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Numerical Solution of Fuzzy Differential Equations of 2nd-Order by Runge-Kutta Method

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Abstract. In this paper, solving fuzzy ordinary differential equations of the n th order by Runge-Kutta method have been done, and the convergence of the proposed method is proved. This method is illustrated by some numerical examples.

AMS Subject Classification: 65L06
Keywords and Phrases: Fuzzy differential equations; Runge-Kutta method; Convergence of numerical method

1. Introduction

The topics of fuzzy differential equations have been rapidly growing in recent years. The theory of fuzzy differential equations was treated by Buckley and Feuring [9], Kaleva [16, 17], Nieto [22], Ouyang and Wu [23], Roman-Flores and Rojas- Medar [26], Seikkala [27], also recently there appeared the papers of Bede [7], Bede and Gal [8], Diamond [10, 11], Georgiou et al., [15] Nieto and Rodriguez-Lopez [21]. In the following, we have mentioned some numerical solution which have proposed by other scientists. Abbasbandy and Allahviranloo have solved fuzzy differential equations by Runge-Kutta and Taylor methods[1, 2]. Also, Allahviranloo et al. solved differential equations by predictor-corrector and transformation methods[4, 5, 6]. Ghazanfari and Shakerami developed Runge-Kutta like formulae of order 4 for solving fuzzy differential

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equations[14]. Nystrom method has been introduced for solving fuzzy differential equations[18]. Mosleh and Otadi (2012) simulated and evaluate fuzzy differential equations by fuzzy neural network[20]. Pederson and Sambandham (2008) applied Runge-Kutta method for solving hybrid fuzzy differential equations [25]. Runge-Kutta method has been used for solving fuzzy differential equations by Palligkinis et al. (2009)[24]. Also, Kim and Sakthivel could solve hybrid fuzzy differential equations using improved predictor-corrector method[19]. The paper is organized as follows. Section 2 includes preliminaries. In Section 3, we can see the main idea of this paper. In Section 4, the proposed method is illustrated by examples. The conclusion is in Section 5.

2. Preliminaries

Definition 2.1. [12] *A fuzzy number is a map* $u : \mathbb{R} \longrightarrow I = [0, 1]$ *which satisfies*

(i) *u is upper semi continuous,*

(ii) $u(x) = 0$ *outside some interval* $[c, d] \subset \mathbb{R}$,

(iii) *There exist real numbers a, b such that $c \leqslant a \leqslant b \leqslant d$ where,*

1. $u(x)$ *is monotonic increasing on* $[c, a]$,

2. $u(x)$ *is monotonic decreasing on* $[b, d]$,

3. $u(x) = 1$, $a \leqslant x \leqslant b$.

The set of all such fuzzy numbers is represented by E^1.

Definition 2.2. [12] *An arbitrary fuzzy number in parametric form is represented by an ordered pair functions* $(\underline{u}(r), \bar{u}(r))$, $0 \leqslant r \leqslant 1$, *which satisfy the following requirements:*

1. $\underline{u}(r)$ *is a bounded left-continuous non-decreasing function over* $[0, 1]$,

2. $\bar{u}(r)$ *is a bounded left -continuous non -increasing function over* $[0, 1]$,

3. $\underline{u}(r) \leqslant \bar{u}(r), \quad 0 \leqslant r \leqslant 1.$

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \leqslant r \leqslant 1$. For arbitrary $u = (\underline{u}(r), \bar{u}(r)), v = (\underline{v}(r), \bar{v}(r))$ and $k \in \mathbb{R}$, we define equality, addition and multiplication by k as

a. $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$,

b. $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,

c. $ku = \begin{cases} (k\underline{u}, k\overline{u}), & k \geqslant 0, \\ (k\overline{u}, k\underline{u}), & k < 0. \end{cases}$

Definition 2.3. [8] *The Hausdorff distance of two fuzzy numbers given by* $D : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+ \cup \{0\}$, *is defined as follows:*

$$D(u, v) = \sup_{r\in[0,1]} \max\{|u^r_- - v^r_-|, |u^r_+ - v^r_+|\} = \sup_{r\in[0,1]} \{d_H([u]^r, [v]^r)\},$$

where $[u]^r = [u^r_-, u^r_+], [v]^r = [v^r_-, v^r_+]$. *We denote* $\|.\| = D(., 0)$.

Definition 2.4. [27] *Let* I *be a real interval. A mapping* $x : I \to E$ *is called a fuzzy process and its r-level set is denoted by*

$$[x(t)]_r = [x_1(t; r), x_2(t; r)], \quad t \in I, r \in (0, 1]$$

the derivative $x'(t)$ *of a fuzzy process* x *is defined by*

$$[x'(t)]_r = [x'_1(t; r), x'_2(t; r)],\ t \in I,\ r \in (0, 1],$$

3. Runge-kutta Mehtods for Solving Fuzzy Differential Equations

Let us consider the second-order fuzzy ordinary differential equations of the form

$$\begin{cases} \frac{d^2x}{dt^2} = f(t, x, \frac{dx}{dt}) \\ \\ x(t_0) = x_0, \quad x'(t_0) = x'_0. \end{cases} \tag{1}$$

Equation (1) can be reduced to 2 first-order simultaneous fuzzy differential equations as follows:

$$\begin{cases} \frac{dx}{dt} = y = f_1(t,x,y), \quad t \in [t_0, T] \\ \frac{dy}{dt} = f_2(t,x,y), \\ x(t_0) = x_0, \quad x'(t_0) = y(t_0) = y_0, \end{cases} \tag{2}$$

where x_0 and y_0 are fuzzy numbers.
Assume that Equation (3) and (4) are the exact and approximate solutions of Equation (2) respectively.

$$\begin{aligned} [x(t)]_r &= [x_1(t;r), x_2(t;r)] \\ [y(t)]_r &= [y_1(t;r), y_2(t;r)] \end{aligned} \tag{3}$$

$$\begin{aligned} [\hat{x}(t)]_r &= [\hat{x}_1(t;r), \hat{x}_2(t;r)] \\ [\hat{y}(t)]_r &= [\hat{y}_1(t;r), \hat{y}_2(t;r)] \end{aligned} \tag{4}$$

by using the fourth-order Runge-Kutta method for $i = 0,1,...,N$ approximate solution is calculated as follows:

$$\begin{aligned} \hat{x}_1(t_{i+1};r) &= \hat{x}_1(t_i;r) + h\sum_{j=1}^{4} w_j k_{j,1}(t_i, \hat{x}(t_i;r), \hat{y}(t_i;r)) \\ \hat{x}_2(t_{i+1};r) &= \hat{x}_2(t_i;r) + h\sum_{j=1}^{4} w_j k_{j,2}(t_i, \hat{x}(t_i;r), \hat{y}(t_i;r)) \\ \hat{y}_1(t_{i+1};r) &= \hat{y}_1(t_i;r) + h\sum_{j=1}^{4} w_j l_{j,1}(t_i, \hat{x}(t_i;r), \hat{y}(t_i;r)) \\ \hat{y}_2(t_{i+1};r) &= \hat{y}_2(t_i;r) + h\sum_{j=1}^{4} w_j l_{j,2}(t_i, \hat{x}(t_i;r), \hat{y}(t_i;r)) \end{aligned} \tag{5}$$

where the w_js are constants. Then, $k_{j,1}$, $k_{j,2}$, $l_{j,1}$ and $l_{j,2}$ for $j = 1,2,3,4$ are defined as follows:

$$k_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \min\{f(t_i,u,v)|$$
$$u \in [\hat{x}_1(t_i;r),\ \hat{x}_2(t_i;r), v \in [\hat{y}_1(t_i;r),\hat{y}_2(t_i;r)]\}$$

$$k_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \max\{f(t_i,u,v)|$$
$$u \in [\hat{x}_1(t_i;r),\hat{x}_2(t_i;r), v \in [\hat{y}_1(t_i;r),\hat{y}_2(t_i;r)]\}$$

$$k_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \min\{f(t_i+\tfrac{h}{2},u,v)|$$
$$u \in [p_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$

$$p_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v \in [q_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)), q_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$

$$k_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \max\{f(t_i+\tfrac{h}{2},u,v)|$$
$$u \in [p_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$

$$p_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v \in [q_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)), q_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$

$$k_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \min\{f(t_i+\tfrac{h}{2},u,v)|$$
$$u \in [p_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$

$$p_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v \in [q_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)), q_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$

$$k_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \max\{f(t_i+\tfrac{h}{2},u,v)|$$
$$u \in [p_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$

$$p_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v \in [q_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)), q_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$

$$k_{4,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \min\{f(t_i+h,u,v)|$$

$$u \in [p_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$

$$p_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$

$$v \in [q_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)), q_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$

$$k_{4,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \max\{f(t_i+h,u,v)|$$
$$u\in[p_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$
$$p_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v\in[q_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),q_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$

$$l_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \min\{g(t_i,u,v)| \tag{6}$$
$$u\in[\hat{x}_1(t_i;r),\hat{x}_2(t_i;r)],\ v\in[\hat{y}_1(t_i;r),\hat{y}_2(t_i;r)]\}$$
$$l_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \max\{g(t_i,u,v)|$$
$$u\in[\hat{x}_1(t_i;r),\hat{x}_2(t_i;r)],\ v\in[\hat{y}_1(t_i;r),\hat{y}_2(t_i;r)]\}$$
$$l_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \min\{g(t_i+\tfrac{h}{2},u,v)|$$
$$u\in[p_{1,1}(t_i;\hat{x}(t_i;r),\hat{y}(t_i;r)),$$
$$p_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v\in[q_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),q_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$
$$l_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \max\{g(t_i+\tfrac{h}{2},u,v)|$$
$$u\in[p_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$
$$p_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v\in[q_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),q_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$
$$l_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \min\{g(t_i+\tfrac{h}{2},u,v)|$$
$$u\in[p_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$
$$p_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v\in[q_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),q_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$
$$l_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \max\{g(t_i+\tfrac{h}{2},u,v)|$$
$$u\in[p_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$
$$p_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v\in[q_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),q_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$

$$l_{4,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \min\{g(t_i+h,u,v)|$$
$$u \in [p_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$
$$p_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v \in [q_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)), q_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$
$$l_{4,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) = \max\{g(t_i+h,u,v)|$$
$$u \in [p_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)),$$
$$p_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))],$$
$$v \in [q_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)), q_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))]\}$$

where

$$\begin{aligned}
p_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{x}_1(t_i;r) + \frac{h}{2}k_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
p_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{x}_2(t_i;r) + \frac{h}{2}k_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
p_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{x}_1(t_i;r) + \frac{h}{2}k_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
p_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{x}_2(t_i;r) + \frac{h}{2}k_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
p_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{x}_1(t_i;r) + hk_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
p_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{x}_2(t_i;r) + hk_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) \qquad (7)\\
q_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{y}_1(t_i;r) + \frac{h}{2}l_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
q_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}_2(t_i;r)) &= \hat{y}_2(t_i;r) + \frac{h}{2}l_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
q_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{y}_1(t_i;r) + \frac{h}{2}l_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
q_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{y}_2(t_i;r) + \frac{h}{2}l_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
q_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{y}_1(t_i;r) + hl_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))\\
q_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) &= \hat{y}_2(t_i;r) + hl_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))
\end{aligned}$$

Now, using the initial conditions x_0, y_0 and the fourth- order Runge-Kutta formula, we compute

$$\begin{aligned}
\hat{x}_1(t_{i+1};r) &= \hat{x}_1(t_i;r)+\frac{h}{6}(k_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))+2k_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))+\\
&\quad 2k_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) + k_{4,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)))\\
\hat{x}_2(t_{i+1};r) &= \hat{x}_2(t_i;r)+\frac{h}{6}(k_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))+2k_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))+\\
&\quad 2k_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) + k_{4,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))) \qquad (8)\\
\hat{y}_1(t_{i+1};r) &= \hat{y}_1(t_i;r)+\frac{h}{6}(l_{1,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))+2l_{2,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))+\\
&\quad 2l_{3,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) + l_{4,1}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)))\\
\hat{y}_2(t_{i+1};r) &= \hat{y}_2(t_i;r)+\frac{h}{6}(l_{1,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))+2l_{2,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r))+\\
&\quad 2l_{3,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)) + l_{4,2}(t_i,\hat{x}(t_i;r),\hat{y}(t_i;r)))
\end{aligned}$$

Then, we illustrated the following some lemmas which express the convergence of the approximate solution to the exact solution as follows.

$$\begin{aligned}
\lim_{h\to 0}\hat{x}_1(t;r) &= x_1(t;r)\\
\lim_{h\to 0}\hat{x}_2(t;r) &= x_2(t;r)\\
\lim_{h\to 0}\hat{y}_1(t;r) &= y_1(t;r)\\
\lim_{h\to 0}\hat{y}_2(t;r) &= y_2(t;r)
\end{aligned} \qquad (9)$$

Lemma 3.1. *Let the sequence of numbers* $\{w_n\}_{n=0}^{N}$ *satisfy*

$$|w_{n+1}| \leqslant A|w_n| + B \quad 0 \leqslant n \leqslant N-1$$

for some given positive constants A and B (proof [12]). Then

$$|w_n| \leqslant A^n|w_0| + B\frac{A^n-1}{A-1} \quad 0 \leqslant n \leqslant N$$

Lemma 3.2. *Let the sequence of numbers* $\{w_n\}_{n=0}^N$, $\{v_n\}_{n=0}^N$ *(proof [12]) satisfy*

$$|w_{n+1}| \leqslant |w_n| + A.\max\{|w_n|, |v_n|\} + B,$$
$$|v_{n+1}| \leqslant |v_n| + A.\max\{|w_n|, |v_n|\} + B,$$

for some given positive constants A *and* B, *and denote*

$$|u_n| = |w_n| + |v_n|, \quad 0 \leqslant n \leqslant N$$

Then

$$|u_n| \leqslant \bar{A}^n |u_0| + \bar{B}\frac{\bar{A}^n - 1}{\bar{A} - 1} \quad 0 \leqslant n \leqslant N,$$

where $\bar{A} = 1 + 2A$ *and* $\bar{B} = 2B$.

Proof. [12] Generally, if we replace the values of t_is with t and after that replace $\hat{x}_1(t;r)$, $\hat{x}_2(t;r)$, $\hat{y}_1(t;r)$ and $\hat{y}_2(t;r)$ with $\hat{u}, \hat{v}, \hat{w}, \hat{z}$, then $F_1[t,\hat{u},\hat{v},\hat{w},\hat{z}]$, $G_1[t,\hat{u},\hat{v},\hat{w},\hat{z}]$, $F_2[t,\hat{u},\hat{v},\hat{w},\hat{z}]$, $G_2[t,\hat{u},\hat{v},\hat{w},\hat{z}]$ are defined as follows:

$$F_1[t,\hat{u},\hat{v},\hat{w},\hat{z}] = k_{1,1}[t,\hat{u},\hat{v},\hat{w},\hat{z}]$$
$$+2k_{2,1}[t,\hat{u},\hat{v},\hat{w},\hat{z}] + 2k_{3,1}[t,\hat{u},\hat{v},\hat{w},\hat{z}] + k_{4,1}[t,\hat{u},\hat{v},\hat{w},\hat{z}]$$
$$G_1[t,\hat{u},\hat{v},\hat{w},\hat{z}] = k_{1,2}[t,\hat{u},\hat{v},\hat{w},\hat{z}]$$
$$+2k_{2,2}[t,\hat{u},\hat{v},\hat{w},\hat{z}] + 2k_{3,2}[t,\hat{u},\hat{v},\hat{w},\hat{z}] + k_{4,2}[t,\hat{u},\hat{v},\hat{w},\hat{z}]$$
$$F_2[t,\hat{u},\hat{v},\hat{w},\hat{z}] = l_{1,1}[t,\hat{u},\hat{v},\hat{w},\hat{z}]$$
$$+2l_{2,1}[t,\hat{u},\hat{v},\hat{w},\hat{z}] + 2l_{3,1}[t,\hat{u},\hat{v},\hat{w},\hat{z}] + l_{4,1}[t,\hat{u},\hat{v},\hat{w},\hat{z}]$$
$$G_2[t,\hat{u},\hat{v},\hat{w},\hat{z}] = l_{1,2}[t,\hat{u},\hat{v},\hat{w},\hat{z}]$$
$$+2l_{2,2}[t,\hat{u},\hat{v},\hat{w},\hat{z}] + 2l_{3,2}[t,\hat{u},\hat{v},\hat{w},\hat{z}] + l_{4,2}[t,\hat{u},\hat{v},\hat{w},\hat{z}]$$

The domain where F_1, G_1, F_2, G_2 are defined is therefore

$$K = \{[t,\hat{u},\hat{v},\hat{w},\hat{z}] \mid 0 \leqslant t \leqslant T, \ -\infty < \hat{u} < \infty, \ -\infty < \hat{v} < \infty,$$
$$-\infty < \hat{w} < \infty, \ -\infty < \hat{z} < \infty\}.$$

Theorem 3.3. *Let* $F_1[t,\hat{u},\hat{v},\hat{w},\hat{z}]$, $G_1[t,\hat{u},\hat{v},\hat{w},\hat{z}]$, $F_2[t,\hat{u},\hat{v},\hat{w},\hat{z}]$ *and* $G_2[t,\hat{u},\hat{v},\hat{w},\hat{z}]$ *belongs to* $C^4(K)$ *and let the partial derivatives of* F_1, G_1, F_2, G_2 *be bounded over* K. *Then for arbitrary fixed* $r: 0 \leqslant r \leqslant 1$, *the approximate solutions converge uniformly in* t *to the exact solutions. this theorem is simply proved (see proof theorem 4.1 in [1]).*

4. Examples

Example 4.1. Consider the following fuzzy differential equation with fuzzy initial value problem:

$$\begin{cases} x'' - 2x' = 0 \quad , \quad t \in [0, 0.5] \\ \\ x(0) = (r, 2-r), \quad x'(0) = (3+r, 4) \end{cases}$$

where the exact solution is as follow:

$$\underline{x}(t) = \frac{1}{2}r - \frac{3}{2} + \frac{1}{2}(3+r)e^{2t}$$

$$\bar{x}(t) = -r + 2e^{2t}$$

In this example, the exact and the approximate solution of the equation and the first differential for t=0.1 have been shown in Tables 1 and 2 respectively. Also, Tables 3, 4 have presented the exact and the approximate solutions of equation and the first differential for t=0.2.

Table 1: Comparison between the exact solution and the approximate solution for $t = 0.1$

r	$\underline{x}_{exa.}(0.1)$	$\underline{x}_{app.}(0.1)$	$\bar{x}_{exa.}(0.1)$	$\bar{x}_{app.}(0.1)$
0.0	0.3321	0.3315	2.4428	2.4420
0.1	0.4432	0.4426	2.3428	2.3420
0.2	0.5542	0.5536	2.2428	2.2420
0.3	0.6653	0.6647	2.1428	2.1420
0.4	0.7764	0.7757	2.0428	2.0420
0.5	0.8875	0.8868	1.9428	1.9420
0.6	0.9985	0.9978	1.8428	1.8420
0.7	1.1096	1.1088	1.7428	1.7420
0.8	1.2207	1.2199	1.6428	1.6420
0.9	1.3317	1.3310	1.5428	1.5420
1.0	1.4428	1.4420	1.4428	1.4420

Table 2: Comparison between the exact solution and the approximate solution of the first differential for $t = 0.1$

r	$\underline{x'}_{exa.}(0.1)$	$\underline{x'}_{app.}(0.1)$	$\overline{x'}_{exa.}(0.1)$	$\overline{x'}_{app.}(0.1)$
0.0	3.6642	3.6416	4.8856	4.8554
0.1	3.7863	3.7630	4.8856	4.8554
0.2	3.9085	3.8844	4.8856	4.8554
0.3	4.0306	4.0057	4.8856	4.8554
0.4	4.1528	4.1272	4.8856	4.8554
0.5	4.2749	4.2485	4.8856	4.8554
0.6	4.3970	4.3700	4.8856	4.8554
0.7	4.5192	4.4913	4.8856	4.8554
0.8	4.6413	4.6128	4.8856	4.8554
0.9	4.7635	4.7341	4.8856	4.8554
1.0	4.8856	4.8556	4.8856	4.8554

Table 3: Comparison between the exact solution and the approximate solution for $t = 0.2$

r	$\underline{x}_{exa.}(0.2)$	$\underline{x}_{app.}(0.2)$	$\bar{x}_{exa.}(0.2)$	$\bar{x}_{app.}(0.2)$
0.0	0.7377	0.7339	2.9836	2.9786
0.1	0.8623	0.8584	2.8836	2.8786
0.2	0.9869	0.9829	2.7836	2.7786
0.3	1.1115	1.1073	2.6836	2.6786
0.4	1.2361	1.2318	2.5836	2.5786
0.5	1.3607	1.3563	2.4836	2.4786
0.6	1.4853	1.4807	2.3836	2.3786
0.7	1.6099	1.6052	2.2836	2.2786
0.8	1.7345	1.7296	2.1836	2.1786
0.9	1.8591	1.8541	2.0836	2.0786
1.0	1.9836	1.9786	1.9836	1.9786

Table 4: Comparison between the exact solution and the approximate solution of the first differential for $t = 0.2$

r	$\underline{x}'_{exa.}(0.2)$	$\underline{x}'_{app.}(0.2)$	$\bar{x}'_{exa.}(0.2)$	$\bar{x}'_{app.}(0.2)$
0.0	4.4755	4.4327	5.9673	5.9103
0.1	4.6247	4.5805	5.9673	5.9103
0.2	4.7738	4.7283	5.9673	5.9103
0.3	4.9230	4.8760	5.9673	5.9103
0.4	5.0722	5.0238	5.9673	5.9103
0.5	5.2214	5.1716	5.9673	5.9103
0.6	5.3706	5.3194	5.9673	5.9103
0.7	5.5198	5.4671	5.9673	5.9103
0.8	5.6689	5.6149	5.9673	5.9103
0.9	5.8181	5.7627	5.9673	5.9103
1.0	5.9673	5.9105	5.9673	5.9103

Example 4.2. Consider the following fuzzy differential equation with fuzzy initial value problem:

$$\begin{cases} x'' - 4x' + 4x = 0 \quad , \quad t \in [0,1] \\ x(0) = (2+r, 4-r), \quad x'(0) = (5+r, 7-r) \end{cases}$$

where the exact solution is as follow:

$$\underline{x}(t) = (2+r)e^{2t} + (1-r)te^{2t}$$

$$\bar{x}(t) = (4-r)e^{2t} + (r-1)te^{2t}$$

In the following table, the error of the proposed method(PM) is compared with the method introduced by [5]. The results shows that the proposed method is better than the method presented by [5].

Table 5: The Comparison between the error of the proposed method(PM) and method presented by [5] in $t = 0.01$

r	$\underline{error}_{[5]}$	$\underline{error}_{PM}$	$\overline{error}_{[5]}$	$\overline{error}_{PM}$
0.0	0.461836e-3	4.0369e-4	0.746204e-3	4.0166e-4
0.1	0.476055e-3	3.6342e-4	0.731985e-3	3.6139e-4
0.2	0.490273e-3	3.2316e-4	0.717767e-3	3.2113e-4
0.3	0.504491e-3	2.8289e-4	0.703549e-3	2.8086e-4
0.4	0.518710e-3	2.4262e-4	0.689330e-3	2.4059e-4
0.5	0.532928e-3	2.0235e-4	0.675112e-3	2.0032e-4
0.6	0.547146e-3	1.6209e-4	0.660894e-3	1.6006e-4
0.7	0.561365e-3	1.2182e-4	0.646675e-3	1.1979e-4
0.8	0.575584e-3	8.1550e-5	0.632456e-3	7.9520e-5
0.9	0.589801e-3	4.1283e-5	0.618239e-3	3.9253e-5
1.0	0.604020e-4	1.0151e-6	0.604020e-3	1.0151e-6

5. Conclusion

So far, many methods have been proposed for solving the first-order fuzzy differential equations in comparison with the other order of fuzzy differential equations. In this paper the forth-order Runge-Kutta method numerically extended to solve the second-order fuzzy ordinary differential equations. In this method, the second-order of fuzzy differential convert to two first-order fuzzy differential equations. Then, by Runge-Kutta method, the solution of equations is calculated. Also, the proposed method have good and acceptable precise.

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