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Existence and Uniqueness Results for Integro Fractional Differential Equations with Atangana-Baleanu Fractional Derivative

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Abstract. In this article, we proposed integro fractional differential equations(FDEs) in the form of Atangana-Baleanu-Caputo (ABC) fractional derivative approach. The studied problem is considered with non-local integral initial condition. The existence of solution is investigated for proposed equations by using Krasnoselskii's fixed point theorem. The uniqueness of the result is derived with the help of the Banach contraction mapping principle. In the end, an example is presented to smooth the understanding of the derived results.

AMS Subject Classification: 26A33; 34A08; 34A12

Keywords and Phrases: Atangana-Baleanu-Caputo fractional derivative, Banach contraction mapping principle, Fractional differential equation, Krasnoselskii's fixed point theorem, initial condition.

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1 Introduction

Fractional calculus is an expeditiously outpacing branch of mathematical analysis that unifies the integer-order derivatives and integrals to random order. Many FDEs involve the fractional in- integral and derivative because the fractional order model can describe several real-world phenomena more realistically. FDEs has many applications in various field of real-world problems. Recently, it has been noticed that the cardiac tissues, ultra waves propagation, speech signals and tautochrone problems are studied in the form of FDEs [24]. Many researchers also stated that the fractional integrals and derivatives are more convenient for modelling some disorder regions and hereditary properties of several complex phenomena than the integer order integrals and derivatives[31, 50]. For more applications of FDEs, the reader can see [44, 36, 6].
Many mathematicians define different type fractional derivatives(FD). This task facilitates the researchers to take the most appropriate FD to obtain the batter description of results to model various fields' additional problem. Leibnitz proposed the fractional order derivative [47, 23]. After that various type of FD are established by the many authors [34, 41, 35, 32, 29, 30]. The more generally used derivatives are Riemann-Liouville and Caputo derivatives. But Riemann-Liouville and Caputo derivatives have the singular kernels i.e. the kernels used in these derivatives contains the singularities. The singular kernels creates many difficulties in applying these fractional operators. To overcome this problem Caputo and Fabrizio innovate the Caputo-Fabrizio(CF) fractional derivative. A fractional model for hearing loss with optimal control is established by Mohammadi et al. using the CF derivative[42]. Aligadeh et al. studied the RLC circuit by using the CF derivative [9]. The applications of CF derivative can be see in [18, 15, 16]. This FD contains the non-singular kernels, but this fractional derivative still conserves the non-locality property. However, CF fractional derivative gives a better description as compared to the other derivatives with the singular kernels. The associated integral with the CF derivative is in the term of classical order. To reduce this problem, Atangana and Baleanu [13] proposed a derivative with generalised Mittag Leffler(ML) function. This so-called is known as Atangana-Baleanu(AB-derivative) fractional derivative. The various model of dissipative phenomena are described

by this derivative due to the nonsingular and nonlocal behaviour of the kernel of AB-derivative. The nonlocality and nonsingularity of the kernel give a more general solution of the memory under the formation of the various scales. Many researchers provided their contribution in the evolution of FDEs related to ABC-derivative see [5, 4, 7, 1]. The rubella disease is analysed by Koca with FDEs involving AB-derivative [37]. Alka- htani and Atangana established a model by using ML function for a mixture of groundwater and chemical waste with decay objects [10]. A general model with ML & the exponential laws was proposed by Atangana and Gomez-Aguilar[14]. Koca and Atangana established the results for the elastic heat conduction equation with ML kernels [**?**]. Analytical solutions for the fractional diffusion equation with fractional derivative help are discussed by Morales-Delgado et al.[43]. Abdel- jawad [2, 3] proposed the fractional difference operators in both senses ABR and ABC derivatives with generalised ML kernels and established the fractional integrals of general order with the help of the infinite binomial theorem. Khan et al. [33] demonstrated the HU stability and existence of FDEs involving AB-derivative with the p-Laplacian operator. Alqahtani et al. [12]. derived the EU of the solutions for non-linear F-contractions involving AB-derivative in the structure of b-metric spaces. Chua's circuit model is proposed by Alkahtani [11]. with the help of the new derivative. The HU stability and existence are analysed by Devi et al. for general FDEs via the fixed point technique [28]. Prakasha et al. [45] analysed the hepatitis E virus model by using AB-derivative. AB-derivative is used by Ullah et al.[51] to establish the fractional HBV model. For more about the FDEs reader can see [26, 25, 21, 39, 27, 22, 49, 19, 20, 48, 17]. Recently, Ravichandran et al. [46] derived the HU and existence stability for a integro FDEs by using explored AB-fractional derivative. Logeswari and Ravichandarn established the EU results for netural integro-differential equations via AB-fractional derivative [40]. Abdo et al. [8] analysed the fractional boundary problem involving AB-derivative with non-linear integral conditions. Motivated by the afore-mentioned work, in this manuscript, we discuss the EU results for integro fractional differential equations involving ABC-fractional derivatives with the non-local initial condition:

$$\begin{cases} {}^{\mathcal{ABC}}_{\mathit{0}}\mathcal{D}^{\omega}\Big[\mathfrak{u}(\varkappa)+\psi^*(\varkappa,\mathfrak{u}(\varkappa))\Big]=\mathfrak{f}\Big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\Big), \quad \varkappa\in[0,1],\\ \mathfrak{u}(0)=\int_0^1 \hbar(\varsigma,\mathfrak{u}(\varsigma))d\varsigma. \end{cases} \tag{1}$$

where ${}^{\mathcal{ABC}}_{\mathit{0}}\mathcal{D}^{\omega}$ be the left Caputo AB-derivative of fractional order $\omega,\ 0<\omega\leq 1,\ \varkappa,\varsigma,T\in[0,1]$. $\psi^*,\hbar:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\mathfrak{f}:[0,1]\times\mathbb{R}^3\longrightarrow\mathbb{R}$ are continuous functions and $I_1\mathfrak{u}(\varkappa)=\int_0^{\varkappa} g(\varkappa,\varsigma,\mathfrak{u}(\varsigma))d\varsigma$ and $I_2\mathfrak{u}(\varkappa)=\int_0^{T}\varphi(\varkappa,\varsigma,\mathfrak{u}(\varsigma))d\varsigma$.
This manuscript is outlined as: in the 2^{nd} section, we serve some definitions, theorems, and lemmas necessary for problem evolution. We discuss the existence of solution in the 3^{rd} section. In tne next section uniquenees is established. An example discussed numerically in the 4^{th} part to illustrate the derived results. The last section includes the conclusion of the solved problem.

2 Basic Results and Preliminaries

Here, we contemplate some definition, lemmas and actual results.

Definition 2.1. [36] *For $\omega>0$, Riemann-Liouville (R-L) fractional integral of order $\omega\in\mathbb{R}$ is defined as*

$$I^{\omega}\mathfrak{u}(\varkappa)=\frac{1}{\Gamma(\omega)}\int_0^{\varkappa}(t-\chi)^{\omega-1}\mathfrak{u}(\chi)d\chi.$$

Definition 2.2. [36] *For $0<\omega\leq 1$, the R-L fractional derivative and Caputo fractional derivative are defined as*

$$\mathcal{D}^{\omega}\mathfrak{u}(t)=\frac{1}{\Gamma(1-\omega)}\frac{d}{dt}\bigg(\int_{\mathit{0}}^{t}(t-\chi)^{-\omega}\mathfrak{u}(\chi)d\chi\bigg),$$

$${}^{c}\mathcal{D}^{\omega}\mathfrak{u}(\varkappa)=\frac{1}{\Gamma(\mathit{1}-\omega)}\int_0^{\varkappa}(t-\chi)^{-\omega}\mathfrak{u}'(\chi)d\chi,$$

respectively.

Definition 2.3. [13] *Let* $0 < \omega \leq 1$ *and* $\mathfrak{u} \in C^1[a,b], \mathfrak{u}' \in L^1[a,b]$, *where* $0 \leq a < b$, *the Caputo AB-fractional derivative and the R-L AB-fractional derivative of order* ω *are defined by*

$$^{\mathcal{ABC}}\mathcal{D}^{\omega}\mathfrak{u}(\varkappa) = \frac{B(\omega)}{1-\omega}\int_0^{\varkappa} \mathfrak{u}'(\chi)E_{\omega}\Big[-\omega\frac{(t-\chi)^{\omega}}{1-\omega}\Big]d\chi,$$

and

$$^{\mathcal{ABR}}\mathcal{D}^{\omega}\mathfrak{u}(\varkappa) = \frac{B(\omega)}{1-\omega}\frac{d}{dt}\bigg(\int_0^{\varkappa} \mathfrak{u}(\chi)E_{\omega}\Big[-\omega\frac{(t-\chi)^{\omega}}{1-\omega}\Big]d\chi\bigg),$$

respectively, where E_{ω} *is called the Mittag-Leffler function and given by*

$$E_{\omega}(\mathfrak{u}) = \sum_{k=0}^{\infty}\frac{\mathfrak{u}^k}{\Gamma(k\omega+1)},$$

and $B(\omega)$ *is a normalizing positive function satisfying* $B(0) = B(1) = 1$.

Definition 2.4. [13] *Let* $0 < \omega \leq 1$ *and* $\mathfrak{u} \in C^1[a,b]$, *where* $0 \leq a < b$, *the associated AB-fractional integral is*

$$^{\mathcal{AB}}I^{\omega}\mathfrak{u}(\varkappa) = \frac{(1-\omega)}{B(\omega)}\mathfrak{u}(\varkappa) + \frac{\omega}{B(\omega)}I^{\omega}\mathfrak{u}(\varkappa),$$

where I^{ω} *is the R-L fractional integral defined in (2.1).*
The following results are based on the fixed point technique for equ. (1).

The following assumptions are needed for establish the EU results. Let $\mathcal{Y} = \mathcal{C}\big([0,1],\mathbb{R}\big)$ be the Banach space of continuous functions $\mathfrak{u} : [0,1] \longrightarrow \mathbb{R}$, with the norm $\|\mathfrak{u}\| = \sup_{\varkappa\in[0,1]}|\mathfrak{u}(\varkappa)|$.

- $(\mathcal{R}_1)$ Suppose that $\mathfrak{f} \in \big([0,1]\times\mathbb{R}^3,\mathbb{R}\big)$ $\exists$ positive constants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

  $$|\mathfrak{f}(\varkappa,\mathfrak{u}_1,\mathfrak{v}_1,\mathfrak{z}_1)-\mathfrak{f}(\varkappa,\mathfrak{u}_2,\mathfrak{v}_2,\mathfrak{z}_2)| \leq \mathcal{L}_1\big(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|\big)$$

  for all $\mathfrak{u}_1,\mathfrak{v}_1,\mathfrak{z}_1,\mathfrak{u}_2,\mathfrak{v}_2,\mathfrak{z}_2 \in \mathcal{Y}$, $\varkappa \in [0,1]$ and $\mathcal{L}_2 = \max_{\varkappa\in[0,1]}\|f(\varkappa,0,0,0)\|$.

- $(\mathscr{R}_2)$ Let $\varkappa \in [0,1]$ and $\psi^* \in \big([0,1]\times\mathbb{R},\mathbb{R}\big)$, there exist positive constants μ_1 and μ_2 such that
$$|\psi^*(\varkappa,\mathfrak{u}_1)-\psi^*(\varkappa,\mathfrak{u}_2|\leq \mu_1\big(\|\mathfrak{u}_1-\mathfrak{u}_2)\|\big),$$
for all $\mathfrak{u}_1,\mathfrak{u}_2\in\mathscr{Y}$ and $\mu_2=\max\limits_{\varkappa\in[0,1]}\|\psi^*(\varkappa,0)\|$.

- $(\mathscr{R}_3)$ If $\varkappa,\varsigma\in[0,1]$ then $\mathscr{g}\in\big([0,1]\times[0,1]\times\mathbb{R},\mathbb{R}\big)$, $\exists$ positive constants λ_1 and λ_2 such that
$$|\mathscr{g}(\varkappa,\varsigma,\mathfrak{u}_1)-\mathscr{g}(\varkappa,\varsigma,\mathfrak{u}_2|\leq\lambda_1\big(\|\mathfrak{u}_1-\mathfrak{u}_2)\|\big),$$
for all $\mathfrak{u}_1,\mathfrak{u}_2\in\mathscr{Y}$ and $\lambda_2=\max\limits_{\varkappa,\varsigma\in[0,1]}\|\mathscr{g}(\varkappa,\varsigma,0)\|$.

- $(\mathscr{R}_4)$ There exist positive constants $\mathscr{M}_1$ and $\mathscr{M}_2$ for $\varphi\in\big([0,1]\times[0,1]\times\mathbb{R},\mathbb{R}\big)$, such that
$$|\varphi(\varkappa,\varsigma,\mathfrak{u}_1)-\varphi(\varkappa,\varsigma,\mathfrak{u}_2|\leq\mathscr{M}_1\big(\|\mathfrak{u}_1-\mathfrak{u}_2)\|\big),$$
for all $\varkappa,\varsigma\in[0,1],\mathfrak{u}_1,\mathfrak{u}_2\in\mathscr{Y}$ and $\mathscr{M}_2=\max\limits_{\varkappa,\varsigma\in[0,1]}\|\varphi(\varkappa,\varsigma,0)\|$.

- $(\mathscr{R}_5)$ Let $\mathscr{h}\in\big([0,1]\times\mathbb{R},\mathbb{R}\big)$, $\exists$ positive constants $\mathscr{N}_1$ and $\mathscr{N}_2$ such that
$$|\mathscr{h}(\varkappa,\mathfrak{u}_1)-\mathscr{h}(\varkappa,\mathfrak{u}_2)|\leq\mathscr{N}_1\big(\|\mathfrak{u}_1-\mathfrak{u}_2)\|\big),$$
for all $\mathfrak{u}_1,\mathfrak{u}_2\in\mathscr{Y}$ and $\mathscr{N}_2=\max\limits_{\varkappa\in[0,1]}\|\mathscr{h}(\varkappa,0)\|$.

- $(\mathscr{R}_6)$ For any positive $\bar{r}$, we take $\mathscr{B}_{\bar{r}}=\{\mathfrak{u}\in\mathscr{Y}:\|\mathfrak{u}\|\leq\bar{r}\}\subset\mathscr{Y}$, where $\bar{r}\geq\dfrac{\mathscr{Q}}{(1-\mathscr{P})}$, where
$$\mathscr{P}=\mu_1+\mathscr{N}_1+\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{\Gamma(\omega)B(\omega)}\right)\mathscr{L}_1\big(1+\lambda_1+\mathscr{T}\ \mathscr{M}_1\big)$$
and
$$\mathscr{Q}=\mu_2+\mathscr{N}_2+\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{\Gamma(\omega)B(\omega)}\right)\mathscr{L}_2\big(1+\lambda_2+\mathscr{T}\ \mathscr{M}_2\big),$$
then $\mathscr{B}_{\bar{r}}$ is bounded, closed and convex subset in $\mathscr{C}([0,1],\mathbb{R})$.

Lemma 2.5. *If* $(\mathcal{R}_3)$ *and* $(\mathcal{R}_4)$ *are satisfied, then the estimate*

$$\|I_1\mathfrak{u}(\varkappa)\| \leq \varkappa\big(\lambda_1\|\mathfrak{u}\| + \lambda_2\big),$$

and

$$\|I_2\mathfrak{u}(\varkappa)\| \leq \mathcal{T}\big(\mathcal{M}_1\|\mathfrak{u}\| + \mathcal{M}_2\big),$$

are hold true for any $\varkappa \in [0,1]$ *and* $\mathfrak{u} \in \mathcal{Y}$.

Proposition 2.6[6, 4] *If* $0 < \omega \leq 1$, *then*

$$\begin{aligned}\big({}^{\mathcal{AB}}_{\ 0}I^{\omega}\big({}^{\mathcal{ABC}}_{\ \ 0}\mathcal{D}^{\omega}\big)\mathfrak{u}\big)(\varkappa) &= \mathfrak{u}(\varkappa) - \mathfrak{u}(0)E_{\omega}(\lambda\varkappa^{\omega}) - \frac{\omega}{1-\omega}\mathfrak{u}(0)E_{\omega,\omega+1}(\lambda\varkappa^{\omega}) \\ &= \mathfrak{u}(\varkappa) - \mathfrak{u}(0).\end{aligned} \tag{2}$$

Theorem 2.6. *(Krasnoselkii's fixed point theorem)[38] Let* $\mathcal{S}$ *is a nonempty, closed, bounded and convex subset of a Banach space* E. *Let* A_1, A_2 *be the operators from* Ω *to* E *such that:*
(i) $A_1\chi + A_2 y \in \Omega$ *whenever* $\chi, y \in \Omega$;
(ii) A_1 *is continuous and compact;*
(iii) A_2 *is a contraction map.*
Then there exists $z \in \Omega$ *such that* $z = A_1 z + A_2 z$.

3 Main Results

First, we observe the existence of the problem (1) by using the fixed point technique.

Theorem 3.1. *Let* $0 < \omega \leq 1$ *and there exists* $\mathfrak{f} \in \big([0,1] \times \mathbb{R}^3, \mathbb{R}\big)$ *with* $f^*(0, \mathfrak{u}(0), 0, \int_0^T \varphi(0, \varsigma, \mathfrak{u}(\varsigma)d\varsigma)) = \psi^*(0, \mathfrak{u}(0)) = 0$. *A function* $\mathfrak{u} \in \mathcal{C}[0,1]$ *be a solution of the integral equation*

$$\begin{aligned}\mathfrak{u}(\varkappa) = \psi^*(\varkappa, \mathfrak{u}(\varkappa)) - \psi^*(0, \mathfrak{u}(0)) + \int_0^1 \hbar(\varsigma, \chi(\varsigma))d\varsigma \\ + \ {}^{AB}_{\ 0}I^{\omega}\mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa)),\end{aligned} \tag{3}$$

iff $\mathfrak{u}(\varkappa)$ *is a solution of the ABC-problem (1).*

Proof. Let $\mathfrak{u}(\varkappa)$ satisfy (1). Applying the AB-fractional integarl of (1),
we get

$$\left({}^{\mathcal{AB}}_{\ 0}I^{\omega}\left({}^{\mathcal{ABC}}_{\ \ 0}\mathcal{D}^{\omega}\right)[\mathfrak{u}(t)-\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))]\right) = {}^{\mathcal{AB}}_{\ 0}I^{\omega}\mathfrak{f}\Big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\Big).$$

By using **Proposition (2.6)**, we obtain

$$\mathfrak{u}(\varkappa)-\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))-(\mathfrak{u}(0)-\psi^{*}(0,\mathfrak{u}(0))) = {}^{\mathcal{AB}}_{\ 0}I^{\omega}\mathfrak{f}\Big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\Big).$$

Since $\mathfrak{u}(0)=\int_0^1 \hbar(\varsigma,\mathfrak{u}(\varsigma))d\varsigma$, then (3) is satisfied.
Now, consider $\mathfrak{u}(\varkappa)$ satisfies the (3), then by

$$f^{*}(0,\mathfrak{u}(0),0,\int_0^T \varphi(0,\varsigma,\mathfrak{u}(\varsigma)d\varsigma)) = \psi^{*}(0,\mathfrak{u}(0)) = 0,$$

it is visible that $\mathfrak{u}(0)=\int_0^1 \hbar(\varsigma,\mathfrak{u}(\varsigma))d\varsigma$.
Applying AB-derivative in R-L sense of (3) and by using $\left({}^{\mathcal{AB}}_{\ 0}\mathcal{D}^{\omega}\left({}^{\mathcal{AB}}_{\ 0}I^{\omega}\right)\mathfrak{u}\right)(\varkappa)=\mathfrak{u}(\varkappa)$, we get

$$\begin{aligned}\left({}^{\mathcal{ABR}}_{\ \ 0}\mathcal{D}^{\omega}\mathfrak{u}\right)(\varkappa) &= \left(\int_0^1 \hbar(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right)\left({}^{\mathcal{ABR}}_{\ \ 0}\mathcal{D}^{\omega}1\right)(\varkappa)+\left({}^{\mathcal{ABR}}_{\ \ 0}\mathcal{D}^{\omega}\right)\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))\\ &\quad+\left({}^{\mathcal{ABR}}_{\ \ 0}\mathcal{D}^{\omega}\ {}^{\mathcal{AB}}_{\ 0}I^{\omega}\right)\mathfrak{f}\Big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\left({}^{\mathcal{ABR}}_{\ \ 0}\mathcal{D}^{\omega}\right)\big(\mathfrak{u}(\varkappa)-\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))\big) &= \left(\int_0^1 \hbar(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right)E_{\omega}\left(\frac{-\omega}{1-\omega}\varkappa^{\omega}\right)\\ &\quad+\mathfrak{f}\Big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\Big).\end{aligned}$$

Hence, the equation (1) can be obtained by the Theorem (1) in [13].
Now, let us define the operator $\mathcal{F}$ on $\mathcal{B}_{\bar{r}}$ as follows

$$\mathcal{F}\mathfrak{u}(\varkappa)=\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))+\int_0^1 \hbar(\varsigma,x(\varsigma))d\varsigma+{}^{AB}_{\ 0}I^{\omega}\mathfrak{f}(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)).$$

It is obeserved that $\mathfrak{u}(\varkappa)$ is the solution of (1) iff the operator $\mathcal{F}$ has a fixed point.

Theorem 3.2. *Assume that $\mathcal{R}_1 - \mathcal{R}_6$ are satisfied and*

$$q(\varkappa_2 - \varkappa_1) = \mathcal{L}_1\big[\|(\varkappa, \upsilon(\varkappa), I_1\upsilon(\varkappa), I_2\upsilon(\varkappa))\| + \varkappa(\lambda_1\|(\varkappa, \upsilon(\varkappa), I_1\upsilon(\varkappa), I_2\upsilon(\varkappa))\|) \\ + \mathcal{T}(\mathcal{M}_1\|(\varkappa, \upsilon(\varkappa), I_1\upsilon(\varkappa), I_2\upsilon(\varkappa))\|)\big].$$

If $\Lambda = \mu_1 + \mathcal{N} \leq 1$, then problem (1) has a solution.

Proof. Let us define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{B}_{\bar{r}}$ such that

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2,$$

$$\mathcal{F}_1\mathfrak{u}(\varkappa) = \psi^*(\varkappa, \mathfrak{u}(\varkappa)) + \int_0^1 \hbar(\varsigma, x(\varsigma))d\varsigma,$$

$$\mathcal{F}_2\mathfrak{u}(\varkappa) = {}_0^{AB}I^\omega \mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa)).$$

The following three steps are required to apply the Theorem (2.6).

1. $\|\mathcal{F}_1\mathfrak{u} + \mathcal{F}_2\mathfrak{z}\| \leq \bar{r}$ where $\bar{r} \in \mathcal{B}_{\bar{r}}$,
   For any $\mathfrak{u}, \mathfrak{z} \in \mathcal{B}_{\bar{r}}$,

$$\begin{aligned}
\|\mathcal{F}_1\mathfrak{u} + \mathcal{F}_2\mathfrak{z}\| &= \sup_{\varkappa\in[0,1]} \Bigg\{ \Bigg| \psi^*(\varkappa, \mathfrak{u}(\varkappa)) + \int_0^1 \hbar(\varsigma, \mathfrak{u}(\varsigma))d\varsigma \\
&\quad + \frac{(1-\omega)}{B(\omega)} \mathfrak{f}(\varkappa, \mathfrak{z}(\varkappa), I_1\mathfrak{z}(\varkappa), I_2\mathfrak{z}(\varkappa)) \\
&\quad + \frac{\omega}{B(\omega)}\ {}_0I^\omega \mathfrak{f}(\varkappa, \mathfrak{z}(\varkappa), I_1\mathfrak{z}(\varkappa), I_2\mathfrak{z}(\varkappa)) \Bigg| \Bigg\} \\
&\leq \sup_{\varkappa\in[0,1]} \Bigg\{ \big|\psi^*(\varkappa, \mathfrak{u}(\varkappa))\big| + \left| \int_0^1 \hbar(\varsigma, \mathfrak{z}(\varsigma))d\varsigma \right| \\
&\quad + \frac{(1-\omega)}{B(\omega)} \big|\mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{z}(t), I_2\mathfrak{z}(\varkappa))\big| \\
&\quad + \frac{\omega}{B(\omega)}\ {}_0I^\omega \big|\mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{z}(\varkappa), I_2\mathfrak{z}(\varkappa))\big| \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\varkappa\in[0,1]} \Big\{ \big| \psi^*(\varkappa, \mathfrak{u}(\varkappa)) - \psi^*(\varkappa, \mathfrak{u}(0)) + \psi^*(\varkappa, \mathfrak{u}(0)) \big| \\
&\quad + \int_0^1 \big| \hbar(\varsigma, \mathfrak{u}(\varsigma)) d\varsigma - \hbar(\varsigma, \mathfrak{u}(0)) + \hbar(\varsigma, \mathfrak{u}(0)) \big| d\varsigma \\
&\quad + \frac{(1-\omega)}{B(\omega)} \big| \mathfrak{f}(\varkappa, \mathfrak{z}(\varkappa), I_1\mathfrak{z}(\varkappa), I_2\mathfrak{z}(\varkappa)) \\
&\quad - \mathfrak{f}(\varkappa, \mathfrak{z}(0), I_1\mathfrak{z}(0), I_2\mathfrak{z}(0)) + \mathfrak{f}(\varkappa, \mathfrak{z}(0), I_1\mathfrak{z}(0), I_2\mathfrak{z}(0)) \big| \\
&\quad + \frac{\omega}{B(\omega)} \ {}_0I^{\omega} \big| \mathfrak{f}(\varkappa, \mathfrak{z}(\varkappa), I_1\mathfrak{z}(\varkappa), I_2\mathfrak{z}(\varkappa)) \\
&\quad - \mathfrak{f}(\varkappa, \mathfrak{z}(0), I_1\mathfrak{z}(0), I_2\mathfrak{z}(0)) + \mathfrak{f}(\varkappa, \mathfrak{z}(0), I_1\mathfrak{z}(0), I_2\mathfrak{z}(0)) \big| \Big\} \\
&\leq \mu_1\|\mathfrak{u}\| + \mu_2 + \mathcal{N}_1\|\mathfrak{u}\| + \mathcal{N}_2 \\
&\quad + \frac{(1-\omega)}{B(\omega)} \Big(\mathcal{L}_1 \big[\|\mathfrak{u}\| + \varkappa(\lambda_1\|\mathfrak{u}\| + \lambda_2) + \mathcal{T}(\mathcal{M}_1\|\mathfrak{z}\| + \mathcal{M}_2) \big] \Big) \\
&\quad + \frac{1}{B(\omega)\Gamma(\omega)} \Big(\mathcal{L}_1 \big[\|\mathfrak{z}\| + \varkappa(\lambda_1\|\mathfrak{u}\| + \lambda_2) + \mathcal{T}(\mathcal{M}_1\|\mathfrak{z}\| + \mathcal{M}_2) \big] \Big) \\
&\quad + \frac{(1-\omega)}{B(\omega)} \mathcal{L}_2 + \Big(\frac{\omega}{B(\omega)} \mathcal{L}_2 \Big) \frac{(1)^{\omega}}{\omega\Gamma(\omega)} \\
&\leq \big(\mu_1 + \mathcal{N}_1\big)\|\mathfrak{u}\| + \Big[\Big(\frac{(1-\omega)}{B(\omega)} + \frac{1}{\Gamma(\omega)B(\omega)} \Big) \mathcal{L}_1 \big[1 + \lambda_1 + \mathcal{T}\ \mathcal{M}_1 \big] \Big] \|\mathfrak{z}\| \\
&\quad + \Big[\mu_2 + \mathcal{N}_2 + \Big(\frac{(1-\omega)}{B(\omega)} + \frac{1}{\Gamma(\omega)B(\omega)} \Big) \mathcal{L}_2 \big[1 + \lambda_2 + \mathcal{T}\ \mathcal{M}_2 \big] \Big] \\
&= \mathcal{P}\|\mathfrak{u}\| + \mathcal{Q} \\
&\leq \mathcal{P}\bar{r} + \mathcal{Q} \leq \bar{r}.
\end{aligned}
$$

2. $\mathcal{F}_1$ is the contraction on $\mathcal{B}_{\bar{r}}$.
   For each $\mathfrak{u}, \mathfrak{z} \in \mathcal{B}_{\bar{r}}$, by using $\mathcal{R}_2$ and $\mathcal{R}_5$.

$$
\begin{aligned}
\|\mathcal{F}_1\mathfrak{u}(\varkappa) - \mathcal{F}_1\mathfrak{z}(\varkappa)\| = \sup_{\varkappa\in[0,1]} \Big| \Big\{ & \psi^*(\varkappa, \mathfrak{u}(\varkappa)) + \int_0^1 \hbar(\varsigma, \mathfrak{u}(\varsigma)) d\varsigma \\
& - \psi^*(\varkappa, \mathfrak{z}(\varkappa)) - \int_0^1 \hbar(\varsigma, \mathfrak{z}(\varsigma)) d\varsigma \Big| \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{\varkappa\in[0,1]}\Big\{\big|\psi^*(\varkappa,\mathfrak{u}(\varkappa))-\psi^*(\varkappa,\mathfrak{z}(\varkappa))\big| \\
&\quad+\int_0^1\big|\hbar(\varsigma,\mathfrak{u}(\varsigma))-\hbar(\varsigma,\mathfrak{z}(\varsigma))\big|d\varsigma\Big\} \\
&\leq \mu_1\|\mathfrak{u}-\mathfrak{z}\|+\mathcal{N}_1\|\mathfrak{u}-\mathfrak{z}\| \\
&\leq \Lambda\|\mathfrak{u}-\mathfrak{z}\|,
\end{aligned}
$$

$\implies \quad \|\mathcal{F}_1\mathfrak{u}(\varkappa)-\mathcal{F}_1\mathfrak{z}(\varkappa)\|\leq\Lambda\|\mathfrak{u}-\mathfrak{z}\|, \quad$ where $\Lambda=\mu_1+\mathcal{N}_1$.
As $\Lambda<1$. Thus $\mathcal{F}_1$ is a contraction operator.

3. We prove that $\mathcal{F}_2$ is completely continuous operator.
For completeness of $\mathcal{F}_2$, firstly we prove that $\mathcal{F}_2$ is continuous.
With $\lim_{n\to\infty}\|\mathfrak{u}_n-\mathfrak{u}\|=0$ for any $\mathfrak{u}_n,\mathfrak{u}\in\mathcal{B}_{\bar{r}}$, $n=1,2,....$
Then $\lim_{n\to\infty}\mathfrak{u}_n(\varkappa)=\mathfrak{u}(\varkappa) \quad \forall\varkappa\in[0,1]$.
Therefore

$$
\lim_{n\to\infty}\mathfrak{f}\Big(\varkappa,\mathfrak{u}_n(\varkappa),I_1\mathfrak{u}_n(\varkappa),I_2\mathfrak{u}_n(\varkappa)\Big)=\mathfrak{f}\Big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\Big).
$$

Now, for $\varkappa\in[0,1]$

$$
\begin{aligned}
\|\mathcal{F}_2\mathfrak{u}_n(\varkappa)-\mathcal{F}_2\mathfrak{u}(\varkappa)\| &= \sup_{\varkappa\in[0,1]}\Big\{\Big|\frac{(1-\omega)}{B(\omega)}\big(\mathfrak{f}\varkappa,\mathfrak{u}_n(\varkappa),I_1\mathfrak{u}_n(\varkappa),I_2\mathfrak{u}_n(\varkappa)) \\
&\quad-\mathfrak{f}(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa))\big) \\
&\quad+\frac{\omega}{B(\omega)}\ {}_0I^{\omega}\big(\mathfrak{f}\varkappa,\mathfrak{u}_n(\varkappa),I_1\mathfrak{u}_n(\varkappa),I_2\mathfrak{u}_n(\varkappa)) \\
&\quad-\mathfrak{f}(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa))\big)\Big|\Big\} \\
&\leq\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{B(\omega)\Gamma(\omega)}\right) \\
&\quad\times\sup_{\varkappa\in[0,1]}\|\mathfrak{f}\varkappa,\mathfrak{u}_n(\varkappa),I_1\mathfrak{u}_n(\varkappa),I_2\mathfrak{u}_n(\varkappa)) \\
&\quad-\mathfrak{f}(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa))\|.
\end{aligned}
$$

Thus $\|\mathcal{F}_2\mathfrak{u}_n(\varkappa)-\mathcal{F}_2\mathfrak{u}(t)\|\longrightarrow0$ as $n\longrightarrow\infty$.
Hence $\mathcal{F}_2$ is continuous.

Now, we prove that $\mathcal{F}_2$ is compact.

$$\begin{aligned}
\|\mathcal{F}_2\mathfrak{u}(\varkappa)\| &= \sup_{\varkappa\in[0,1]}\bigg\{\bigg|\frac{(1-\omega)}{B(\omega)}\mathfrak{f}(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa))\\
&\quad+\frac{\omega}{B(\omega)}\ {}_0I^{\omega}\mathfrak{f}(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa))\bigg|\bigg\}\\
&\le\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{B(\omega)\Gamma(\omega)}\right)\sup_{\varkappa\in[0,1]}\|\mathfrak{f}(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa))\|\\
&\le\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{B(\omega)\Gamma(\omega)}\right)\\
&\quad\times\Big(\mathcal{L}_1\big[\|\mathfrak{u}\|+\varkappa(\lambda_1\|\mathfrak{u}\|+\lambda_2)+\mathcal{T}(\mathcal{M}_1\|\mathfrak{u}\|+\mathcal{M}_2)\big]+\mathcal{L}_2\Big)\\
&\le\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{B(\omega)\Gamma(\omega)}\right)\\
&\quad\times\Big[\mathcal{L}_1\big[1+\lambda_1+\mathcal{T}\ \mathcal{M}_1\big]\|\mathfrak{u}\|+\mathcal{L}_2\big[1+\lambda_2+\mathcal{T}\ \mathcal{M}_2\big]\Big]\\
&\le\Big[\big(\mathcal{P}-\mu_1-\mathcal{N}_1\big)\bar{r}+\big(\mathcal{Q}-\mu_2-\mathcal{N}_2\big)\Big]<\infty,
\end{aligned}$$

which shows that $\mathcal{F}_2$ is bounded on $\mathcal{B}_{\bar{r}}$.
Next we prove that $\mathcal{F}_2$ is equicontinuous.
For any $0<\varkappa_1<\varkappa_2<\varkappa<1$, we have

$$\begin{aligned}
\|\mathcal{F}_2\mathfrak{u}(\varkappa_2)-\mathcal{F}_2\mathfrak{u}(\varkappa_1)\| &= \sup_{\varkappa\in[0,1]}\bigg\{\bigg|\frac{(1-\omega)}{B(\omega)}\mathfrak{f}(\varkappa_2,\mathfrak{u}(\varkappa_2),I_1\mathfrak{u}(\varkappa_2),I_2\mathfrak{u}(\varkappa_2))\\
&\quad+\frac{\omega}{B(\omega)}\ {}_0I^{\omega}\mathfrak{f}(\varkappa_2,\mathfrak{u}(\varkappa_2),I_1\mathfrak{u}(\varkappa_2),I_2\mathfrak{u}(\varkappa_2))\\
&\quad-\frac{(1-\omega)}{B(\omega)}\mathfrak{f}(\varkappa_1,\mathfrak{u}(\varkappa_1),I_1\mathfrak{u}(\varkappa_1),I_2\mathfrak{u}(\varkappa_1))\\
&\quad-\frac{\omega}{B(\omega)}\ {}_0I^{\omega}\mathfrak{f}(\varkappa_1,\mathfrak{u}(\varkappa_1),I_1\mathfrak{u}(\varkappa_1),I_2\mathfrak{u}(\varkappa_1))\bigg|\bigg\}\\
&\le\frac{(1-\omega)}{B(\omega)}\|\mathfrak{f}(\varkappa_2,\mathfrak{u}(\varkappa_2),I_1\mathfrak{u}(\varkappa_2),I_2\mathfrak{u}(\varkappa_2))\\
&\quad-\mathfrak{f}(\varkappa_1,\mathfrak{u}(\varkappa_1),I_1\mathfrak{u}(\varkappa_1),I_2\mathfrak{u}(\varkappa_1))\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\omega}{B(\omega)}\ {}_0I^{\omega}\|\mathfrak{f}(\varkappa_2,\mathfrak{u}(\varkappa_2),I_1\mathfrak{u}(\varkappa_2),I_2\mathfrak{u}(\varkappa_2))\\
&-\mathfrak{f}(\varkappa_1,\mathfrak{u}(\varkappa_1),I_1\mathfrak{u}(\varkappa_1),I_2\mathfrak{u}(\varkappa_1))\|\\
\le\ &\frac{(1-\omega)}{B(\omega)}\Big(\mathcal{L}_1\big[\|(\varkappa,\mathfrak{z}(\varkappa),I_1\mathfrak{z}(\varkappa),I_2\mathfrak{z}(\varkappa))\|\\
&+\varkappa(\lambda_1\|(\varkappa,\mathfrak{z}(\varkappa),I_1\mathfrak{z}(\varkappa),I_2\mathfrak{z}(\varkappa))\|)\\
&+\mathcal{T}(\mathcal{M}_1\|(\varkappa,\mathfrak{z}(\varkappa),I_1\mathfrak{z}(\varkappa),I_2\mathfrak{z}(\varkappa))\|)\big]\Big)\\
&+\frac{\omega}{B(\omega)}\Big(\mathcal{L}_1\big[\|(\varkappa,\mathfrak{z}(\varkappa),I_1\mathfrak{z}(\varkappa),I_2\mathfrak{z}(\varkappa))\|\\
&+\varkappa(\lambda_1\|(\varkappa,\mathfrak{z}(\varkappa),I_1\mathfrak{z}(\varkappa),I_2\mathfrak{z}(\varkappa))\|)\\
&+\mathcal{T}(\mathcal{M}_1\|(\varkappa,\mathfrak{z}(\varkappa),I_1\mathfrak{z}(\varkappa),I_2\mathfrak{z}(\varkappa))\|)\big]\Big)\frac{(\varkappa_2-\varkappa_1)^{\omega}}{\omega\Gamma(\omega)}\\
\le\ &\frac{(1-\omega)}{B(\omega)}q(\varkappa_2-\varkappa_1)+\frac{\omega}{B(\omega)}q(\varkappa_2-\varkappa_1)\frac{(\varkappa_2-\varkappa_1)^{\omega}}{\omega\Gamma(\omega)}\\
\le\ &q\left(\frac{(1-\omega)}{B(\omega)}+\frac{(\varkappa_2-\varkappa_1)^{\omega}}{B(\omega)\Gamma(\omega)}\right)(\varkappa_2-\varkappa_1),
\end{aligned}
$$

$\|\mathcal{F}_2\mathfrak{u}(\varkappa_2)-\mathcal{F}_2\mathfrak{u}(\varkappa_1)\|\longrightarrow 0$ as $\varkappa_2\longrightarrow\varkappa_1$. Consequently, $\mathcal{F}_2$ is equicontinuous operator on $\mathcal{B}_{\bar{r}}$. Therefore by the Arzela-Ascoli theorem $\mathcal{F}_2$ is relatively compact on $\mathcal{B}_{\bar{r}}$. Hence by the Theorem (2.6) $\mathcal{F}$ has at least one fixed point. Thus $\mathfrak{u}$ is that fixed point of $\mathcal{F}$. Consequently, $\mathfrak{u}$ is solution of equ. (1).

4 Uniqueness Result

Theorem 4.1. *Assume that $\mathcal{R}_1-\mathcal{R}_6$ are satisfied. If*

$$
f^*(0,\mathfrak{u}(0),0,\int_0^T\varphi(0,\varsigma,\mathfrak{u}(\varsigma)d\varsigma))=\psi^*(0,\mathfrak{u}(0))=0
$$

and

$$
\mu_1+\mathcal{N}_1+\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{\Gamma(\omega)B(\omega)}\right)\mathcal{L}_1\big(1+\lambda_1+\mathcal{T}\ \mathcal{M}_1\big)\le 1
$$

. Then problem (1) has unique solution on $[0,1]$.

Proof. For any $\mathfrak{u} \in \mathcal{B}_{\bar{r}}$.

$$\begin{aligned}
\|\mathcal{F}\mathfrak{u}\| = & \sup_{\varkappa\in[0,1]} \Bigg\{ \Bigg| \psi^*(\varkappa, \mathfrak{u}(\varkappa)) + \int_0^1 \hbar(\varsigma, \mathfrak{u}(\varsigma)) d\varsigma \\
& + \frac{(1-\omega)}{B(\omega)} \mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa)) \\
& + \frac{\omega}{B(\omega)} \, {}_0I^{\omega} \mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa)) \Bigg| \Bigg\} \\
\leq & \sup_{\varkappa\in[0,1]} \Bigg\{ |\psi^*(\varkappa, \mathfrak{u}(\varkappa))| + \Bigg| \int_0^1 \hbar(\varsigma, \mathfrak{u}(\varsigma)) d\varsigma \Bigg| \\
& + \frac{(1-\omega)}{B(\omega)} |\mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa))| \\
& + \frac{\omega}{B(\omega)} \, {}_0I^{\omega} |\mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa))| \Bigg\} \\
\leq & \sup_{\varkappa\in[0,1]} \Bigg\{ |\psi^*(\varkappa, \mathfrak{u}(\varkappa)) - \psi^*(\varkappa, \mathfrak{u}(0)) + \psi^*(\varkappa, \mathfrak{u}(0))| \\
& + \int_0^1 |\hbar(\varsigma, \mathfrak{u}(\varsigma)) d\varsigma - \hbar(\varsigma, \mathfrak{u}(0)) + \hbar(\varsigma, \mathfrak{u}(0))| d\varsigma \\
& + \frac{(1-\omega)}{B(\omega)} |\mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa)) \\
& - \mathfrak{f}(\varkappa, \mathfrak{u}(0), I_1\mathfrak{u}(0), I_2\mathfrak{u}(0)) + \mathfrak{f}(\varkappa, \mathfrak{u}(0), I_1\mathfrak{u}(0), I_2\mathfrak{u}(0))| \\
& + \frac{\omega}{B(\omega)} \, {}_0I^{\omega} |\mathfrak{f}(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa)) \\
& - \mathfrak{f}(\varkappa, \mathfrak{u}(0), I_1\mathfrak{u}(0), I_2\mathfrak{u}(0)) + \mathfrak{f}(\varkappa, \mathfrak{u}(0), I_1\mathfrak{u}(0), I_2\mathfrak{u}(0))| \Bigg\} \\
\leq & \mu_1\|\mathfrak{u}\| + \mu_2 + \mathcal{N}_1\|\mathfrak{u}\| + \mathcal{N}_2 \\
& + \frac{(1-\omega)}{B(\omega)} \Big(\mathcal{L}_1 \big[\|\mathfrak{u}\| + \varkappa\|(\lambda_1\|\mathfrak{u}\| + \lambda_2) + \mathcal{T}(\mathcal{M}_1\|\mathfrak{u}\| + \mathcal{M}_2) \big] \Big) \\
& + \frac{\omega}{B(\omega)} \Big(\mathcal{L}_1 \big[\|\mathfrak{u}\| + \varkappa\|(\lambda_1\|\mathfrak{u}\| + \lambda_2) + \mathcal{T}(\mathcal{M}_1\|\mathfrak{u}\| + \mathcal{M}_2) \big] \Big) \frac{(1)^{\omega}}{\omega\Gamma(\omega)} \\
& + \frac{(1-\omega)}{B(\omega)} \mathcal{L}_2 + \left(\frac{\omega}{B(\omega)} \mathcal{L}_2 \right) \frac{(1)^{\omega}}{\omega\Gamma(\omega)} \\
\leq & \left[\mu + \mathcal{N} + \left(\frac{(1-\omega)}{B(\omega)} + \frac{1}{\Gamma(\omega)B(\omega)} \right) \mathcal{L} \big[1 + \lambda + \mathcal{T}\ \mathcal{M} \big] \right] \|\mathfrak{u}\| \\
& + \left[\mu + \mathcal{N} + \left(\frac{(1-\omega)}{B(\omega)} + \frac{1}{\Gamma(\omega)B(\omega)} \right) \mathcal{L} \big[1 + \lambda + \mathcal{T}\ \mathcal{M} \big] \right] \\
= & \mathcal{Q}\|\mathfrak{u}\| + \mathcal{Q} \\
\leq & \mathcal{Q}\bar{r} + \mathcal{Q} \leq \bar{r},
\end{aligned}$$

which shows that $\mathcal{F}$ is bounded on $\mathcal{B}_{\bar{r}}$. Now to prove uniqueness

$$\begin{aligned}
\|\mathcal{F}\mathfrak{u}_1(\varkappa)-\mathcal{F}\mathfrak{u}_2(\varkappa)\| = \sup_{\varkappa\in[0,1]}\Bigg|\Bigg\{&\psi^*(\varkappa,\mathfrak{u}_1(\varkappa))+\int_0^1 \hbar(\varsigma,\mathfrak{u}_1(\varsigma))d\varsigma\\
&+\frac{(1-\omega)}{B(\omega)}\mathfrak{f}(\varkappa,\mathfrak{u}_1(\varkappa),I_1\mathfrak{u}_1(\varkappa),I_2\mathfrak{u}_1(\varkappa))\\
&+\frac{\omega}{B(\omega)}\ {}_0I^{\omega}\mathfrak{f}(\varkappa,\mathfrak{u}_1(\varkappa),I_1\mathfrak{u}_1(\varkappa),I_2\mathfrak{u}_1(\varkappa))\\
&-\psi^*(\varkappa,\mathfrak{u}_2(\varkappa))-\int_0^1 \hbar(\varsigma,\mathfrak{u}_2(\varsigma))d\varsigma\\
&-\frac{(1-\omega)}{B(\omega)}\mathfrak{f}(\varkappa,\mathfrak{u}_2(\varkappa),I_1\mathfrak{u}_2(\varkappa),I_2\mathfrak{u}_2(\varkappa))\\
&-\frac{\omega}{B(\omega)}\ {}_0I^{\omega}\mathfrak{f}(\varkappa,\mathfrak{u}_2(\varkappa),I_1\mathfrak{u}_2(\varkappa),I_2\mathfrak{u}_2(\varkappa))\Bigg|\Bigg\}\\
\le{}& \mu_1\|\mathfrak{u}_1-\mathfrak{u}_2\|+\mathcal{N}_1\|\mathfrak{u}_1-\mathfrak{u}_2\|\\
&+\frac{(1-\omega)}{B(\omega)}\Big(\mathcal{L}_1\big[\|\mathfrak{u}_1(\varkappa)-\mathfrak{u}_2(\varkappa)\|+\varkappa(\lambda_1\|\mathfrak{u}_1(\varkappa)-\mathfrak{u}_2(\varkappa)\|)\\
&+\mathcal{T}(\mathcal{M}_1\|\mathfrak{u}_1(\varkappa)-\mathfrak{u}_2(\varkappa)\|)\big]\Big)\\
&+\frac{(\omega)}{B(\omega)}\Big(\mathcal{L}_1\big[\|\mathfrak{u}_1(\varkappa)-\mathfrak{u}_2(\varkappa)\|+\varkappa(\lambda_1\|\mathfrak{u}_1(\varkappa)-\mathfrak{u}_2(\varkappa)\|)\\
&+\mathcal{T}\big(\mathcal{M}_1\|\mathfrak{u}_1(\varkappa)-\mathfrak{u}_2(\varkappa)\|\big)\big]\Big)\frac{(1)^{\omega}}{\omega\Gamma(\omega)}\\
\le{}& \Bigg[\mu_1+\mathcal{N}_1+\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{\Gamma(\omega)B(\omega)}\right)\\
&\times\mathcal{L}_1\big(1+\lambda_1+\mathcal{T}\ \mathcal{M}_1\big)\Bigg]\|\mathfrak{u}_1-\mathfrak{u}_2\|.
\end{aligned}$$

Since

$$\mu_1+\mathcal{N}_1+\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{\Gamma(\omega)B(\omega)}\right)\mathcal{L}_1\big(1+\lambda_1+\mathcal{T}\ \mathcal{M}_1\big)\le 1.$$

Consequently, $\mathcal{F}$ is a contraction mapping. Therefore by the Banach contraction principle, the operator $\mathcal{F}$ has a unique fixed point. Hence equ. (1) has a unique solution.

5 Example

This section of the article produce examples related to EU of solutions of the discussed problem.

Example 4.1. Let us analyse the given below FDEs:

$$\begin{cases} {}^{\mathcal{ABC}}_{\ \ 0}\mathcal{D}^{\omega}\Big[\mathfrak{u}(\varkappa)+\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))\Big]=\mathfrak{f}\Big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\Big), \quad \varkappa\in[0,1],\\ \mathfrak{u}(0)=\int_0^1 \hbar(\varsigma,\mathfrak{u}(\varsigma))d\varsigma, \end{cases} \tag{4}$$

where $\omega=\frac{1}{2}, \quad \mathcal{T}=\frac{\pi}{4},$

$$\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))=\frac{1}{100+\varkappa^2}\mathfrak{u}(\varkappa).$$

$$\mathfrak{f}\left(\varkappa,\mathfrak{u}(\varkappa),\int_0^{\varkappa} g(\varkappa,\varsigma,\mathfrak{u}(\varsigma))d\varsigma,\int_0^{T}\varphi(\varkappa,\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right)$$
$$=\frac{1}{\varkappa^2+20}\left(\frac{1}{50}\int_0^{\varkappa}(\varkappa^2+\varsigma^2)\mathfrak{u}(\varsigma)d\varsigma+\frac{1}{10}\int_0^{\mathcal{T}}(\varkappa^2\sin\varsigma)\ \mathfrak{u}(\varsigma)d\varsigma\right),$$

$$g(\varkappa,\varsigma,\mathfrak{u}(\varsigma))=\frac{1}{50}(\varkappa^2+\varsigma^2)\mathfrak{u}(\varsigma),$$
$$\varphi(\varkappa,\varsigma,\mathfrak{u}(\varsigma))=\frac{1}{10}(\varkappa^2\sin\varsigma)\mathfrak{u}(\varsigma),$$
$$\hbar(\varsigma,\mathfrak{u}(\varsigma))=\frac{1}{30}\varsigma^2\mathfrak{u}(\varsigma).$$

Now,

$$|\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))-\psi^{*}(\varkappa,\mathfrak{u}(\varkappa))|\le\frac{1}{100}\|\mathfrak{u}-\mathfrak{u}_1\|,$$
$$|g(\varkappa,\varsigma,\mathfrak{u})-g(\varkappa,\varsigma,\mathfrak{u}_1|\le\frac{1}{25}\big(\|\mathfrak{u}-\mathfrak{u}_1)\|\big),$$
$$|\varphi(\varkappa,\varsigma,\mathfrak{u})-\varphi(\varkappa,\varsigma,\mathfrak{u}_1|\le\frac{1}{10}\big(\|\mathfrak{u}-\mathfrak{u}_1)\|\big),$$

$$\Big|\mathfrak{f}\big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\big)-\mathfrak{f}\big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}_1(\varkappa),I_2\mathfrak{u}_1(\varkappa)\big)\Big|$$
$$\le\frac{1}{20}\left(\frac{2}{75}+\frac{1-\cos T}{10}\right)\|\mathfrak{u}-\mathfrak{u}_1)\|,$$

thus assumptions $(\mathcal{R}_1), (\mathcal{R}_2), (\mathcal{R}_3)$ and $(\mathcal{R}_4)$ are hold true. Hence, $\Lambda = \mu_1 + \mathcal{N}_1 \approx 0.021111 < 1$. Consequently, the Theorem (3.2) implies that equ.(4) has a solution.
In addition, $\mu_1 + \mathcal{N}_1 + \left(\dfrac{(1-\omega)}{B(\omega)} + \dfrac{1}{\Gamma(\omega)B(\omega)}\right)\mathcal{L}_1\big(1 + \lambda_1 + \varkappa\ \mathcal{M}_1\big) \approx 0.024441 < 1$, hence using the Theorem (4.1) the equ. (4) has a unique.
Example 4.2. Let us consider the following FDEs:

$$\begin{cases} {}^{\mathcal{ABC}}_{\ 0}\mathcal{D}^{\omega}\Big[\mathfrak{u}(\varkappa) + \psi^*(\varkappa, \mathfrak{u}(\varkappa))\Big] = \mathfrak{f}\Big(\varkappa, \mathfrak{u}(\varkappa), I_1\mathfrak{u}(\varkappa), I_2\mathfrak{u}(\varkappa)\Big), \quad \varkappa \in [0,1], \\ \mathfrak{u}(0) = \int_0^1 \mathcal{h}(\varsigma, \mathfrak{u}(\varsigma))\,d\varsigma, \end{cases} \tag{5}$$

where $\quad \omega = \dfrac{1}{2}, \quad \mathcal{T} = 0.25,$

$$\psi^*(\varkappa, \mathfrak{u}(\varkappa)) = \frac{1}{200}e^{T}\mathfrak{u}(\varkappa).$$

$$\begin{aligned} \mathfrak{f}\Big(\varkappa, \mathfrak{u}(\varkappa), \int_0^{\varkappa} g(\varkappa,\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma, \int_0^{T} \varphi(\varkappa,\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma\Big) \\ = \frac{\varsigma^2+1}{25}\Big(\frac{1}{40}\int_0^{\varkappa}(1+\varkappa^2)\frac{|\mathfrak{u}(\varsigma)|}{|1+\mathfrak{u}(\varsigma)|}\,d\varsigma \\ + \frac{1}{60}\int_0^{\mathcal{T}}(e^{\varkappa+\varsigma}\cos(\mathfrak{u}(\varsigma))d\varsigma\Big), \end{aligned}$$

$$g(\varkappa,\varsigma,\mathfrak{u}(\varsigma)) = \frac{1}{40}(1+\varkappa^2)\frac{|\mathfrak{u}(\varsigma)|}{|1+\mathfrak{u}(\varsigma)|},$$

$$\varphi(\varkappa,\varsigma,\mathfrak{u}(\varsigma)) = \frac{1}{60}(e^{\varkappa+\varsigma}\cos(\mathfrak{u}(\varsigma))\,d\varsigma,$$

$$\mathcal{h}(\varsigma,\mathfrak{u}(\varsigma)) = \frac{1}{20}\Big(\varsigma^2 + \frac{1}{2}\Big)\mathfrak{u}(\varsigma).$$

Now,

$$|\psi^*(\varkappa, \mathfrak{u}(\varkappa)) - \psi^*(\varkappa, \mathfrak{u}(\varkappa))| \le \frac{e^{T}}{200}\|\mathfrak{u} - \mathfrak{u}_1\|,$$

$$|g(\varkappa,\varsigma,\mathfrak{u}) - g(\varkappa,\varsigma,\mathfrak{u}_1| \le \frac{1}{20}\big(\|\mathfrak{u} - \mathfrak{u}_1)\|\big),$$

$$|\varphi(\varkappa,\varsigma,\mathfrak{u})-\varphi(\varkappa,\varsigma,\mathfrak{u}_1|\leq\frac{e^2}{60}\big(\|\mathfrak{u}-\mathfrak{u}_1)\|\big),$$

$$\begin{aligned}&\Big|\mathfrak{f}\big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}(\varkappa),I_2\mathfrak{u}(\varkappa)\big)-\mathfrak{f}\big(\varkappa,\mathfrak{u}(\varkappa),I_1\mathfrak{u}_1(\varkappa),I_2\mathfrak{u}_1(\varkappa)\big)\Big|\\&\qquad\leq\frac{3}{25}\left(\frac{1}{20}+e(e^T-1)\right)\|\mathfrak{u}-\mathfrak{u}_1)\|,\end{aligned}$$

thus assumptions $(\mathcal{R}_1),(\mathcal{R}_2),(\mathcal{R}_3)$ and $(\mathcal{R}_4)$ are hold true. Hence, $\Lambda=\mu_1+\mathcal{N}_1\approx 0.04808<1$. Consequently, the Theorem (3.2) implies that equ.(5) has a solution.

In addition, $\mu_1+\mathcal{N}_1+\left(\frac{(1-\omega)}{B(\omega)}+\frac{1}{\Gamma(\omega)B(\omega)}\right)\mathcal{L}_1\big(1+\lambda_1+\varkappa\,\mathcal{M}_1\big)\approx 0.655852<1$, hence using the Theorem (4.1) the equ. (5) has a unique.

6 Conclusion

In this paper, we examined the EU of solutions for integro FDEs involving ABC-derivative and non-local initial condition. Recently, AB-derivative gained much attention due to the non-singular property of the kernels. Although EU results for different types of FDEs in terms of ABC derivative have been investigated with non-local conditions, but this type of problem is not studied yet. The existence of solution is investigated for proposed equations by using Krasnoselskii's fixed point theorem. The uniqueness of the result is derived with the help of the Banach contraction mapping principle.

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