,cv}(\mathcal{A}) &= \{C \subset \mathcal{A}, C \text{ is nonempty, compact and convex}\}. \end{aligned}$$

A correspondence $G : \mathcal{A} \to P_l(\mathcal{A})$ is said to be a multivalued map on $\mathcal{A}$. A point $z \in \mathcal{A}$ is called a fixed point of G if $z \in G(z)$. The multivalued

map G is closed (convex) valued if $G(z)$ is closed (convex) for each $z \in \mathcal{A}$. The multivalued map G is bounded if $G(A) = \bigcup_{z\in A} G(z)$ is bounded for any $A \in P_{bd}(\mathcal{A})$, more precisely $\sup_{z\in A}\big\{ \sup\{\|y\| : y \in G(z)\}\big\} < \infty$.
Let $C_1, C_2 \in P_l(\mathcal{A})$. The Hausdorff-Pompeiu metric H_d on $P_l(\mathcal{A})$ is defined by

$$H_d(C_1, C_2) = \max\big\{ \sup_{a\in C_1} D(a, C_2), \sup_{b\in C_2} D(b, C_1)\},$$

where $D(a, C_2) = \inf\{\|a - b\| : b \in C_2\}$. The metric space $(P_{cl}(\mathcal{A}), H_d)$ is complete; see [9].
In the sequel, we mean by $\mathfrak{M}_Y$, the family of bounded subsets of the metric space (Y, d).

Definition 2.4. *[8] Suppose that (Y, d) is a metric space. Also, suppose that $\mathcal{P} \in \mathfrak{M}_Y$. The Kuratowski measure of noncompactness of $\mathcal{P}$ which is denoted by $\chi(\mathcal{P})$ is defined by*

$$\chi(\mathcal{P}) = \inf\Big\{\varepsilon > 0 : \mathcal{P} \subset \bigcup_{i=1}^{n} K_i, K_i \subset Y, diam(K_i) < \varepsilon \ (i = 1, \dots, n);\ n \in \mathbb{N}\Big\},$$

where $diam(K_i) = \sup\{d(\varsigma, \nu) : \varsigma, \nu \in K_i\}$.

Definition 2.5. *[2] Suppose that $\mathcal{A}$ is a separable Banach space, $\emptyset \neq Z \subseteq \mathcal{A}$, and $G : Z \to P_l(\mathcal{A})$ is a multivalued map. We have*
(1) G is upper semi-continuous on Z if for any $z \in Z$, $\emptyset \neq G(z) \subseteq Z$ is closed, and if for each open subset U of Z containing $G(z)$, an open neighborhood V of z exists such that $G(V) \subset U$.
(2) If for each $B \in P_{bd}(Z)$, $G(B)$ is relatively compact, then G is completely continuous.
(3) G is a condensing map if it is upper semi-continuous, and for each $B \subset Z$ with $\chi(B) \neq 0$, we have

$$\chi(G(B)) < \chi(B).$$

A completely continuous multivalued map is a condensing map; see [2]. If $G : \mathcal{A} \to P_l(\mathcal{A})$ is a multivalued operator, then graph of the operator G is defined by the set of $(z, y) \in \mathcal{A}\times\mathcal{A}$ in which $y \in G(z)$. The completely continuous map $G : \mathcal{A} \to P_{cp}(\mathcal{A})$ is upper semi-continuous

if and only if it possesses a closed graph *i.e.*, $z_n \to z$ and $w_n \to w$, $w_n \in G(z_n)$ imply $w \in G(z)$; see [2].
Let $K = [a,b]$, and $G : K \times \mathbb{R} \to P_{cp}(\mathbb{R})$. The multivalued operator $S_G : C(K,\mathbb{R}) \to P(L^1(K,\mathbb{R}))$ is defined by

$$S_G(z) = \{ f \in L^1(K,\mathbb{R}) : f(\nu) \in G(\nu, z(\nu)) \text{ for } a.e.\ \nu \in K \}$$

for any $z \in C(K,\mathbb{R})$. The operator S_G is said to be the Niemytzki operator associated with G.
A multivalued mapping $G : [0,\rho] \times \mathbb{R} \to P_{cl}(\mathbb{R})$ is called measurable if for any real number x, the function

$$s \mapsto d(z, G(s)) = \inf\{|z - x| : \ x \in G(s)\}$$

is measurable.

Definition 2.6. *[1] A mapping $G : [0,\rho] \times \mathbb{R} \to 2^{\mathbb{R}}$ is Carathéodory if*
(i) $s \mapsto G(s,z)$ is measurable for any real number z;
(ii) $z \mapsto G(s,z)$ is upper semi-continuous for almost all $0 \le s \le \rho$.
Besides, G is L^1- Carathéodory if
(iii) For any $l > 0$, $h_l \in L^1([0,\rho], \mathbb{R}_+)$ exists such that

$$\|G(s,z)\| = \sup\{\|y\| : y \in G(s,z)\} \le h_l(t)$$

for any $|z| \le l$ and for a.e. $s \in [0,\rho]$.

Now we are in position to state some important lemmata.

Lemma 2.7. *[23] Let $K = [a,b]$. Also, let $\mathcal{A}$ be a finite dimensional separable Banach space. If $G : K \times \mathcal{A} \to P(\mathcal{A})$ is compact and convex, then $S_G(z) \neq \emptyset$ for each $z \in \mathcal{A}$.*

Lemma 2.8. *[23] Assume that $K = [a,b]$, $G : K \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ is L^1-Carathéodory, and $S_G(z) \neq \emptyset$, then for each continuous linear map $\Lambda : L^1(K,\mathcal{A}) \to C(K,\mathcal{A})$, the compact convex multifunction $\Lambda o S_G : C(K,\mathcal{A}) \to P(C(K,\mathcal{A}))$ possesses a closed graph.*

Lemma 2.9. *[24] Assume that $G : \mathcal{A} \to P(\mathcal{A})$ is a compact convex condensing multivalued map and the set $\Omega := \{z \in \mathcal{A} : \eta z \in G(z) \text{ for some } \eta > 1\}$ is bounded. Then G has a fixed point.*

We end this section by dealing with the solution of IBVP (1).

Definition 2.10. *Let $\kappa \in [1,2]$ and let $u_i \in C([0,\rho],\mathbb{R})$ $(i \in \mathbb{N})$ so that its derivative of order κ, exists on the interval $[0,\rho]$. Then $u(t) = (u_i(t))$ is a solution of the system (1) if for each natural number i, functions $g_i \in L^1([0,\rho],\mathbb{R})$ exist such that $g_i(t) \in G_i(t, (\sum_{j=i}^{i+k} |u_j(t)|^2)^{\frac{1}{2}})$ a.e. on $[0,\rho]$ and*

$$
\begin{aligned}
u_i(t) \quad = \quad & \int_0^t \big[(t-\upsilon)^{\kappa-1} - \frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1} - a\zeta^{\kappa-1}}\big] \frac{g_i(\upsilon)}{\Gamma(\kappa)} d\upsilon \\
& - \frac{1}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)} \int_t^\rho (t(\rho-\upsilon))^{\kappa-1} g_i(\upsilon) d\upsilon \\
& + \frac{a}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)} \int_0^\zeta (t(\zeta-\upsilon))^{\kappa-1} g_i(\upsilon) d\upsilon.
\end{aligned}
$$

3 Main Results

In this section, we first make some sufficient conditions to discuss the solvability of IBVP (1). Eventually, we demonstrate an example to present the effectiveness of the obtained result.
Here, we consider some assumptions.
(H_1) Assume that $I = [0,1]$, $G_i : I \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$, $(i = 1, 2, ...)$ is an L^1-Carathéodory multivalued map, and the set $S_{G_i}(z) \neq \emptyset$, where $z \in C(I,\mathbb{R})$.
(H_2) For each positive number r, functions $m_{i,r} \in L^1([0,\rho],\mathbb{R}_+)$ exist such that for any $i = 1, 2, \ldots$

$$\sup_{i \in \mathbb{N}} \|m_{i,r}\|_{L^1} < \infty \text{ and } \|G_i(t, (\sum_{j=i}^{i+k} |x_j|^2)^{\frac{1}{2}})\| \leq m_{i,r}(t)$$

for almost all $t \in I$, $z = (z_i) \in \mathbb{R}^\infty$, with $(\sum_{j=i}^{i+k} |z_j|^2)^{\frac{1}{2}} \leq r$ and

$$\omega_i := \liminf_{r \to \infty} (\frac{\int_0^\rho m_{i,r}(t)dt}{r}) < \infty, \ i = 1, 2, \ldots. \tag{2}$$

regarding to (1), we define

$$\Upsilon := \frac{1}{\sqrt{k+1}} \Big(\frac{(a+4)\rho^{2\kappa-2}}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)}\Big)^{-1}. \tag{3}$$

Theorem 3.1. *Suppose that the assumptions* (H_1) *and* (H_2) *are satisfied, and*

$$\omega = \sup_{i\in\mathbb{N}} \omega_i < \Upsilon, \tag{4}$$

where ω_i *and* Υ *are respectively given by (2) and (3). Then IBVP (1) has at least one solution on* $[0,\rho]$.

Proof. Let us convert this problem into a fixed point problem. Define the multivalued map $\Psi : (C(I,\mathbb{R}))^\infty \to P((C(I,\mathbb{R}))^\infty)$ as follows:
For $y = (y_i) \in (C(I,\mathbb{R}))^\infty$, $\Psi(y)$ is the set of $(u_i) \in (C(I,\mathbb{R}))^\infty$ such that

$$u_i(t) = \int_0^t \big[(t-\upsilon)^{\kappa-1} - \frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1} - a\zeta^{\kappa-1}}\big]\frac{g_i(\upsilon)}{\Gamma(\kappa)}d\upsilon - \frac{1}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)}\times$$
$$\int_t^\rho (t(\rho-\upsilon))^{\kappa-1} g_i(\upsilon)d\upsilon + \frac{a}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta (t(\zeta-\upsilon))^{\kappa-1} g_i(\upsilon)d\upsilon,$$

where $g_i \in S_{G_i}((\sum_{j=i}^{i+k} |y_j|^2)^{\frac{1}{2}})$. We show that the hypotheses of Lemma 2.9 hold. The proof is presented in five steps:

Step 1: We show that Ψ is convex. For, assume that $y = (y_i) \in (C(I,\mathbb{R}))^\infty$. It is proved that $\Psi(y)$ is convex. Suppose that $(u_i^\gamma) \in \Psi(y)$ $(\gamma = 1,2)$. Thus we have

$$u_i^\gamma(t) = \int_0^t \big[(t-\upsilon)^{\kappa-1} - \frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1} - a\zeta^{\kappa-1}}\big]\frac{g_j^\gamma(\upsilon)}{\Gamma(\kappa)}d\upsilon - \frac{1}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)}\times$$
$$\int_t^\rho (t(\rho-\upsilon))^{\kappa-1} g_j^\gamma(\upsilon)d\upsilon + \frac{a}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta (t(\zeta-\upsilon))^{\kappa-1} g_j^\gamma(\upsilon)d\upsilon,$$

when $g_i^\gamma \in S_{G_i}((\sum_{j=i}^{i+k} |y_j|^2)^{\frac{1}{2}})$, $i \in \mathbb{N}$. Let $0 \le \nu \le 1$. For each $t \in I$ and $i \in \mathbb{N}$ we have

$$\begin{aligned}
&\nu u_i^1(t) + (1-\nu)u_i^2(t)\\
&\quad= \int_0^t \big[(t-\upsilon)^{\kappa-1} - \frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1} - a\zeta^{\kappa-1}}\big]\frac{\nu g_i^1(\upsilon) + (1-\nu)g_i^2(\upsilon)}{\Gamma(\kappa)}d\upsilon\\
&\qquad - \frac{1}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)}\int_t^\rho (t(\rho-\upsilon))^{\kappa-1}(\nu g_i^1(\upsilon) + (1-\nu)g_i^2(\upsilon))d\upsilon\\
&\qquad + \frac{a}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta (t(\zeta-\upsilon))^{\kappa-1}(\nu g_i^1(\upsilon) + (1-\nu)g_i^2(\upsilon))d\upsilon.
\end{aligned}$$

Now since G_i is convex, then the set $S_{G_i}((\sum_{j=i}^{i+k}|y_j|^2)^{\frac{1}{2}})$ is convex. Thus $\nu g_i^1+(1-\nu)g_i^2\in S_{G_i}((\sum_{j=i}^{i+k}|y_j|^2)^{\frac{1}{2}})$. It follows that $(\nu u_i^1+(1-\nu)u_i^2)\in \Psi(y)$.

Step 2: Ψ is bounded on bounded sets of $(C(I,\mathbb{R}))^\infty$. For this, let r be an arbitrary positive number and

$$B_r=\{y=(y_i)\in(C(I,\mathbb{R}))^\infty:\sup_{i\in\mathbb{N}}\sup_{t\in I}(\sum_{j=i}^{i+k}|y_j(t)|^2)^{\frac{1}{2}}\le r\}.$$

Clearly, B_r is a bounded closed convex set in $(C(I,\mathbb{R}))^\infty$. We are going to show that $\Psi(B_r)\subseteq B_r$. If it is false, then a real number $r>0$, a sequence $y^r=(y_i^r)\in B_r$, and a sequence $h_r=(h_{i,r})\in\Psi(y^r)$ exist with $\sup_{i\in\mathbb{N}}\sup_{t\in I}(\sum_{j=i}^{i+k}|h_{j,r}(t)|^2)^{\frac{1}{2}}>r$. Therefore $\eta\in\{0,1,\dots,k\}$, $i_0\in\mathbb{N}$ and $t_0\in I$ exist so that $|h_{i_0+\eta,r}(t_0)|>\frac{r}{\sqrt{k+1}}$. We get

$$\begin{aligned}h_{i_0+\eta,r}(t_0)=&\int_0^t\big[(t-\upsilon)^{\kappa-1}-\frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1}-a\zeta^{\kappa-1}}\big]\frac{g_{i_0+\eta,r}}{\Gamma(\kappa)}d\upsilon\\&-\frac{1}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_t^\rho(t(\rho-\upsilon))^{\kappa-1}g_{i_0+\eta,r}(\upsilon)d\upsilon\\&+\frac{a}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta(t(\zeta-\upsilon))^{\kappa-1}g_{i_0+\eta,r}(\upsilon)d\upsilon,\end{aligned}$$

where $g_{i_0+\eta,r}\in S_{G_{i_0+\eta}}((\sum_{j=i_0+\eta}^{i_0+\eta+k}|y_j^r|^2)^{\frac{1}{2}})$. Applying assumption (H_2), we deduce that

$$\begin{aligned}\frac{r}{\sqrt{k+1}}&<|h_{i_0+\eta,r}(t_0)|\\&\le\int_0^\rho\big|[(\rho-s)^{\kappa-1}+\frac{(\rho(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1}-a\zeta^{\kappa-1}}]\big|\frac{g_{i_0+\eta,r}(\upsilon)}{\Gamma(\kappa)}d\upsilon\\&\quad+\frac{1}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\rho(\rho(\rho-\upsilon))^{\kappa-1}g_{i_0+\eta,r}(\upsilon)d\upsilon\\&\quad+\frac{a}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta(\rho(\zeta-\upsilon))^{\kappa-1}g_{i_0+\eta,r}(\upsilon)d\upsilon\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^\rho [\rho^{\kappa-1}+\frac{\rho^{2\kappa-2}}{\rho^{\kappa-1}-a\zeta^{\kappa-1}}]\frac{m_{i_0+\eta,r}(\upsilon)}{\Gamma(\kappa)}d\upsilon\\
&\quad+\frac{1}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\rho \rho^{2\kappa-2}m_{i_0+\eta,r}(\upsilon)d\upsilon\\
&\quad+\frac{a}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\rho \rho^{2\kappa-2}m_{i_0+\eta,r}(\upsilon)d\upsilon\\
&= \frac{(a+4)\rho^{2\kappa-2}}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\rho m_{i_0+\eta,r}(\upsilon)d\upsilon.
\end{aligned}
$$

when $g_{i_0+\eta,r}\in S_{G_{i_0+\eta}}((\sum_{j=i_0+\eta}^{i_0+\eta+k}|y_j^r|^2)^{\frac{1}{2}})$. If we divide both sides by r and take the lower limit as $r\to\infty$, then we obtain

$$
\Upsilon=\frac{1}{\sqrt{k+1}}(\frac{(a+4)\rho^{2\kappa-2}}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)})^{-1}\leq \liminf_{r\to\infty}(\frac{\int_0^\rho m_{i_0+\eta,r}(\upsilon)d\upsilon}{r})\leq \sup_{i\in\mathbb{N}}\omega_i=\omega.
$$

It contradicts (4). Thus, for any real number $r>0$, $\Psi(B_r)\subseteq B_r$.

Step 3: The multivalued operator Ψ maps bounded sets of $(C(I,\mathbb{R}))^\infty$ into equicontinuous sets. For, suppose that $r>0$ is an arbitrary fixed number, $h=(h_i)\in\Psi(y)$, and $y\in B_r$. Then there exists $g_i\in S_{G_i}((\sum_{j=i}^{i+k}|y_j|^2)^{\frac{1}{2}})$ such that for any $t\in I$, we have

$$
h_i(t)=\int_0^t [(t-\upsilon)^{\kappa-1}-\frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1}-a\zeta^{\kappa-1}}]\frac{g_i(\upsilon)}{\Gamma(\kappa)}d\upsilon-\frac{1}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\times
$$
$$
\int_t^\rho (t(\rho-\upsilon))^{\kappa-1}g_i(\upsilon)d\upsilon+\frac{a}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta (t(\zeta-\upsilon))^{\kappa-1}g_i(\upsilon)d\upsilon,
$$

Let $t,\upsilon\in[0,\rho]$ be such that $t<\upsilon$, we can write

$|\pi_i(h(\upsilon))-\pi_i(h(t))|$

$$
\begin{aligned}
&\leq \int_t^\upsilon \Big|[(\upsilon-\tau)^{\kappa-1}-\frac{(\upsilon(\rho-\tau))^{\kappa-1}}{\rho^{\kappa-1}-a\zeta^{\kappa-1}}]\Big|\frac{m_{i,r}(\tau)}{\Gamma(\kappa)}d\tau\\
&\quad+\int_0^\rho \Big|[(\upsilon-\tau)^{\kappa-1}-(t-\tau)^{\kappa-1}-\frac{(\upsilon(\rho-\tau))^{\kappa-1}-(t(\rho-\tau))^{\kappa-1}}{\rho^{\kappa-1}-a\zeta^{\kappa-1}}]\Big|\frac{m_{i,r}(\tau)}{\Gamma(\kappa)}d\tau\\
&\quad+\frac{1}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\Big|\int_\upsilon^\rho (t(\rho-\tau))^{\kappa-1}m_{i,r}(\tau)d\tau-\int_t^\rho (\upsilon(\rho-\tau))^{\kappa-1}m_{i,r}(\tau)d\tau\Big|\\
&\quad+\frac{a}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta \Big|\big((t(\zeta-\tau))^{\kappa-1}-(\upsilon(\zeta-\tau))^{\kappa-1}\big)\Big|m_{i,r}(\tau)d\tau, \qquad (5)
\end{aligned}
$$

where $\pi_i(h(t)) = h_i(t)$ for each $i \in \mathbb{N}$. Now if υ tends to t, then the right side of (5) converges to zero. Therefore, Ψ is equicontinuous. In view of Steps 2 and 3 and the Arzelá-Ascoli Theorem, we infer that Ψ is completely continuous.

Step 4: Ψ possesses a closed graph. For, suppose that $(y_n) = ((y_n^m)_{m=1}^{\infty}) \in (C(I,\mathbb{R}))^{\infty}$ converges to y as $n \to \infty$ and take $h_n = (h_n^m)_{m=1}^{\infty}$ in which $h_n \in \Psi(y_n)$ (or $(h_n^m)_{m=1}^{\infty} \in \Psi((y_n^m)_{m=1}^{\infty})$) converges to $h = (h^m)$. It is shown that $h \in \Psi(y)$. For any $g_n^m \in S_{G_m}((\sum_{j=m}^{m+k} |y_{n,j}|^2)^{\frac{1}{2}})$, we have

$$\begin{aligned} h_n(t) = (h_n^m(t))_{m=1}^{\infty} &= \int_0^t \big[(t-\upsilon)^{\kappa-1} - \frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1} - a\zeta^{\kappa-1}}\big] \frac{g_n^m(\upsilon)}{\Gamma(\kappa)} d\upsilon \\ &\quad - \frac{1}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)} \int_t^{\rho} (t(\rho-\upsilon))^{\kappa-1} g_n^m(\upsilon) d\upsilon \\ &\quad + \frac{a}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)} \int_0^{\zeta} (t(\zeta-\upsilon))^{\kappa-1} g_n^m(\upsilon) d\upsilon. \end{aligned}$$

Now, we define the continuous linear operator $\Lambda : L^1(I,\mathbb{R}) \to C(I,\mathbb{R})$ by

$$\begin{aligned} g \mapsto \Lambda(g)(t) &= \int_0^t \big[(t-\upsilon)^{\kappa-1} - \frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1} - a\zeta^{\kappa-1}}\big] \frac{g(\upsilon)}{\Gamma(\kappa)} d\upsilon \\ &\quad - \frac{1}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)} \int_t^{\rho} (t(\rho-\upsilon))^{\kappa-1} g(\upsilon) d\upsilon \\ &\quad + \frac{a}{(\rho^{\kappa-1} - a\zeta^{\kappa-1})\Gamma(\kappa)} \int_0^{\zeta} (t(\zeta-\upsilon))^{\kappa-1} g(\upsilon) d\upsilon. \end{aligned}$$

We have $h_n^m \in \Lambda o S_{G_m}((\sum_{j=m}^{m+k} |y_{n,j}|^2)^{\frac{1}{2}})$. Thus, it follows by Lemma 2.8, $\Lambda o S_G$ has a closed graph. Besides, for each natural numbers m and n, $h_n^m \in \Lambda o S_{G_m}((\sum_{j=m}^{m+k} |y_{n,j}|^2)^{\frac{1}{2}})$ and $h_n^m \to h^m$ as $n \to \infty$. Then there exists $g^m \in S_{G_m}((\sum_{j=m}^{m+k} |y_{n,j}|^2)^{\frac{1}{2}})$ such that $h^m(t) = \Lambda(g^m)(t)$ or equivalently $(h^m)_{m=1}^{\infty} = (\Lambda(g^m))_{m=1}^{\infty} \in \Psi(y) = \Psi((y_m)_{m=1}^{\infty})$. Hence $h = (h^m)_{m=1}^{\infty} \in \Psi(y)$ and therefore Ψ is upper semi-continuous.

Step 5: We prove that for each arbitrary number r, the set

$$\Omega = \{y = (y_i) \in B_r : \lambda y \in \Psi(y) \text{ for some } \lambda > 1\},$$

is bounded. For this aim, let $y \in \Omega$. Then

$$\begin{aligned} y_i(t) &= \lambda^{-1}\int_0^t \big[(t-\upsilon)^{\kappa-1} - \frac{(t(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1}-a\zeta^{\kappa-1}}\big]\frac{g_i(\upsilon)}{\Gamma(\kappa)}d\upsilon \\ &\quad -\frac{\lambda^{-1}}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_t^\rho (t(\rho-\upsilon))^{\kappa-1}g_i(\upsilon)d\upsilon \\ &\quad +\frac{\lambda^{-1}a}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta (t(\zeta-\upsilon))^{\kappa-1}g_i(\upsilon)d\upsilon, \end{aligned}$$

where $g_i \in S_{G_i}((\sum_{j=i}^{i+k}|y_j|^2)^{\frac{1}{2}})$. Regarding to (H_2), we get

$$\begin{aligned} \|y\| &= \sup_{i\in\mathbb{N}}\sup_{t\in[0,1]}|y_i(t)| \\ &\leq \sup_{i\in\mathbb{N}}\lambda^{-1}\int_0^\rho \big[(\rho-\upsilon)^{\kappa-1} + \frac{(\rho(\rho-\upsilon))^{\kappa-1}}{\rho^{\kappa-1}-a\zeta^{\kappa-1}}\big]\frac{m_{i,r}(\upsilon)}{\Gamma(\kappa)}d\upsilon \\ &\quad +\frac{\lambda^{-1}}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\rho (\rho(\rho-\upsilon))^{\kappa-1}m_{i,r}(\upsilon)d\upsilon \\ &\quad +\frac{\lambda^{-1}a}{(\rho^{\kappa-1}-a\zeta^{\kappa-1})\Gamma(\kappa)}\int_0^\zeta (\rho(\zeta-\upsilon))^{\kappa-1}m_{i,r}(\upsilon)d\upsilon \\ &\leq \sup_{i\in\mathbb{N}}\frac{\|m_{i,r}\|_{L^1}}{(\sqrt{k+1})\lambda\Upsilon}. \end{aligned}$$

This shows that Ω is bounded. Hence, all the conditions of Lemma 2.9 hold. Thus, Ψ possesses a fixed point (u_i) in B_r which is a solution of (1). □

Example 3.2. Consider IBVP

$$D^{\frac{5}{3}}u_i(t) \in \big[e^{-5t}, \Big(\frac{\sin((\sum_{j=i}^{i+1}|u_j(t)|^2)^{\frac{1}{2}}) + \cos((\sum_{j=i}^{i+1}|u_j(t)|^2)^{\frac{1}{2}})}{6} + t^3 + 2\Big)e^{-t}\big],\ i\in\mathbb{N}, 0\leq t\leq 1 \tag{6}$$

with the boundary value conditions

$$u_i(0) = 0,\ u_i(1) = u_i(\frac{1}{3}) \quad (i\in\mathbb{N}).$$

Put $k = 1$, $\kappa = \frac{5}{3}$, $\zeta = \frac{1}{3}$, $a = \rho = 1$, and for any $i \in \mathbb{N}$ consider the multivalued mapping $G_i : [0,1]\times\mathbb{R} \to P(\mathbb{R})$ defined by

$$G_i(t, (\sum_{j=i}^{i+1}|u_j(t)|^2)^{\frac{1}{2}}) = \big[e^{-5t}, \Big(\frac{\sin((\sum_{j=i}^{i+1}|u_j(t)|^2)^{\frac{1}{2}}) + \cos((\sum_{j=i}^{i+1}|u_j(t)|^2)^{\frac{1}{2}})}{6} + t^3 + 2\Big)e^{-t}\big],$$

where $t \in [0,1]$ and $(u_i) \in (C([0,1],\mathbb{R}))^\infty$. Thus, (6) is a special case of system (1). Note that

$$\|G_i(t,(\sum_{j=i}^{i+1}|u_j(t)|^2)^{\frac{1}{2}})\| = \sup\left\{|y|, y \in G_i(t,(\sum_{j=i}^{i+1}|u_j(t)|^2)^{\frac{1}{2}})\right\} \leq \frac{10}{3}.$$

If for each $i \in \mathbb{N}$ and $r > 0$ take $m_{i,r}(t) = \frac{10}{3}$ ($t \in [0,1]$), then the hypotheses of Theorem 3.1 are true. Therefore IBVP (6) has a solution.

4 Conclusion

Martelli [24] constructed a fixed point theorem for condensing maps. Also, a Caputo fractional differential equation was studied by Mursaleen et al. [29]. Now, in this work, we discuss the existence of solutions of infinite systems for a boundary value problem of Caputo fractional differential inclusions involving convex-compact multivalued maps. As a future and next project, one can investigate the solvability of this infinite system for multivalued case with the help of other fixed point theorems for condensing maps.

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