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Hybrid Functions Method Based on Radial Basis Functions for Solving Nonlinear Fredholm Integral Equations

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Abstract. A numerical technique based on hybrid of radial basis functions including Guassians (GAs) and Multiquadrics (MQs) is proposed to obtain the solution of nonlinear Fredholm integral equations. Zeros of the shifted Legendre polynomials are used as the collocation points. The integral involved in the formulation of the problems are approximated based on Legendre-Gauss-Lobatto integration rule. Some numerical examples illustrate the accuracy and validity of the proposed method.

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1. Introduction

In recent decades, the meshless methods have been extensivly used to find the approximate solution of various types of linear and nonlinear equations [1]. A meshless method does not require a structured grid and only make use of a scattered set of collocation points regardless of the connectivity information between the collocation points. The radial basis functions (RBFs) method was known as a powerful tool for the

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scattered data interpolation problem. The main advantage of numerical methods which use RBFs is meshless characteristic of these methods. All of the radial basis functions have global support, and in fact many of them, such as MQs, do not even have isolated zeros [2-4]. RBFs can be compactly and globally supported, infinitely differentiable, and contain a free parameter c called the shape parameter [3-5]. These functions can be used to approximate the solution of integral equations. Parand *et al.* [6] used a collocation method based on radial basis functions to obtain the solution of nonlinear Volterra-Fredholm-Hammerstein integral equations.

In this paper, we use a hybrid of GAs and MQs, which is called HMGFs, to approximate the solution of nonlinear Fredholm integral equations. MQs was ranked as the best based on its accuracy, visual aspects, sensitivity to parameters, execution time, storage requirements, and ease of implementation [6], and HMGFs give desired results. GAs and MQs depend on shape parameter and the shape parameter is more important in accuracy solution. In this paper, a powerful technique based on the zeros of the shifted Legendre polynomials as the collocation points is applied. Thus, we organized this paper as follows:

In Section 2, we briefly review of radial basis functions, HMGFs, Legendre-Gauss-Lobatto nodes and weights and shape parameter c. In Section 3, we implement the problem with the proposed method. Finally, we report our numerical finding and demonstrate the accuracy of the proposed method.

2. Radial Basis Functions

2.1 Definition

Let $\mathbb{R}^+ = \{x \in \mathbb{R},\ x \geqslant 0\}$ be the non-negative half-line and let $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $\psi(0) \geqslant 0$. Radial basis functions on $\mathbb{R}^d$ are functions of the form

$$\psi(||x - x_j||), \tag{1}$$

in which $x, x_j \in \mathbb{R}^d$, and $||.||$ denotes the Euclidean distance between x and x_js. If one chooses N points $\{x_j\}_{j=1}^{N}$ in $\mathbb{R}^d$ then by custom

$$f(x) = \sum_{j=1}^{N} \lambda_j \psi(||x - x_j||), \ \lambda_j \in \mathbb{R}, \tag{2}$$

is called the radial basis functions as well [6].
The standard radial functions are categorized into two major classes,

Class1. Infinitely smooth RBFs.
These basis functions are infinitely differentiable and heavily depend on the shape parameter c e.g. Hardy multiquadric (MQ, $\sqrt{r^2+c^2}$), Gaussian (GA, $exp(-cr^2)$), inverse multiquadric (IMQ, $(\sqrt{r^2+c^2})^{-1}$) and inverse quadric (IQ, $(r^2+c^2)^{-1}$).

Class 2. Infinitely smooth (except at centers) RBFs.
The basic functions of this category are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than the basis functions discussed in the Class 1. For example, thin plate spline ($r^{2n} logr$, $n = 1, 2, 3, ...$), cubic r^3 and linear r, etc.

2.2 Function approximation using HMGFs

A one dimensional function $f(x)$ can be approximated by HMGFs as follows

$$f(x) \simeq \sum_{j=1}^{N} [\lambda_j \psi_j(x) + \mu_j \gamma_j(x)] = \mathbf{C}^T \mathbf{\Phi}(x) = \mathbf{\Phi}^T(x)\mathbf{C}, \tag{3}$$

where

$$\begin{aligned}
\psi_j(x) &= \psi(||x - x_j||) = \sqrt{||x - x_j||^2 + c^2}, \\
\gamma_j(x) &= \gamma(||x - x_j||) = e^{-c||x-x_j||^2}, \\
\mathbf{\Psi}(x) &= \begin{bmatrix} \psi_1(x) & \psi_2(x) & ... & \psi_N(x) \end{bmatrix}^T, \\
\mathbf{\Gamma}(x) &= \begin{bmatrix} \gamma_1(x) & \gamma_2(x) & ... & \gamma_N(x) \end{bmatrix}^T, \\
\mathbf{\Phi}(x) &= \begin{bmatrix} \mathbf{\Psi}(x) & \mathbf{\Gamma}(x) \end{bmatrix}^T, \\
\mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & \lambda_2 & ... & \lambda_N \end{bmatrix}^T, \\
\mathbf{M} &= \begin{bmatrix} \mu_1 & \mu_2 & ... & \mu_N \end{bmatrix}^T, \\
\mathbf{C} &= \begin{bmatrix} \mathbf{\Lambda} & \mathbf{M} \end{bmatrix}^T,
\end{aligned}$$

where $[.]^T$ denotes transpose, x is input and $\{\lambda_j\}_{j=1}^N$, $\{\mu_j\}_{j=1}^N$ are the set of constant coefficient of ψ_j's and γ_j's, respectively. ψ_j's and γ_j's are the jth element of the n-vectors $\mathbf{\Psi}(x)$ and $\mathbf{\Gamma}(x)$, respectively.

2.3 Legendre-Gauss-Lobatto Nodes and Weights

Let $S_N[-1,1]$ denote the space of algebraic polynomials of degree $\leqslant N$, let $(L_k)_{k\geqslant 0}$ denote the family of well-known Legendre polynomials of order i in this space as

$$(L_i, L_j) = \frac{2}{2j+1}\delta_{ij}, \tag{4}$$

where $(.,.)$ represents the usual $L^2[-1,1]$ inner product and norm. Legendre polynomials are orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-1,1]$, and satisfy the following formula

$$L_0(x) = 1, \qquad L_1(x) = x,$$
$$L_{i+1}(x) = (\frac{2i+1}{i+1})xL_i(x) - (\frac{i}{i+1})L_{i-1}, \quad i = 1, 2, ... \tag{5}$$

Then, let x_j, $j = 0, 1, ..., N$ denote the zeros of

$$(1-x^2)\dot{L}(x), \tag{6}$$

with

$$x_0 = -1 < x_1 < x_2 < ... < x_N = 1,$$

where $\dot{L}(x)$ is the first derivative of $L(x)$.
No explicit formula for the nodes x_i's, $1 \leqslant j \leqslant N-1$ is known. However, they are computed numerically using the existing subroutines [7,8].
Now, assume $g \in S_{2N-1}[-1,1]$, we have the Legendre-Gauss-Lobatto quadrature rule as the following

$$\int_{-1}^{1} g(t)\ dt \simeq \sum_{j=0}^{N} w_j g(x_j), \tag{7}$$

where the weights are given in [9]

$$w_j = \frac{2}{N(N+1)} \times \frac{1}{(L_N(x_j))^2}.$$

2.4 Shape Parameter c

We have a sharp noise for a small change to the shape parameter c. Thus, the problem of how to select a good value for the parameter c appears in front of us. Several methods for selecting c where suggested in the literature . Hardy [10] used $c = 0.815\,d$, where $d = (1/N)\sum_{i=1}^{N} d_i$ and d_i is the distance between the ith node and its neighbour node. d is replaced by $D/\sqrt{N}$, where D is the diameter of the minimal circle enclosing all supporting points and suggested to use $c = 1.25\,(D/\sqrt{N})$, [11]. Up to now, optimization of shape parameter and its distribution are still under research (see [6] and references in it). We can say the interval of stability for any problem is difference which is observed in numerical examples.

3. Problem Statement

Consider a nonlinear Fredholm integral equations as follows:

$$f(x) = g(x) + \lambda \int_0^1 k(x,t)G(t, f(t))\; dt, \quad 0 \leqslant x \leqslant 1, \tag{8}$$

where λ is constant and $g(x)$ and the kernel $k(x,t)$ are known functions assumed to have nth derivatives on the interval $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 1$. $G(t, f(t)) = F(f(t))$, where $F(f(t))$ is given as a continuous function which is nonlinear with respect to $f(t)$. In this paper, we propose a meshless collocation method based on HMGFs to obtain the solution of nonlinear Fredholm integral equations given in Eq. (8), Using Eq. (3), it yields

$$\mathbf{\Phi}^T(x)\mathbf{C} = g(x) + \lambda \int_0^1 k(x,t)G(t, \mathbf{\Phi}^T(t)\mathbf{C})\; dt,\; 0 \leqslant x \leqslant 1. \tag{9}$$

Now, we collocate Eq. (9) at points $\{x_i\}_{i=1}^{N}$ as the zeros of Legendre polynomials

$$\mathbf{\Phi}^T(x_i)\mathbf{C} = g(x_i) + \lambda \int_0^1 k(x_i,t)G(t, \mathbf{\Phi}^T(t)\mathbf{C})\; dt,\; 0 \leqslant x \leqslant 1. \tag{10}$$

We first transform the integral over $[0,1]$ into the integral over $[-1,1]$ by using the following transformations,

$$\eta = 2t - 1, \ t \in [0,1]. \tag{11}$$

Let

$$H(x_i, t) = k(x_i, t)G(t, \Phi^T(t)C), \tag{12}$$

by substituting Eq. (12) in Eq. (10), we get

$$\mathbf{\Phi}^T(x_i)\mathbf{C} = g(x_i) + \frac{\lambda}{2}\int_{-1}^{1} H(x_i, \frac{1}{2}(\eta + 1)) \ d\eta. \tag{13}$$

By using the Legendre-Gauss-Lobatto integration rule, we can rewrite Eq. (13) as follows

$$\mathbf{\Phi}^T(x_i)\mathbf{C} = g(x_i) + \frac{\lambda}{2}\sum_{j=0}^{r_1} w_{1j} H(x_i, \frac{1}{2}(\eta_j + 1)), \ i = 1, \ldots, N. \tag{14}$$

Eq. (14) generates a system of $2N$ equations and $2N$ unknowns which can be solved by MATLAB software for the constant coefficients $\{\lambda_j\}_{j=1}^N$ and $\{\mu_j\}_{j=1}^N$, respectively.

4. Numerical Examples

In order to illustrate the performance of the proposed method to obtain the numerical solution of IEs and justify the accuracy and efficiency of the proposed method in this paper, we consider absolute error between the exact solution and the presented solution defined as

$$e(x) = |y(x) - \bar{y}(x)|, \ x \in [0,1], \tag{15}$$

where $y(x)$ is exact and $\bar{y}(x)$ is approximate solution, respectively. The obtained errors of examples listed in Tables 1 and 2 for different values of shape parameter, where x_i's are the zeros of the shifted Legendre polynomials.

The Maximum errors of the proposed method compared with the method in [12] which were calculated using three RBFs, are considered as follows

$$E_i = max\{|y(x) - \bar{y}_i(x)| \ : x \in [0,1]\}, \quad i = 1,2,3.$$

In addition, Maximum errors for Example 4.2 are listed in Table 4 which are calculated in zeros of shifted Legendre polynomials of degree 10. All the computations associated with the examples were performed using **MATLAB** on a **PC**.

Example 4.1. Consider a nonlinear integral equation given in [12,13]

$$y(x) = \sinh(x) - \frac{1}{2} + \frac{1}{2}\cosh(1)\sinh(1) - \int_0^1 y^2(x)\ dx,$$

where the exact solution is $y(x) = \sinh(x)$.

Example 4.2. Consider a nonlinear integral equation given in [14]

$$y(x) = \sin(\pi x) + \frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi t)(y(t))^3\ dt,$$

where the exact solution is $y(x) = \sin(\pi x) - \frac{20-\sqrt{391}}{3}\cos(\pi x)$.

5. Conclusion

In this paper, we applied an approximation technique to solve the nonlinear Fredholm integral equations. This method is based on the collocation method and the hybrid of radial basis functions including MQs and GAs, which is called HMGFs. We applied powerful search technique and obtained the best value of c in any example. As, it is shown in Tables, the numerical results which were obtained by HMGFs have more accuracy than those of MQs and GAs. The interval of stability for c is $[0, 2]$ and Table 4 showed that the Maximum error of HMGFs is less than those of MQs and GAs. Choosing the parameter shape is a fundamental point to calculate less errors. Therefore, the comparison of numerical results with the exact solution is shown the good reliability and efficiency of HMGFs method.

Table 1: Errors for Examples 4.1 and 4.2 with $N = 5$.

x	Example1, $c = 1$			Example 2, $c = 0.1$		
	e_{GA}	e_{MQ}	e_{HMGF}	e_{GA}	e_{MQ}	e_{HMGF}
0.0130	1.9556E-3	1.2947E-3	1.2556E-9	1.0212E-3	4.8898E-2	8.2192E-5
0.0675	7.0164E-4	4.7476E-4	1.2557E-9	8.7940E-4	1.8901E-2	8.0407E-5
0.1603	1.0355E-3	7.2488E-4	1.2556E-9	9.5286E-4	2.1450E-2	7.2038E-5
0.2833	6.9573E-4	5.0509E-4	1.2557E-9	3.8334E-5	1.1574E-3	5.1750E-5
0.4256	7.7640E-4	5.7948E-4	1.2556E-9	1.1857E-4	7.5407E-3	1.9059E-5
0.5744	7.4096E-4	5.5922E-4	1.2557E-9	1.7580E-5	8.6619E-3	1.9065E-5
0.7167	6.0739E-4	4.5607E-4	1.2556E-9	3.0791E-4	1.0653E-3	5.1775E-5
0.8397	8.3748E-4	6.1987E-4	1.2557E-9	1.5347E-4	2.1684E-2	7.2034E-5
0.9325	5.3584E-4	3.9066E-4	1.2557E-9	1.8983E-4	1.8692E-2	8.0401E-5
0.9870	1.4445E-3	1.0428E-3	1.2557E-9	1.1492E-3	5.1168E-2	8.2164E-5

Table 2: Errors for Examples 4.1 and 4.2 with $N = 10$.

x	Example1	Example2	
	$c = 1.7$	$c = 0.5$	$c = 1.7$
0.0130	9.4325E-1	3.7009E-2	3.2875E-2
0.0675	3.3012E-1	3.1299E-2	2.7752E-2
0.1603	3.3012E-1	3.1299E-2	2.7752E-2
0.2833	3.5868E-1	2.1991E-2	2.1723E-2
0.4256	3.5868E-1	2.1991E-2	2.1723E-2
0.5744	4.0346E-1	2.7479E-2	3.0981E-2
0.7167	4.0346E-1	2.7479E-2	3.0981E-2
0.8397	4.4662E-1	3.7332E-2	4.1465E-2
0.9325	4.4662E-1	5.8385E-2	2.2518E-2
0.9870	4.7418E-1	1.6997E-1	1.6584E-1

Table 3: Muximum errors for Example 4.1, using HMGF method and the method in [12] with $N = 5$.

HMGF method		Method in [12]		
$c = 1$	$c = 1.9$	$E_1 : \phi_1(x) = e^{-\|x\|}$	$E_2 : \phi_2(x) = e^{\frac{-x^2}{4}}$	$E_3 : \phi_3(x) = \frac{1}{1+x^2}$
1.255722E-9	1.157696E-9	6.0E-3	1.1E-5	2.0E-3

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Table 4: Maximum errors for Example 4.2 with $N = 5$.

c	E_{GA}	E_{MQ}	E_{HMGF}
0.0	9.9026E-1	1.1718E-1	9.9026E-1
0.2	8.5630E-4	2.9839E-2	3.8879E-4
0.4	5.0159E-4	0.0167E-2	4.1647E-4
0.6	6.3526E-4	1.9737E-3	4.1668E-4
0.8	9.3549E-4	2.6729E-3	4.1714E-4
1.0	1.0637E-3	3.1340E-3	4.1717E-4
1.2	1.0403E-3	2.8694E-3	4.1588E-4
1.4	8.8347E-4	2.3882E-3	4.1700E-4
1.6	7.8687E-4	1.8846E-3	4.1703E-4
1.8	1.1987E-3	1.4261E-3	4.1695E-4
2.0	6.2430E-4	1.0299E-3	4.1698E-4

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