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Essential Submodules with respect to an Arbitrary Submodule

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Abstract. The concept of *essential submodules* is a well known concept. In this paper we try to replace an arbitrary submodule of M, say T, instead of 0 in the definition of essential submodules. By this, essential submodules are precisely $\{0\}$-essential submodules. For a submodule K of right R-module M, we have $K \subseteq_{\text{ess}} M$ if and only if $(K : m)$ is $\text{ann}_{\text{r}}(m)$-essential right ideal of R, for each $m \in M \setminus \{0\}$. Among other things, this generalization of essential submodules gives a necessary and sufficient condition for $\frac{M}{T}$ being finitely co-generated.

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1. Introduction

Throughout this article, all rings are associative with identity and all modules are unitary right modules. We know that the submodule K of right R-module M is called essential, denoted by $K \subseteq_{\text{ess}} M$, provided that for each submodule L of M, $K \bigcap L = 0$ implies that $L = 0$. The right R-module M is called *uniform* provided that every non-zero submodule of M is an essential submodule. If K is a submodule of right R-module M, then by Zorn's Lemma, $\mathcal{S} = \{L |\ L \leqslant M \ and\ K \bigcap L = 0\}$

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has a maximal element which is called the complement of K in M and is denoted by K^c. For each $m \in M$, $(K : m) = \{r \in R|\ mr \in K\}$. In Section 2, first, the essentiality with respect to a submodule is defined and is shown, this concept is different from the concept of essentiality (Example 2.9). After that, for a submodule T of right R-module M, the relationship between essential submodules of M with respect to T and essential right ideals of R with respect to $(T : m)$, for each $m \in M \setminus \{0\}$, will be investigated (Theorem 2.7). Moreover, it will be answered, for a submodule K of M, when is K^c the largest submodule of M which has zero intersection with K?

In Section 3, for a submodule T of right R-module$M$, the intersection of all submodules of M which containing T and also are essential with respect to T will be investigated. All unexplained terminologies and basic results on modules that are used in the sequel can be found in [3], [4] and [5].

2. {}-essential submodules

The reader is reminded that a submodule K of right R-module M is essential provided that K has non-zero intersection to every non-zero submodule.

Definition 2.1. *Let R be a ring and T be a proper submodule of right R-module M. The submodule K of M is called T-essential provided that $K \nsubseteq T$ and for each submodule L of M, $K \bigcap L \subseteq T$ implies that $L \subseteq T$. In this case K is denoted by $K \trianglelefteq_T M$.*

Proposition 2.2. *For each $m, n \in \mathbb{Z}$, $m\mathbb{Z} \trianglelefteq_{n\mathbb{Z}} m\mathbb{Z} + n\mathbb{Z}$.*

Proof. Put $(n, m) = d$, $[n, m] = l$. Assume that $k\mathbb{Z} \subseteq m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}$ such that $m\mathbb{Z} \bigcap k\mathbb{Z} \subseteq n\mathbb{Z}$. Put $(k, m) = g$, $[k, m] = e$. It is clear that $nm = dl$ and $km = ge$. Since both $d|k$ and $d|m$, then $d|(k, m) = g$. On the other hand, since $n|e$ and $m|e$, then $l = [n, m]|e$. Therefore $d|g$ and $l|e$ imply that $dl|ge$. Thus $nm|km$ and hence $n|k$ which implies that $k\mathbb{Z} \subseteq n\mathbb{Z}$. $\square$

At first glance, it seems that for submodules K and $T(\neq M)$ of M, $K \trianglelefteq_T M$ if and only if $\frac{K+T}{T} \subseteq_{\text{ess}} M$. But it is not true, generally. For this, we need some assertions.

Lemma 2.3. *Let $T \subseteq K \subseteq M$ be submodules of right R-module M. Then $K \trianglelefteq TM$ if and only if $\frac{k}{T} \subseteq_{\text{ess}} \frac{M}{T}$.*

Proof. The verification is immediate. □

Proposition 2.4. *Let K and T be submodules of right R-module M. Then $\trianglelefteq TM$ implies that $\frac{K+T}{T} \subseteq_{\text{ess}} \frac{M}{T}$.*

Proof. Let $\frac{A}{T}$ be a non-zero submodule of $\frac{M}{T}$ such that $\frac{A}{T} \bigcap \frac{K+T}{T} = 0$. Therefore $K \bigcap A \subseteq T$ and hence the T-essentiality of K in M implies that $A \subseteq T$, as desired. □

Definition 2.5. *Let K be a submodule and T be a proper submodule of right R-module M. A submodule K' of M is called T-complement to K if K' is maximal with respect to the property that $K \bigcap K' \subseteq T$.*

Proposition 2.6. *Let C and S be submodules of right R-module M and $T = C \bigcap S$. Then C is T-complement to S if and only if $\frac{S+C}{C} \subseteq_{\text{ess}} \frac{M}{C}$.*

Proof. Let $\frac{S+C}{C} \subseteq_{\text{ess}} \frac{M}{C}$ and D be a submodule of M such that $C \subseteq D$ and $D \bigcap S \subseteq T$. It is clear that $\frac{D}{C} \bigcap \frac{(S+C)}{c} = 0_{\frac{M}{C}}$ because $d+C = s+C$, for $d \in D$ and $s \in S$, implies that $s \in D \bigcap S \subseteq T = C \bigcap S \subseteq C$. The essentiality $\frac{S+C}{C}$ in $\frac{M}{C}$ implies that $C = D$. Conversely, assume that D is a submodule of M containing C such that $\frac{D}{C} \bigcap \frac{S+C}{C} = 0$. If $x \in D \bigcap S$, then $x + C \in \frac{D}{C} \bigcap \frac{S+C}{C}$ and hence $x + C = C$. Therefore $D \bigcap S \subseteq C \bigcap S = T$. By assumption, $D = C$. □

By the above definition, it is easy to see that K is an essential submodule of right R-module M if and only if $K \trianglelefteq_{\{0\}} M$. It is well known that if $K \subseteq_{\text{ess}} M$, then $(K : m) \subseteq_{\text{ess}} R$, for each $m \in M$. But the converse is not true. For an example $K = \{\bar{0}, \bar{2}, \bar{4}\}$ is not essential in $\mathbb{Z}_6$ as a $\mathbb{Z}$-module but for each $\bar{x} \in \mathbb{Z}_6$, $(K : \bar{x}) \subseteq_{\text{ess}} \mathbb{Z}$ because $\mathbb{Z}$ is uniform. Now consider the following theorem.

Theorem 2.7. *Let M be an R-module and K, T be submodules of M. The following assertions are equivalent*

1. $K\Delta_T M$;
2. *For each $m \in M \setminus T$, there exists $r \in R$ such that $mr \in K \setminus T$.*
3. $(K : m) \trianglelefteq_{(T:m)} R$, *for each $m \in M \setminus T$.*

Proof. 1⇒ 2 Let $m \in M \setminus T$. Since $K\Delta_T M$, then $K \bigcap mR \nsubseteq T$. Hence there exists $r \in R$ such that $mr \in K \setminus T$.
2⇒1 By hypotheses, $K \nsubseteq T$. Assume that L is a submodule of M such that $K \bigcap L \subseteq T$. If $L \nsubseteq T$, there exists $a \in L \setminus T$. By assumption, there is an $r \in R$ such that $ar \in K \setminus T$. On the other hand $ar \in K \bigcap L \subseteq T$, a contradiction.
1⇒3 Assume that $K\Delta_T M$ and $m \in M \setminus T$. By 2, there exists $r \in R$ such that $mr \in K \setminus T$ or equivalently $(K : m) \nsubseteq (T : m)$. Suppose that I is a right ideal of R such that $(K : m) \bigcap I \subseteq (T : m)$. It is clear that $K \bigcap mI \subseteq T$ and hence $mI \subseteq T$ because $K\Delta_T M$. Now, $mI \subseteq T$ implies that $I \subseteq (T : m)$, as desired.
3⇒ 1 Suppose that L is a submodule of M such that $K \bigcap L \subseteq T$. If $L \nsubseteq T$, there exists $x \in L \setminus T$. By hypotheses, there exists $r \in R$ such that $xr \in K \setminus T$. It is a contradiction because $xr \in K \bigcap L \subseteq T$. □

Proposition 2.8. *Let $\{N_i\}_{i\in I}$, $\{M_i\}_{i\in I}$ and T be submodules of right R-module M such that $N_i \trianglelefteq_T M_i$ for every $i \in I$. Then $\oplus_{i\in I} N_i \trianglelefteq_{\oplus_{i\in I} T} \oplus M_i$.*

Proof. By Theorem 2.7, assume that $\{m_i\}_{i\in I} \in \oplus M_i \setminus \oplus T$. Since $N_i \trianglelefteq_T M_i$ for every $i \in I$, there exists an $r \in R$ such that $\{m_i r\} \in \oplus N_i \setminus \oplus T$, as desired. □

The following example shows that the converse of Proposition 2.4, is not true, generally.

Example 2.9. It is easy to check that $\frac{6\mathbb{Z}+12\mathbb{Z}}{12\mathbb{Z}} = \frac{6\mathbb{Z}}{12\mathbb{Z}}$ is an essential $\mathbb{Z}$-submodule of $\frac{\mathbb{Z}}{12\mathbb{Z}}$, but $6\mathbb{Z}$ is not $12\mathbb{Z}$-essential $\mathbb{Z}$-submodule of $\mathbb{Z}$. To the contrary, assume that $6\mathbb{Z} \trianglelefteq_{12\mathbb{Z}} \mathbb{Z}$. By Theorem 2.7, for $8 \in \mathbb{Z} \setminus 12\mathbb{Z}$ there exists an $n \in \mathbb{Z}$ such that $8n \in 6\mathbb{Z}$. Therefore $3|n$ and hence $8n \in 12\mathbb{Z}$, a contradiction.

Corollary 2.10. *Let K be a submodule of right R-module M. Then $N \subseteq_{\text{ess}} M$ if and only if $(K : m) \trianglelefteq_{\text{ann}_\text{r}(m)} R$, for each $m \in M \setminus \{0\}$.*

Proof. It is clear that for each $m \in M$, $\text{ann}_\text{r}(m) = (\{0\} : m)$. By Theorem 2.7, we have $N \subseteq_{\text{ess}} M$ if and only if $N \trianglelefteq_{\{0\}} M$ if and only if $(N : m) \trianglelefteq_{(\{0\}:m)} R$, for each $m \in M$. $\square$

Let R be a ring. An element $x \in R$ is said to be *regular* provided that $\text{ann}_\text{r}(x) = \text{ann}_l(x) = 0$ and the set of all regular elements of R is denoted by $\mathcal{C}_R$. For a right R-module M, put $\text{T}(M) = \{m \in M |\, \text{ann}_\text{r}(m) \bigcap \mathcal{C}_R \neq \emptyset\}$. If $\text{T}(M) = 0$, M is called torsion free and if $\text{T}(M) = M$, M is called torsion R-module. See [4, §10, Exercise 19].

Corollary 2.11. *Let R be a domain, M be a right R-module and K be a non-zero submodule of M. Then K is an essential submodule of M if and only if $\frac{M}{K}$ is a torsion R-module.*

Proof. For each $0 \neq m \in M$, we have $\text{ann}_\text{r}(m) = 0$ because $\mathcal{C}_R = R\setminus\{0\}$ and

$$\text{T}(M) = \{x \in M |\, \text{ann}(x) \bigcap (R\setminus\{0\}) \neq \emptyset\} = \{x \in M |\, \text{ann}_\text{r}(x) \neq 0\} = \{0\}.$$

By Theorem 2.7, $K \subseteq_{\text{ess}} M$ if and only if $K \trianglelefteq_{\{0\}} M$ if and only if $(K : m) \nsubseteq (0 : m), \forall m \in M \setminus \{0\}$ if and only if $(K : m) \nsubseteq \text{ann}_\text{r}(m) = 0, \forall m \in M \setminus \{0\}$ if and only if $\frac{M}{K}$ is a torsion R-module. $\square$

Proposition 2.12. *Let K, L and T be submodules of right R- module. Then*
1. If K and L are T-essential submodules of M, then $K \bigcap L$ is T-essential too.
2. Let $K \subseteq L \subseteq M$. Then $K \trianglelefteq_T M$ if and only if $K \trianglelefteq_T L$ and $L \trianglelefteq_T M$.

Proof. The verification is immediate. $\square$

Theorem 2.13. *Let $T_1 \leqslant K_1 \leqslant M_1 \leqslant M$ and $T_2 \leqslant K_2 \leqslant M_2 \leqslant M$*

such that $M_1 \bigcap M_2 = T_1 \bigcap T_2$. *Then,* $K_1 + K_2 \trianglelefteq_{(T_1+T_2)} M_1 + M_2$ *if and only if* $K_1 \trianglelefteq_{T_1} M_1$ *and* $K_2 \trianglelefteq_{T_2} M_2$.

Proof. Assume that $K_1 + K_2 \trianglelefteq_{(T_1+T_2)} M_1 + M_2$ and L_1 is a submodule of M_1 such that $K_1 \bigcap L_1 \subseteq T_1$. It is clear that $(K_1 + K_2) \bigcap L_1 \subseteq T_1 + T_2$($\because$ If $x \in K_1$, $y \in K_2$ and $z \in L_1$ such that $x + y = z$, then $x - z = -y \in M_1 \bigcap M_2 = T_1 \bigcap T_2$. Hence $y \in T_1 \subseteq K_1$. Therefore $z = x + y \in K_1 \bigcap L_1 \subseteq T_1$. In the other hand $x - z \in T_1$ implies that $x \in T_1$. Thus $x + y \in T_1 + T_2$). By hypothesis, $L_1 \subseteq T_1 + T_2$. It implies that $L_1 \subseteq T_1$. Similarly, we can show that $K_2 \trianglelefteq_{T_2} M_2$. Conversely, suppose that $x + y \in M_1 + M_2 \setminus T_1 + T_2$, where $x \in M_1$ and $y \in M_2$. Either $x \notin T_1$ or $y \notin T_2$. Assume that $x \in M_1 \setminus T_1$. There exists $r \in R$ such that $xr \in K_1 \setminus T_1$. If $yr \in K_2$, then the proof is completed($\because$ $(x+y)r \in K_1 + K_2 \setminus T_1 + T_2$). If $yr \in M_2 \setminus K_2 \subseteq M_2 \setminus T_2$, then there exists $s \in R$ such that $yrs \in K_2 \setminus T_2$. Hence $(x + y)rs \in K_1 + K_2 \setminus T_1 + T_2$. $\square$

Theorem 2.14. *Let* M *and* N *be* R*-modules,* $T \leqslant N$ *and* $f \in \mathrm{Hom}_R(M, N)$ *such that* $\mathrm{Im}f \nsubseteq T$. *Then* $\mathrm{Im}f \trianglelefteq_T N$ *if and only if, for each homomorphism* h*, if* $\ker h \bigcap \mathrm{Im}f \subseteq T$*, then* $\ker h \subseteq T$.

Proof. The "only if" part is clear. Conversely, let K be a submodule of N such that $\mathrm{Im}f \bigcap K \subseteq T$. Define the map $h : (\mathrm{Im}f + K) \longrightarrow \frac{M}{f^{-1}(T)}$, with $h(f(m) + k) = m + f^{-1}(T)$, for each $m \in M$ and $k \in K$. It is clear that h is an R-homomorphism such that $\ker h \bigcap \mathrm{Im}f \subseteq T$. By hypotheses, $K \subseteq \ker h \subseteq T$. $\square$

Lemma 2.15. *Let* M *and* N *be right* R*-modules,* T *and* K *be submodules of* N *and* $f \in \mathrm{Hom}_R(M, N)$. *If* $\trianglelefteq_T N$, *then* $f^{-1}(K) \trianglelefteq_{f^{-1}(T)} M$.

Proof. Assume that L be a submodule of M such that $f^{-1}(K) \bigcap L \subseteq f^{-1}(T)$. It is clear that $K \bigcap f(L) \subseteq T$ and hence $f(L) \subseteq T$. Thus $L \subseteq f^{-1}(T)$, as desired. $\square$

Corollary 2.16. *Let* M *and* N *be right* R*-modules,* K *be a submodule of* N *and* $f \in \mathrm{Hom}_R(M, N)$. *If* $K \subseteq_{\mathrm{ess}} N$, *then* $f^{-1}(K) \trianglelefteq_{\ker f} M$. *Moreover, if* f *is an epimorphism, then* $K \subseteq_{\mathrm{ess}} N$ *if and only if* $f^{-1}(K) \trianglelefteq_{\ker f}$ M.

Proof. The first part is immediate consequence of Lemma 2.15, because $f^{-1}(0) = \ker f$. Now suppose that L be a submodule of N such that $K \bigcap L = 0$. It is obvious that $f^{-1}(K) \bigcap f^{-1}(L) \subseteq \ker f$. Thus $f^{-1}(L) \subseteq \ker f$ since $f^{-1}(K) \trianglelefteq_{\ker f} M$. If $y \in L$, there exists $x \in M$ such that $y = f(x)$. Therefore $x \in f^{-1}(L) \subseteq \ker f$ and hence $y = f(x) = 0$. $\square$

Lemma 2.17. *Let K and T be submodules of right R-module M. If $K \trianglelefteq_T M$, then $K^c \subseteq T$. Moreover, if $K \trianglelefteq_T M$ and $K \bigcap T = 0$, then $K^c = T$.*

Proof. The verification is immediate. $\square$

The following proposition shows that when the complement of the submodule K of a right R-module M, is the largest submodule which has zero intersection with K.

Proposition 2.18. *Let K be a submodule of right R-module M. The following assertions are equivalent.*

1. *K is K^c-essential in M;*
2. *For each submodule N of M, $K \bigcap N = 0$ implies that $N \subseteq K^c$;*
3. *For each $x \in M \setminus K^c$ there exists $r \in R$ such that $0 \neq xr \in K$.*

Proof. 1⇒2 It is clear by definition.
1⇒3 By Theorem 2.7, For each $x \in M \setminus K^c$ there exists $r \in R$ such that $xr \in K \setminus K^c = K \setminus \{0\}$.
2⇒1 Let N be a submodule of M such that $K \bigcap N \subseteq K^c$. Then $K \bigcap N \subseteq K \bigcap K^c = \{0\}$ and by hypotheses $N \subseteq K^c$.
3⇒1 it is clear by Theorem 2.7. $\square$

As an application of the Proposition 2.18, we have the following theorem.

Theorem 2.19. *Let R be a commutative ring and $M = \oplus_{i \in F} M_i$ be an R-module, where M_i's are non-isomorphic simple submodules of M and $F = \{1, 2, \cdots, n\}$. Then, for each $I \subseteq F$, $\oplus_{i \in I} M_i \trianglelefteq_T M$, where $T = \oplus_{j \in F \setminus I} M_j$*

Proof. Let K be a submodule of M such that $(\oplus_{i\in I}M_i)\bigcap K=0$. We must show that $K\subseteq T$. By [1, Lemma 9.2], there exists a subset $J\subseteq F$ such that $M=(\oplus_{i\in I}M_i)\oplus K\oplus(\oplus_{j\in J}M_j)$. Hence

$$\mathrm{ann}(K)=\mathrm{ann}(\oplus_{t\in F\setminus(I\cup J)}M_t)\supseteq\mathrm{ann}(\oplus_{t\in F\setminus I}M_t)=\bigcap_{t\in F\setminus I}\mathrm{ann}(M_t).$$

In the other hand for each disjoint $i,j\in F\setminus I$, $\mathrm{ann}(M_i)$ and $\mathrm{ann}(M_j)$ are coprime and hence

$$\bigcap_{t\in F\setminus I}\mathrm{ann}(M_t)=\prod_{t\in F\setminus I}\mathrm{ann}(M_t),$$

by [2, Proposition 1.10]. Therefore for each $x\in K$, $x=m_1+m_2+\cdots+m_r$, where $0\neq m_i\in M_{j_i}$. Hence

$$\prod_{t\in F\setminus I}\mathrm{ann}(M_t)\subseteq\mathrm{ann}(x)\subseteq\mathrm{ann}(m_i)\ (\forall i),$$

therefore there exists $t_i\in F\setminus I$ such that $\mathrm{ann}(M_{t_i})\subseteq\mathrm{ann}(m_i)=\mathrm{ann}(M_{j_i})$. By maximality of $\mathrm{ann}(M_t)$'s we have $\mathrm{ann}(M_{t_i})=\mathrm{ann}(M_{j_i})$. Thus $M_{t_i}\cong M_{j_i}$ and hence $M_{t_i}=M_{j_i}$. Therefore $x\in\oplus_{i\in F\setminus I}M_i$, as desired. □

3. The {}-Socle

In this section, for a proper submodule T of right R-module M, the intersection of all submodules of M which containing T and simultaneously are T-essential is investigated.

Lemma 3.1. *Let K and $T(\neq M)$ be submodules of right R-module M such that $T\subseteq K$. Then there exists a submodule K' of M such that $K+K'\trianglelefteq_T M$ and $\frac{K+K'}{T}=\frac{K}{T}\oplus\frac{K'+T}{T}$.*

Proof. Define $\mathcal{S}=\{N|\ N$ *is a submodule of* M *and* $N\bigcap K\subseteq T\}$. By *Zorn's Lemma*, $\mathcal{S}$ has a maximal element, say K'. Assume that L is a submodule of M such that $(K+K')\bigcap L\subseteq T$. We clime that $K\bigcap(K'+L)\subseteq T$. For, suppose that $x\in K$, $y\in K'$, and $z\in L$ such that

$x = y + z$. Thus $x - y = z \in (K + K') \bigcap L \subseteq T \subseteq K$. Hence $y = x - z \in K \bigcap K' \subseteq T$ and hence $x \in T$, as desired. The maximality of K' in $\mathcal{S}$ implies that $L \subseteq K'$ and hence $L \subseteq T$. For the second part it is enough to show that $\frac{K}{T} \bigcap \frac{K'+T}{T} = 0$. Assume that $x \in K$ and $y \in K'$ such that $x + T = y + T$. Thus $x - y \in T \subseteq K$ and hence $y \in K \bigcap K' \subseteq T$, as desired. $\square$

Definition 3.2. Let K and T be submodules of right R-module M. K is called T-simple submodule of M provided that $\frac{K+T}{T}$ is a simple R-module. Moreover,

$$\mathrm{Soc}_T(M) = \sum\{K : K \textit{ is a } T - \textit{simple submodule of } M\}.$$

Lemma 3.3. *Let T be a submodule of right R-module M and*

$$\mathrm{S}_T(M) = \bigcap\{L : T \subseteq L \textit{ and } L \trianglelefteq_T M\}.$$

Then $\frac{\mathrm{S}_T(M)}{T}$ is a semisimple right R-module.

Proof. Let $\frac{H}{T}$ be a submodule of $\frac{\mathrm{S}_T(M)}{T}$. By Lemma 3.1, there exists a submodule H' of M such that $H + H' \trianglelefteq_T M$. Then $\frac{H}{T} \subseteq \frac{\mathrm{S}_T(M)}{T} \subseteq \frac{H+H'}{T} = \frac{H}{T} \oplus \frac{H'+T}{T}$. Then

$$\frac{\mathrm{S}_T(M)}{T} = \frac{\mathrm{S}_T(M)}{T} \bigcap (\frac{H}{T} \oplus \frac{H'+T}{T}) = \frac{H}{T} \oplus (\frac{\mathrm{S}_T(M)}{T} \bigcap \frac{H'+T}{T}). \quad \square$$

Proposition 3.4. *Let T be a submodule of right R-module M. Then*

$$\mathrm{Soc}_T(M) = \bigcap\{L : T \subseteq L \textit{ and } L \trianglelefteq_T M\}.$$

Proof. Let S be a T-simple submodule of M and L be a submodule of M containing T such that $L \trianglelefteq_T M$. Since $\frac{(S \bigcap L)+T}{T}$ is a submodule of $\frac{S+T}{T}$, then either $(S \bigcap L) + T = T$ or $(S \bigcap L) + T = S + T$. But $(S \bigcap L) + T = T$ and $L \trianglelefteq_T M$ imply that $S \subseteq T$, a contradiction. Thus $(S \bigcap L) + T = S + T$. At the other hand $L \bigcap (T + S) = T + (L \bigcap S)$ and hence $S+T \subseteq L$. Therefore $S \subseteq L$ and hence $\mathrm{Soc}_T(M) \subseteq \bigcap\{L : T \subseteq L \textit{ and } L \trianglelefteq_T M\} = \mathrm{S}_T(M)$. In the other hand by Lemma 3.3,

$$\frac{\mathrm{S}_T(M)}{T} = \sum_{i\in I} \frac{S_i}{T} = \frac{\sum_{i\in I} S_i}{T},$$

where $\frac{S_i}{T}$'s are simple R-modules. Then for each $i \in I$, S_i is a T-simple submodule of M and hence $\mathrm{S}_T(M) \subseteq \mathrm{Soc}_T(M)$. $\square$

The following theorem gives a necessary and sufficient condition under which $\frac{M}{T}$ is finitely co-generated.

Theorem 3.5. *Let T be a submodule of right R-module M. Then $\frac{M}{T}$ is finitely co-generated if and only if $\frac{\mathrm{Soc}_T(M)}{T}$ is finitely co-generated and $\mathrm{Soc}_T(M) \trianglelefteq_T M$.*

Proof. Let $\{\frac{L_i}{T}\}_{i\in I}$ be a family of submodules of $\frac{M}{T}$ such that $\bigcap_{i\in I} \frac{L_i}{T} = 0$. Then $\bigcap_{i\in I} \frac{L_i \bigcap \mathrm{Soc}_T(M)}{T} = 0$. since $\frac{\mathrm{Soc}_T(M)}{T}$ is finitely co-generated, then $\bigcap_{i\in I_0} \frac{L_i \bigcap \mathrm{Soc}_T(M)}{T} = 0$, for some finite subset I_0 of I. Therefore $(\cap_{i\in I_0} L_i) \cap \mathrm{Soc}_T(M) \subseteq T$. Since $\mathrm{Soc}_T(M) \trianglelefteq_T M$, then $(\cap_{i\in I_0} L_i) \subseteq T$ or equivalently $\cap_{i\in I_0} \frac{L_i}{T} = 0$. Conversely, assume that K be a submodule of M such that $\mathrm{Soc}_T(M) \cap K \subseteq T$. By Proposition 3.4, we have $(\bigcap\{L : T \subseteq L \textit{ and } L \trianglelefteq_T M\}) \cap K \subseteq T$. Since $\frac{M}{T}$ is finitely co-generated, then so $(\cap_{i=1}^n L_i) \cap K \subseteq T$ for finite number $L_i \in \{L : T \subseteq L \textit{ and } L \trianglelefteq_T M\}$. By Proposition 2.12, $\cap_{i=1}^n L_i \trianglelefteq_T M$ and hence $K \subseteq T$. $\square$

Corollary 3.6. *Let T be a submodule of right R-module M. Then $\frac{M}{T}$ is finitely co-generated if and only if $\frac{\mathrm{Soc}_T(M)}{T}$ is finitely generated and $\mathrm{Soc}_T(M) \trianglelefteq_T M$.*

Proof. By [1, Corllary 10.16], finitely co-generated semisimple R-modules are precisely finitely generated semisimple R-modules. Now by Lemma 3.3 and Proposition 3.4, $\frac{\mathrm{Soc}_T(M)}{T}$ is semisimple, hence $\frac{\mathrm{Soc}_T(M)}{T}$ is finitely co-generated if and only if it is finitely generated. $\square$

Definition 3.7. *Let T be a proper submodule of right R-module M. M is called T-uniform provided that for each submodule K of M, if $K \nsubseteq T$, then $K \trianglelefteq_T M$.*

Lemma 3.8. *Let T be a proper submodule of right R-module M. Then M is T-uniform if and only if for each two submodules K and N of M, $K \bigcap N \subseteq T$ implies that either $K \subseteq T$ or $N \subseteq T$.*

Proof. Let K and N be two submodules of M such that $K \bigcap N \subseteq T$ and $K \nsubseteq T$. By hypotheses, $K \trianglelefteq_T M$ and hence $L \subseteq T$. Conversely, assume that K and N are submodules of M such that $K \nsubseteq T$ and $K \bigcap L \subseteq T$, Then $L \subseteq T$, as desired. $\square$

The right R-module M is said to be *uniserial* provided that the lattice of all submodules of M is totally ordered with inclusion.

Proposition 3.9. *The right R-module M is uniserial if and only if for each proper submodule T, M is T-uniform.*

Proof. Let T be proper submodule of M. Assume that N and K are submodules of M such that $K \bigcap N \subseteq T$. Since M is uniserial, either $N \subseteq K$ or $K \subseteq N$. Hence either $K \bigcap N = K$ or $K \bigcap N = N$. Conversely, assume that N and K are submodules of M such that $K \nsubseteq N$. Hence $K \nsubseteq (K \bigcap N)$ and by assumption $K \trianglelefteq_{(K \cap N)} M$. On the other hand $K \bigcap N \subseteq K \bigcap N$. Thus $N \subseteq K \bigcap N$ and hence $N \subseteq K$. $\square$

Note that if R-module M is T-uniform, then $\frac{M}{T}$ is a uniform R-module but the converse is not true. For instance, assume that $R = \mathbb{Z}_2$ and $M = R \oplus R$ as an R-module. We know that $T = \{(x, x) |\ x \in R\}$ is a maximal submodule of M, hence $\frac{M}{T}$ is uniform. But $R \oplus 0 \nsubseteq T$ and $R \oplus 0$ is not T-essential submodule of M because $(0, 1) \in M \setminus T$ and for each $r \in R$, $(0, 1)r \notin (R \oplus 0) \setminus T$.

Example 3.10. 1. Uniform R-modules are precisely 0-uniform R-module.
2. If P is a prime ideal of a commutative ring R, then R is a P-uniform R-module. Moreover, P is a prime ideal of R if and only if R is a P-uniform R-module. Moreover, P is a semi-prime ideal of R if and only if $\frac{R}{P}$ is uniform and P is a semi-prime ideal of R.

Proposition 3.11. *For each positive integer number n, $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is a uniform $\mathbb{Z}$-module if and only if $\mathbb{Z}$ is an $n\mathbb{Z}$-uniform $\mathbb{Z}$-module.*

Proof. The "if" part is always true. For the "only if" part, assume that $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is a uniform $\mathbb{Z}$-module. It is clear that there exist a positive integer number k and a prime number p such that $n = p^k$. Suppose that $m \in \mathbb{Z}$ such that $m\mathbb{Z} \nsubseteq n\mathbb{Z}$ (or equivalently $n \nmid m$). If $t \in \mathbb{Z} \setminus n\mathbb{Z}$, then there exist integer numbers $0 \leqslant r, s < k$ and prime numbers $p_1, p_2, \cdots p_a$ such that

$$m = p^r p_1^{n_1} p_2^{n_2} \ldots p_a^{n_a} \ and$$
$$t = p^s p_1^{m_1} p_2^{m_2} \ldots p_a^{m_a}.$$

It is clear that there exists integer number b such that $tb \in m\mathbb{Z} \setminus n\mathbb{Z}$ and by Lemma 2.7, proof is complete. □

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