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Original Research Paper

On the n-Ary Variable-Order $\boldsymbol{\alpha}(t)$-Derivative Calculus

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Abstract. This paper generalises the definition of an n-ary variable-order of one variable $\boldsymbol{\alpha}(t)$-derivative of multi-variable vector-valued functions. First, we develop the concepts in this new fractional vector calculus. Second, we present only a few aspects of the theory. Finally, the fundamental theorems (Chain Rule, Mean Value) on the n-ary variable-order $\boldsymbol{\alpha}(t)$-derivative calculus are investigated.

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1 Introduction

The "fractional calculus" (FC) has many applications in different fields of science. " Fractional differential equations" (FDE) are very important in many sciences. For example, control theory of dynamical systems, mechanics, biology, chemistry, engineering, and physics, using FDEs

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(see, for example, the works of H. Jafari et al. [5, 11, 17, 21, 9] and references).

Recently many definitions of fractional derivatives (FDs) have been presented by various researchers. There are several concepts of FDs, such as Riemann-Liouville or Caputo definitions. Riemann-Liouville and Caputo FDs satisfy the property that the FD is linear. But all of these definitions do not satisfy the natural properties of the derivative. In 2014, a new definition of FDs was introduced by R. Khalil which is the new definitions of all-natural properties of derivative that is called conformable fractional derivative [16]. For a function $f : [0, \infty) \longrightarrow \mathbb{R}$, the "conformable fractional derivative" (CFD) of order $\alpha \in (0, 1]$ for f at $a \in [0, \infty)$ is defined as follows

$$D^{\alpha} f(a) = \lim_{h \to 0} \frac{f(a + ha^{1-\alpha}) - f(a)}{h}. \tag{1}$$

If there is a CFD for $\alpha \in (0, 1]$ then we say f is α-differentiable. According to the definition CFD, if f is a α-differentiable at $a \in [0, \infty)$, then f is is continuous at a. This FD is improved by Abdeljawad. Properties such as chain rule, the Leibniz rule, Laplace transform, integration by parts, and Taylor power are provided in[1]. Here are some theorems and propositions of CFD.

Proposition 1.1. *[16] Assume that $\alpha \in (0; 1]$ and $f, g : [0; \infty) \longrightarrow \mathbb{R}$ is two α-differentiable functions at $a > 0$. Then*

I) $D^{\alpha}(\lambda f + \mu g) = \lambda D^{\alpha} f + \mu D^{\alpha} g$; each all $\lambda, \mu \in \mathbb{R}$.

II) $D^{\alpha}(x^r) = r x^{r-\alpha}$; for all $r \in \mathbb{R}$.

III) $D^{\alpha}(c) = 0$; for all constant functions $f(x) = c$.

IV) $D^{\alpha}(fg) = f D^{\alpha} g + g D^{\alpha} f$.

V) $D^{\alpha}(\frac{f}{g}) = \frac{g D^{\alpha} f - f D^{\alpha} g}{g^2}$.

VI) If f is differentiable, then f is α-differentiable at t and $D^{\alpha} f(a) = a^{1-\alpha} f'(a)$.

Proposition 1.2. *[1] Suppose that $f, g : [0; \infty) \longrightarrow \mathbb{R}$ is two α-differentiable functions where $\alpha \in (0; 1]$. Then $g \circ f$ is α-differentiable and for each*

$a \neq 0$ and $f(a) \neq 0$, the CFD of $(g \circ f)$ is equal to

$$D^{\alpha}(g \circ f)(a) = D^{\alpha} g(f(a)) D^{\alpha} f(a)(f(a))^{\alpha-1}. \tag{2}$$

Theorem 1.3. ***(Mean Value for CFD)****[16] Assume that $a > 0$ and given $f : [a, b] \longrightarrow \mathbb{R}$ be a function that satisfies*
I) f is continuous on $[a, b]$.
II) f is α-differentiable for $\alpha \in (0, 1)$.
Then, there exists $c \in (a, b)$, such that

$$D^{\alpha} f(c) = \alpha . \frac{f(b) - f(a)}{b^{\alpha} - a^{\alpha}}. \tag{3}$$

The "conformable fractional derivative" (CFD) aims at extension the usual derivative, gives a new solution for some FDE. For details, see [4, 7, 8, 10, 14, 18]. The authors of [20] state that the conformable interpretation of the derivatives gives a larger error than the fractional framework in each case.

Definition 1.4. Suppose that $f : \mathbb{R}^{+} \longrightarrow \mathbb{R}$ be a differentiable function. The "$\alpha(t)$-generalized fractional derivative" ($\alpha(t)$-GFD) for $\alpha(t) \in (0,]$ at pint a is defined by:

$$D_{\delta}^{\alpha(t)} f(a) = \lim_{h \to 0} \frac{\| f(a + h\delta(\alpha(t), a) a^{1-\alpha}) - f(a) \|}{\| h \|}$$

where $\delta(\alpha(t), a)$ is a function that may depend on $\alpha(t)$ and a.

Remark 1.5. As a consequence of the above definition, we have

$$D_{\delta}^{\alpha(t)} f(a) = \delta(\alpha(t), a) a^{1-\alpha(t)} D(t) f(a).$$

Definition 1.6. A differentiable function $f : \mathbb{R}^{+} \longrightarrow \mathbb{R}$ is said to be $\alpha(t)$-GFD function over $\mathbb{R}^{+}$ if it exists $D_{\delta}^{\alpha(t)} f(a)$ for all $a \in \mathbb{R}^{+}$ and $\alpha(t) \in (0, 1]$.

Example 1.7. We now apply our definition to the following one-dimensional $\alpha(t)$-GFD equation of variable-order, $D_{\delta}^{\alpha(t)} f(x) = 7(f(x))^2 x^2$. We mention that when $\alpha(t) = \frac{t}{2} \in (0, 1]$. and $\delta(\alpha(t), x) = x^2 \frac{t}{2}$. One obtains the classical equation $x^2 \frac{t}{2} x^{1-\frac{t}{2}} \frac{df(x)}{dx} = 7(f(x))^2 x^2$. It is easy to check that $f(x) = \frac{-1}{\frac{28}{t^2} x^{\frac{t}{2}} + C}$ for $x \neq 0$ is a solution.

Remark 1.8. In the following, we want to see the relation between $\alpha(t)$-GFD and the others definitions:
I) The FD of Khalil, Al Horani, Yousef and Sababheh in [16] is a particular case of $\alpha(t)$-GFD where $\delta(\alpha(t),x) = 1$ and $\alpha(t) = \alpha$.
II) The FD of Anderson and Ulness in [12] is a particular case of $\alpha(t)$-GFD where $\delta(\alpha(t),x) = \frac{(1-\alpha(t))x^{\alpha(t)}f(x)+\alpha(t)x^{1-\alpha(t)}Df}{\alpha(t)x^{1-\alpha(t)-1}}$ and $\alpha(t) = \alpha$.
II) The FD of Guebbai and Ghiat in [13] and Camrud in [7] are particular cases of $\alpha(t)$-GFD where $\delta(\alpha(t),x) = (\frac{xDf}{f})^{\alpha(t)-1}$ and $\alpha(t) = \alpha$.

Given the recent applications of these fractional derivatives in industry, the authors conclude that they can refine this theory by providing a new definition. In the first step, the basic concepts and results are presented. Further application of this new definition will be postponed to later work.

2 The n-ary variable-order $\boldsymbol{\alpha}(t)$-derivative of a multivariable vector-valued functions

The definitions provided so far have only been real value functions. In 2018 the N.Y. Gözütok et al. introduce CFD definition for the vector valued functions of several variables [12]. In this paper, we define the n-ary variable-order $\boldsymbol{\alpha}(t)$-derivative of multi-variable vector-valued functions. This definition have a many applications for fractional partial equations $\boldsymbol{\alpha}(t)$-differential of several orders.

In this article, we will denote by $\mathbb{R}^{n^+}$ the set of $a = (a_1, ..., a_n) \in \mathbb{R}^n$ where $a_i > 0$ for $i = 1, ..., n$. Also we write $\mathbb{T} = (0, 1]$ and defined $\mathbb{T}^n = \underbrace{\mathbb{T} \times ... \times \mathbb{T}}_{n \text{ times}}$. Note that $\boldsymbol{\alpha}(t) = (\alpha_1(t), ..., \alpha_n(t)) \in \mathbb{T}^n$ such that $\alpha_i(t) \in \mathbb{T}$ for $i = 1, ..., n$.
In this section, we define the new definition n-ary variable-order $\boldsymbol{\alpha}(t)$-derivative of vector valued functions with several variables $((n, \alpha(t)) - VOD)$. In this definition, we consider the $\alpha(t) \in \mathbb{T}$ for each variable of the multi-variable vector-valued function f. In the particular case this definition equivalent to the definition N.Y. Gözütok [12] and Z. Toghani [20]. Let $\mho = (a_1+h_1\delta(\alpha_1(t), a_1)a_1^{1-\alpha_1(t)}, ..., a_n+h_n\delta(\alpha_n(t), a_n)a_n^{1-\alpha_n(t)})$.

Definition 2.1. Assume that $f:\mathbb{R}^{n^+}\longrightarrow\mathbb{R}^m$ be a multi-variable vector-valued function such that

$$f(x_1,...,x_n)=(f_1(x_1,...,x_n),...,f_m(x_1,...,x_n))$$

and $\boldsymbol{\alpha}(t)\in\mathbb{T}^n$. Then the f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a}\in\mathbb{R}^{n^+}$, if there is a linear transformation $L:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ such that

$$\lim_{h\to 0}\frac{\|f(\mho)-f(a_1,...,a_n)-L(h)\|}{\|h\|}=0$$

where $h=(h_1,...,h_n)$. The linear transformation L is denoted by $D_\delta^{\boldsymbol{\alpha}}f(a)$ and called the CD of f of order $\boldsymbol{\alpha}(t)$ ($\boldsymbol{\alpha}(t)$-derivative of f) at $\boldsymbol{a}$.

Remark 2.2. I) For $m=n=1$, Definition 2.1 equivalent to definition R. Khalil of conformable fractional derivative in [16].
II) For $\alpha_1=...=\alpha_n=\alpha$, Definition 2.1 equivalent to definition N.Y. Gözütok of α-derivative of a vector-valued function in [12].

There is a uniqueness proposition here which is as follows:

Proposition 2.3. *Let $f:\mathbb{R}^{n^+}\to\mathbb{R}^m$ be a multi-variable vector-valued function. If f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a}\in\mathbb{R}^{n^+}$ where $\boldsymbol{\alpha}(t)\in\mathbb{T}^n$, then there is a unique linear transformation $L:\mathbb{R}^n\to\mathbb{R}^m$ such that*

$$\lim_{h\to 0}\frac{\|f(\mho)-f(a_1,...,a_n)-L(h)\|}{\|h\|}=0.$$

Proof. Assume that $K:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ satisfies

$$\lim_{h\to 0}\frac{\|f(\mho)-f(a_1,...,a_n)-K(h)\|}{\|h\|}=0.$$

Then,

$$\lim_{h\to 0}\frac{\|L(h)-K(h)\|}{\|h\|}$$

$$\leq\lim_{h\to 0}\frac{\|L(h)-f(\mho)-f(a)\|}{\|h\|}$$

$$+\lim_{h\to 0}\frac{\|f(\mho)-f(a)-K(h)\|}{\|h\|}=0.$$

If $x\in\mathbb{R}^n$, then $tx\to 0$ as $t\to 0$. Therefore for $x\neq 0$ we see that

$$0=\lim_{t\to 0}\frac{\|L(tx)-K(tx)\|}{\|tx\|}=\frac{\|L(x)-K(x)\|}{\|x\|}.$$

Hence $L(x)=K(x)$. $\square$

Example 2.4. Consider the function f defined by $f(x,y)=(e^x,\cos y)$ and the point $(a,b)\in\mathbb{R}^2$ such that $a,b>0$, $\boldsymbol{\alpha}(t)=(\alpha_1(t),\alpha_2(t))\in\mathbb{T}^2$, $\delta(\alpha_1(t),x)=\delta(\alpha_2(t),y)=1$ and $h=(h_1,h_2)$. Then $D^{\boldsymbol{\alpha}(t)}f(a,b)=L$ satisfies $L(x,y)=(xa^{1-\alpha_1(t)}e^a,ya^{1-\alpha_2(t)}\cos b)$.
We prove this, note that

$$\begin{aligned}
&\lim_{h\to 0}\frac{\|f(a+h_1a^{1-\alpha_1(t)},b+h_2b^{1-\alpha_2(t)})-f(a,b)-L(h_1,h_2)\|}{\|h\|}\\
&=\lim_{h\to 0}\frac{\|(e^{a+h_1a^{1-\alpha_1(t)}},\cos(b+h_2b^{1-\alpha_2(t)}))-(e^a,\cos b)-(L(h_1,h_2))\|}{\|h\|}\\
&=\lim_{(h_1,h_2)\to 0}\|\left(\frac{E_1}{h_1},\frac{E_2}{h_2}\right)\|, \qquad (4)
\end{aligned}$$

where

$$E_1=e^{a+h_1a^{1-\alpha_1(t)}}-(1+h_1a^{1-\alpha_1(t)})e^a,$$
$$E_2=\cos(b+h_2b^{1-\alpha_2(t)})-(1+h_2b^{1-\alpha_2(t)})\cos b.$$

Let

$$A=e^{a+h_1a^{1-\alpha_1(t)}}-e^a-h_1a^{1-\alpha_1(t)}e^a,$$
$$B=\cos(b+h_2b^{1-\alpha_2(t)})-\cos b-h_2b^{1-\alpha_2(t)}\cos b.$$

Therefore (4) becomes

$$\lim_{(h_1,h_2)\to 0}\|\left((\frac{A}{h_1},0)+(0,\frac{B}{h_2})\right)\|\leq\lim_{(h_1,h_2)\to 0}\left(\|(\frac{A}{h_1},0)\|+\|(0,\frac{B}{h_2})\|\right)=0,$$

because

$$\lim_{h_1\to 0}\frac{e^{a+h_1a^{1-\alpha_1(t)}}-e^a-h_1a^{1-\alpha_1(t)}e^a}{h_1}=0,$$
$$\lim_{h_2\to 0}\frac{\cos(b+h_2b^{1-\alpha_2(t)})-\cos b-h_2b^{1-\alpha_2(t)}\cos b}{h_2}=0.$$

Definition 2.5. Consider the matrix of the linear transformation $D_\delta^{\boldsymbol{\alpha}(t)}f$ with the standard ordered bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. This $m\times n$ matrix representation of $D_\delta^{\boldsymbol{\alpha}(t)}f$ is called the $\boldsymbol{\alpha}(t)$-derivative Jacobian matrix of f at a and denoted by $J_{\delta,f}^{\boldsymbol{\alpha}(t)}(a)$.

Example 2.6. If $f(x,y)=(e^x,\cos y)$, then

$$J_{\delta,f}^{\boldsymbol{\alpha}}(a,b)=\begin{pmatrix} a^{1-\alpha_1(t)}e^a & 0\\ 0 & b^{1-\alpha_2(t)}\cos b\end{pmatrix}.$$

Proposition 2.7. *Assume that f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a}\in\mathbb{R}^{n^+}$, where $\boldsymbol{\alpha}\in\mathbb{T}^n$. If f is differentiable at $\boldsymbol{a}$, then $D^{\boldsymbol{\alpha}_\delta(t)}f(\boldsymbol{a})=Df(\boldsymbol{a})\circ L_a^{1-\boldsymbol{\delta},\boldsymbol{\alpha}(t)}$ where $Df(\boldsymbol{a})$ is the derivative of f and $L_{\delta,\boldsymbol{a}}^{1-\boldsymbol{\alpha}(t)}$ is the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by*

$$L_{\delta,\boldsymbol{a}}^{1-\boldsymbol{\alpha}(t)}(x_1,...,x_n)=(x_1\delta(\alpha_1(t),x_1)a_1^{1-\alpha_1(t)},...,x_n\delta(\alpha_n(t),x_n)a_n^{1-\alpha_n(t)}).$$

Proof. Let $\mho=(a_1+h_1\delta(\alpha_1(t),a_1)a_1^{1-\alpha_1(t)},...,a_n+h_n\delta(\alpha_n(t),a_n)a_n^{1-\alpha_n(t)})$ It is sufficient to prove that

$$\lim_{h\to 0}\frac{\|f(\mho)-f(\boldsymbol{a})-Df(\boldsymbol{a})\circ L_{\boldsymbol{a}}^{1-\boldsymbol{\alpha}(t)}(h)\|}{\|h\|}=0. \tag{5}$$

Let $\varepsilon=(\varepsilon_1,...,\varepsilon_n)=(h_1\delta(\alpha_1(t),x_1)a_1^{1-\alpha_1(t)},...,h_n\delta(\alpha_n(t),x_n)a_n^{1-\alpha_n(t)})$ then $\varepsilon\to 0$ as $h\to 0$. We set $\mathcal{M}=\max\{((\delta(\alpha_i(t),a_i))a_i^{1-\alpha_i(t)})^2\ ;\ a_i>0, i=1,...,n\}$. Thus,

$$\begin{aligned}\|\varepsilon\|&=\sqrt{(h_1(\delta(\alpha_1(t),a_1)a_1^{1-\alpha_1(t)})^2+...+(h_n\delta(\alpha_n(t),a_n)a_n^{1-\alpha_n(t)})^2}\\ &\le\sqrt{(h_1)^2\mathcal{M}+...+(h_n)^2\mathcal{M}}=\sqrt{n\mathcal{M}}\|h\|.\end{aligned}$$

Consequently $\frac{1}{\sqrt{n\mathcal{M}}}\|\varepsilon\| \leq \|h\|$, and finally, we can write

$$\begin{aligned}
&\lim_{h\to 0}\frac{\|f(\mho)-f(\boldsymbol{a})-Df(\boldsymbol{a})\circ L_{\boldsymbol{a}}^{1-\boldsymbol{\alpha}}(h)\|}{\|h\|}\\
&=\lim_{h\to 0}\frac{\|f(\mho)-f(\boldsymbol{a})-Df(\varepsilon)\|}{\|h\|}\\
&\leq \lim_{\varepsilon\to 0}\frac{\|f(a_1+\varepsilon_1,...,a_n+\varepsilon_n)-f(\boldsymbol{a})-Df(\varepsilon)\|}{\frac{1}{\sqrt{n\mathcal{M}}}\|\varepsilon\|}\\
&=\sqrt{n\mathcal{M}}\lim_{\varepsilon\to 0}\frac{\|f(a_1+\varepsilon_1,...,a_n+\varepsilon_n)-f(\boldsymbol{a})-Df(\varepsilon)\|}{\|\varepsilon\|}=\sqrt{n\mathcal{M}}.0=0.
\end{aligned}$$

The (5) is proved. □

Remark 2.8. Proposition 2.7 is the generalized case of the part slowromancapvi@ of Proposition 1.2.

Proposition 2.9. *Let $f:\mathbb{R}^{n^+}\to\mathbb{R}^m$ be a multi-variable vector-valued function. If f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a}\in\mathbb{R}^{n^+}$ where $\boldsymbol{\alpha}\in\mathbb{T}^n$, then f is continuous at $\boldsymbol{a}$.*

Proof. Let $\mho=(a_1+h_1\delta(\alpha_1(t),a_1)a_1^{1-\alpha_1(t)},...,a_n+h_n\delta(\alpha_n(t),a_n)a_n^{1-\alpha_n(t)})$ Because

$$\begin{aligned}
&\|f(a_1+h_1\delta(\alpha_1(t),a_1)a_1^{1-\alpha_1(t)},...,a_n+h_n\delta(\alpha_n(t),a_n)a_n^{1-\alpha_n(t)})-f(a_1,...,a_n)\|\\
&=\lim_{h\to 0}\frac{f(\mho)-f(a_1,...,a_n)-L(h)+L(h)\|}{\|h\|}\|h\|\\
&\leq \lim_{h\to 0}\frac{f(\mho)-f(a)-L(h)\|}{\|h\|}\|h\|+\|L(h)\|.
\end{aligned}$$

By taking limits of the two sides of the inequality as $h\to 0$, we obtain

$$\begin{aligned}
&\lim_{h\to 0}\|f(\mho)-f(a_1,...,a_n)\|\\
&\leq \lim_{h\to 0}\frac{\|f(\mho)-f(\boldsymbol{a})-L(h)\|}{\|h\|}\|h\|+\|L(h)\|=0.
\end{aligned}$$

Let $\varepsilon = (\varepsilon_1, ..., \varepsilon_n) = (h_1 a_1^{1-\alpha_1(t)}, ..., h_n a_n^{1-\alpha_n(t)})$ so $\varepsilon \to 0$ as $h \to 0$. Since

$$\lim_{\varepsilon \to 0} \|f(a_1 + \varepsilon_1, ..., a_n + \varepsilon_n) - f(a_1, ..., a_n)\| \leq 0,$$

we see $\lim_{\varepsilon \to 0} \|f(a_1 + \varepsilon_1, ..., a_n + \varepsilon_n) - f(a_1, ..., a_n)\| = 0$. Therefore f is continuous at $\boldsymbol{a} \in \mathbb{R}^{n^+}$. $\square$

Theorem 2.10. ***(Chain Rule)*** *Assume that $f : \mathbb{R}^{n^+} \longrightarrow \mathbb{R}^m$ is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a} \in \mathbb{R}^{n^+}$ where $\boldsymbol{\alpha}(t) \in \mathbb{T}^n$, $(\delta(\alpha_1(t), a_1), ..., \delta(\alpha_n(t), a_n)) = (1, ..., 1)$ and function $g : \mathbb{R}^{m^+} \to \mathbb{R}^p$ is $\boldsymbol{\beta}(t)$-differentiable at $f(\boldsymbol{a}) = (f_1(a), ..., f_m(a)) \in \mathbb{R}^{m^+}$ such that $\boldsymbol{\beta} \in \mathbb{T}^m$.*
I) If $n > m$, then $(g \circ f)$ is $\boldsymbol{\gamma}(t)$-differentiable at $\boldsymbol{a} \in \mathbb{R}^{n^+}$ where for $i = 1, ..., n$, $\boldsymbol{\gamma}(t) = (\gamma_1(t), ..., \gamma_n(t))$ defined by

$$\gamma_i(t) = \begin{cases} \alpha_i(t) = \beta_i(t) & i = 1, ..., m \\ \alpha_i(t) & i = m+1, ..., n. \end{cases} \tag{6}$$

II) If $m > n$, then $(g \circ f)$ is $\boldsymbol{\gamma}(t)$-differentiable at $\boldsymbol{a} \in \mathbb{R}^{n^+}$ where for $i = 1, ..., n$, $\boldsymbol{\gamma}(t) = (\gamma_1(t), ..., \gamma_n(t))$ defined by

$$\gamma_i(t) = \alpha_i(t) = \beta_i(t), \qquad i = 1, ..., n. \tag{7}$$

The CFD of $(g \circ f)$ for order $\boldsymbol{\gamma}(t)$ is equal to

$$D^{\boldsymbol{\gamma}(t)}(g \circ f)(\boldsymbol{a}) = D^{\boldsymbol{\beta}(t)} g(f(\boldsymbol{a})) \circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1} \circ D^{\boldsymbol{\alpha}(t)} f(\boldsymbol{a}) \tag{8}$$

where $L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is the linear transformation defined by

$$L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}(x_1, ..., x_m) = (x_1(f_1(\boldsymbol{a}))^{\beta_1(t)-1}, ..., x_n(f_m(\boldsymbol{a}))^{\beta_m(t)-1}).$$

Proof. We prove (I). Similar arguments apply to the case (II). In case (I) since for $i = 1, ..., m$ we have $\alpha_i(t) = \beta_i(t)$. Therefore

$$L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}(x_1, ..., x_m) = (x_1(f_1(\boldsymbol{a}))^{\alpha_1(t)-1}, ..., x_n(f_m(\boldsymbol{a}))^{\alpha_m(t)-1}).$$

For abbreviation, let $L = D^{\boldsymbol{\alpha}(t)} f(\boldsymbol{a})$, $G = D^{\boldsymbol{\beta}(t)} g(f(\boldsymbol{a}))$ and $h = (h_1, ..., h_n) \in \mathbb{R}^n$, $k = (k_1, ..., k_n) \in \mathbb{R}^n$. If we consider the maps

1) $\mu(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)})$
$= f(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)}) - f(a_1, ..., a_n) - L(h)$,

2) $\eta(f_1(\boldsymbol{a}) + k_1(f_1(\boldsymbol{a}))^{1-\alpha_1(t)}, ..., f_m(\boldsymbol{a}) + k_m(f_m(a))^{1-\alpha_m(t)})$
$= g(f_1(\boldsymbol{a}) + k_1(f_1(\boldsymbol{a}))^{1-\alpha_1(t)}, ..., f_m(\boldsymbol{a}) + k_m(f_m(a))^{1-\alpha_m(t)}) - g(f(\boldsymbol{a})) -$
$G \circ (f(\boldsymbol{a}))$

3) $\varphi(a_1 + h_1 a_1^{1-\gamma_1(t)}, ..., a_n + h_n a_n^{1-\gamma_n(t)})$
$= g \circ f(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)}) - g \circ f(\boldsymbol{a}) - G \circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}-1} \circ L(h)$,

then according to the above maps, we have

4) $\lim_{h\to 0} \dfrac{\|\mu(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)})\|}{\|h\|} = 0$,

5) $\lim_{h\to 0} \dfrac{\|\eta(f_1(\boldsymbol{a}) + k_1(f_1(\boldsymbol{a}))^{1-\alpha_1(t)}, ..., f_m(\boldsymbol{a}) + k_m(f_m(\boldsymbol{a}))^{1-\alpha_m(t)})\|}{\|k\|} =$
0,

Let $\Upsilon = a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)}$. We must prove that,

$$\lim_{h\to 0} \frac{\|\varphi(a_1 + h_1 a_1^{1-\gamma_1(t)}, ..., a_n + h_n a_n^{1-\gamma_n(t)})\|}{\|h\|} = 0.$$

Now,

$$\begin{aligned}
&\varphi(a_1 + h_1 a_1^{1-\gamma_1(t)}, ..., a_n + h_n a_n^{1-\gamma_n(t)}) \\
=& g \circ f(\Upsilon) - g \circ f(\boldsymbol{a}) - G \circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1} \circ L(h) \\
=& g\Big(f_1(\Upsilon), ..., f_m(\Upsilon)\Big) \\
&- g(f(\boldsymbol{a})) - G \circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}\Big(f(\Upsilon) - f(\boldsymbol{a}) - \mu(\Upsilon)\Big) \qquad \text{by (1).}
\end{aligned}$$

Hence, the above equation becomes

$$\begin{aligned}
&\left[g\left(f_1(\Upsilon),...,f_m(\Upsilon)\right)-g(f(\boldsymbol{a}))-G\left(L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}(f(\Upsilon)-f(\boldsymbol{a}))\right)\right]\\
&+G\circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}\left(\mu(\Upsilon)\right)=\left[g\left(f_1(\Upsilon),...,f_m(\Upsilon)\right)\right.\\
&\left.-g(f(\boldsymbol{a}))-G\left(L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}(f_1(\Upsilon)-f_1(\boldsymbol{a}),...,f_m(\Upsilon)-f_m(\boldsymbol{a}))\right)\right]\\
&+G\circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}-1}\left(\mu(\Upsilon)\right)=\left[g\left(f_1(\Upsilon),...,f_m(\Upsilon)\right)\right.\\
&-g(f(\boldsymbol{a}))-G\left[\left(f_1(\Upsilon)-f_1(\boldsymbol{a})\right)(f_1(a))^{\alpha_1-1},...\right.\\
&\quad,\left(f_m(\Upsilon)-f_m(\boldsymbol{a})\right)(f_m(\boldsymbol{a}))^{\alpha_m-1}+G\circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}\left(\mu(\Upsilon)\right).
\end{aligned}$$

If $u_i=\left(f_i(\Upsilon)-f_i(\boldsymbol{a})\right)(f_i(\boldsymbol{a}))^{\alpha_i(t)-1}$ for $i=1,...,m$, then we see

$$f_i(\Upsilon)=f_i(\boldsymbol{a})+u_i(f_i(\boldsymbol{a}))^{1-\alpha_i(t)}$$

and $u=(u_1,..,u_m)\to 0$ as $h=(h_1,...,h_n)\to 0$. It follows that,

$$\begin{aligned}
&\varphi(a_1+h_1a_1^{1-\gamma_1(t)},...,a_n+h_na_n^{1-\gamma_n(t)})\\
=&\left[g\left(f_1(\Upsilon),...,f_m(\Upsilon)\right)-g(f(\boldsymbol{a}))-G(u)\right]\\
&+G\circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}\left(\mu(\Upsilon)\right) \qquad (9)
\end{aligned}$$

where by (2), (9) becomes

$$\begin{aligned}
&\eta(f_1(\boldsymbol{a})+u_1(f_1(\boldsymbol{a}))^{1-\alpha_1(t)},...,f_m(\boldsymbol{a})+u_m(f_m(\boldsymbol{a}))^{1-\alpha_m(t)})\\
&+G\circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}\left(\mu(\Upsilon)\right).
\end{aligned}$$

Since

$$\lim_{h\to 0}\frac{\|\eta(f_1(\boldsymbol{a})+k_1(f_1(\boldsymbol{a}))^{1-\alpha_1(t)},...,f_m(\boldsymbol{a})+k_m(f_m(\boldsymbol{a}))^{1-\alpha_m(t)})\|}{\|k\|}=0$$

and the linear transformation satisfies

$$\|G \circ L_{f(\boldsymbol{a})}^{\boldsymbol{\beta}(t)-1}\Big(\mu(\Upsilon)\Big)\| \leq K\|\mu(\Upsilon)\|$$

such that $K \geq 0$, therefore

$$\lim_{h\to 0} \frac{\|G \circ L_{f(a)}^{\boldsymbol{\beta}(t)-1}\Big(\mu(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)})\Big)\|}{\|k\|} = 0.$$

It follows that $\lim_{h\to 0} \dfrac{\|\varphi(a_1 + h_1 a_1^{1-\gamma_1(t)}, ..., a_n + h_n a_n^{1-\gamma_n(t)})\|}{\|h\|} = 0$. We give the proof only for the case (I). □

Remark 2.11. For, $m = n = p = 1$, Theorem 2.10 states that

$$D^{\alpha_1(t)}(g \circ f)(\boldsymbol{a}) = D^{\alpha_1(t)} g(f(\boldsymbol{a})) D^{\alpha_1(t)} f(\boldsymbol{a})(f(\boldsymbol{a}))^{\alpha_1(t)-1}.$$

Let $\alpha_i(t) = \alpha(t) = \beta_j(t)$ for $i = 1, ..., n$ and $j = 1, ..., m$. We can show that the Theorem 2.10 equivalent to the Theorem 3.9 in [12].

Corollary 2.12. *Suppose all conditions of Theorem 2.10 is satisfied. Then* $J_{g\circ f}^{\boldsymbol{\gamma}(t)}(\boldsymbol{a}) = J_g^{\boldsymbol{\beta}(t)}(f(\boldsymbol{a}))\times$

$$\begin{pmatrix} (f_1(\boldsymbol{a}))^{\alpha_1(t)-1} & 0 & \cdots & 0 \\ 0 & (f_2(\boldsymbol{a}))^{\alpha_2(t)-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (f_m(\boldsymbol{a}))^{\alpha_m(t)-1} \end{pmatrix} J_f^{\boldsymbol{\alpha}(t)}(a).$$

Theorem 2.13. *Assume that $f : \mathbb{R}^{n^+} \longrightarrow \mathbb{R}^m$ be a multivariable vector-valued function such that $f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$ and $\boldsymbol{\alpha}(t) \in \mathbb{T}^n$. Then f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a} \in \mathbb{R}^{n^+}$, if and only if each f_i is $\boldsymbol{\alpha}(t)$-differentiable, and $D^{\boldsymbol{\alpha}(t)} f(\boldsymbol{a}) = (D^{\boldsymbol{\alpha}(t)} f_1(\boldsymbol{a}), ..., D^{\boldsymbol{\alpha}(t)} f_m(\boldsymbol{a}))$.*

Proof. Suppose that for each $i = 1, ..., n$, f_i is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a}$. If we take $L = (D^{\boldsymbol{\alpha}(t)} f_1(\boldsymbol{a}), ..., D^{\boldsymbol{\alpha}(t)} f_m(\boldsymbol{a}))$ then

$$f(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)}) - f(a_1, ..., a_n) - L(h)$$
$$= f_1(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)}) - f_1(a_1, ..., a_n) - D^{\boldsymbol{\alpha}} f_1(h), ...$$
$$, f_m(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)}) - f_m(a_1, ..., a_n) - D^{\boldsymbol{\alpha}} f_m(h).$$

Hence,

$$\lim_{h\to 0}\frac{\|f(a_1+h_1a_1^{1-\alpha_1(t)},...,a_n+h_na_n^{1-\alpha_n(t)})-f(a_1,...,a_n)-L(h)\|}{\|h\|}$$
$$\leq \lim_{h\to 0}\sum_{i=1}^{m}\frac{\|f_i(a_1+h_1a_1^{1-\alpha_1(t)},...,a_n+h_na_n^{1-\alpha_n(t)})-f_i(a_1,...,a_n)-D^{\boldsymbol{\alpha}(t)}f_i(h)\|}{\|h\|}.$$

If f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a}\in\mathbb{R}^{n^+}$, then according to Theorem 2.10 $f_i=\pi_i\circ f$ is $\boldsymbol{\alpha}(t)$-differentiable at a. □

Theorem 2.14. *Let $\boldsymbol{\alpha}(t)\in\mathbb{T}^n$ and f,g be two $\boldsymbol{\alpha}(t)$-differentiable multi-variable vector-valued function at $\boldsymbol{a}\in\mathbb{R}^{n^+}$ and $(\delta(\alpha_1(t),x_1),...,\delta(\alpha_n(t),x_n))=(1,...,1)$, then we have*
I) $D^{\boldsymbol{\alpha}(t)}(\lambda f+\mu g)=\lambda D^{\boldsymbol{\alpha}(t)}f+\mu D^{\boldsymbol{\alpha}(t)}g$; for all $\lambda,\mu\in\mathbb{R}$.
II) $D^{\boldsymbol{\alpha}(t)}(fg)=fD^{\boldsymbol{\alpha}(t)}g+gD^{\boldsymbol{\alpha}(t)}f$.

Proof. (I) This proof follows from the definition.
(II) Given $A=(a_1+h_1a_1^{1-\alpha_1(t)},...,a_n+h_na_n^{1-\alpha_n(t)})$, then

$$\lim_{h\to 0}\frac{\|(fg)(A)-(fg)(\boldsymbol{a})-\Big(f(\boldsymbol{a})D^{\boldsymbol{\alpha}(t)}g(\boldsymbol{a})+g(\boldsymbol{a})D^{\boldsymbol{\alpha}(t)}f(\boldsymbol{a})\Big)(h)\|}{\|h\|}$$
$$\leq \lim_{h\to 0}\frac{\|f(A)g(A)-f(\boldsymbol{a})g(A)-g(A)D^{\boldsymbol{\alpha}(t)}f(\boldsymbol{a})(h)\|}{\|h\|}$$
$$+\lim_{h\to 0}\frac{\|f(\boldsymbol{a})g(A)-f(\boldsymbol{a})g(\boldsymbol{a})-f(\boldsymbol{a})D^{\boldsymbol{\alpha}(t)}g(\boldsymbol{a})(h)\|}{\|h\|}$$
$$+\lim_{h\to 0}\frac{\|g(A)D^{\boldsymbol{\alpha}(t)}f(\boldsymbol{a})(h)-g(\boldsymbol{a})D^{\boldsymbol{\alpha}(t)}f(\boldsymbol{a})(h)\|}{\|h\|} \tag{10}$$

Therefore (10) becomes

$$\lim_{h\to 0}\|g(A)\|\frac{\|f(A)-f(\boldsymbol{a})-D^{\boldsymbol{\alpha}(t)}f(\boldsymbol{a})(h)\|}{\|h\|}$$

$$+\lim_{h\to 0}\|f(\boldsymbol{a})\|\frac{\|g(A)-g(\boldsymbol{a})-D^{\boldsymbol{\alpha}(t)}g(\boldsymbol{a})(h)\|}{\|h\|}$$

$$+\lim_{h\to 0}\|D^{\boldsymbol{\alpha}(t)}f(\boldsymbol{a})(h)\|\frac{\|g(A)-g(\boldsymbol{a})\|}{\|h\|}$$
$$\leq \lim_{h\to 0}K\|h\|\frac{\|g(A)-g(\boldsymbol{a})\|}{\|h\|}=0.$$

The proof is complete. □

Theorem 2.15. ***(Mean Value)*** *Assuming that E is convex subset of*

$$\mathbb{R}^{n^+}=\{(x_1,...,x_n)\in\mathbb{R}^n\ ;\ x_i>0\ for\ all\ i=1,...,n\}$$

and $f:E\longrightarrow\mathbb{R}$ is $\boldsymbol{\alpha}(t)$-differentiable at of all E where $\boldsymbol{\alpha}(t)\in\mathbb{T}^n$ and $(\delta(\alpha_1(t),x_1),...,\delta(\alpha_n(t),x_n))=(1,...,1)$. If for any $x,y\in E$, define $[x,y]=\{(1-t)x+ty\ |\ 0\leq t\leq 1\}$, then there is $(1-t_0)x+t_0y=z=(z_1,...,z_n)\in[x,y]\subseteq\mathbb{R}^{n^+}$ such that

$$\alpha_1 t_0^{\alpha_1(t)-1}\Big(f(y)-f(x)\Big)=\Big(D^{\boldsymbol{\alpha}(t)}f(z)\circ L_z^{\boldsymbol{\alpha}(t)-1}\Big)(y-x)$$

where $L_z^{\boldsymbol{\alpha}(t)-1}$ is the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$L_z^{\boldsymbol{\alpha}(t)-1}(w_1,...,w_n)=(w_1z_1^{\alpha_1(t)-1},...,w_nz_n^{\alpha_n(t)-1}).$$

Proof. Let $g(t)=(1-t)x+ty$, then $D^{\alpha_1(t)}g(t)=t^{1-\alpha_1(t)}(y-x)$. According to Theorem (2.10), the function $F(t)=(f\circ g)(t)$ is $\boldsymbol{\gamma}(t)$-differentiable at $t\in[0,1]$. Therefore, we have

$$D^{\alpha_1(t)}F(t)=D^{\gamma}(f\circ g)(t)=D^{\boldsymbol{\alpha}(t)}f(g(t))\circ L_{g(t)}^{\boldsymbol{\alpha}(t)-1}\circ D^{\alpha_1(t)}g(t), \quad (11)$$

where $\boldsymbol{\gamma}(t)$ is obtained from equation (7). Since $F(t)$ is a real valued one variable function, so from mean value Theorem (3) for conformable fractional differentiable functions, there is $t_0\in[0,1]$ such that $\alpha_1(t).\dfrac{F(1)-F(0)}{1^{\alpha_1(t)}-0^{\alpha_1(t)}}=D^{\alpha_1(t)}F(t_0)$. Since

$$F(1)=f(g(1))=f(y)$$
$$F(1)=f(g(0))=f(x)$$
$$g(t_0)=(1-t_0)x+t_0y=z,$$

therefore (11) at t_0 becomes

$$\alpha_1(t)(F(1)-F(0)) = D^{\alpha_1(t)}F(t_0) = D^{\boldsymbol{\alpha}(t)}f(g(t_0)) \circ L_{g(t_0)}^{\boldsymbol{\alpha}(t)-1} \circ D^{\alpha_1(t)}g(t_0)$$

$$\alpha_1(t)\Big(f(y)-f(x)\Big) = \Big(D^{\boldsymbol{\alpha}(t)}f(z) \circ L_z^{\boldsymbol{\alpha}(t)-1}\Big)(t_0^{1-\alpha_1(t)}(y-x))$$

$$\alpha_1(t)t_0^{\alpha_1(t)-1}\Big(f(y)-f(x)\Big) = \Big(D^{\boldsymbol{\alpha}(t)}f(z) \circ L_z^{\boldsymbol{\alpha}(t)-1}\Big)(y-x).$$

□

Theorem 2.16. *Suppose that U is open subset of $\mathbb{R}^{n^+}$. If $f : U \longrightarrow \mathbb{R}$ have a local extremum at $\boldsymbol{a} \in U \subseteq \mathbb{R}^{n^+}$ and f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a}$ where $\boldsymbol{\alpha}(t) \in \mathbb{T}^n$ and $(\delta(\alpha_1(t), x_1), ..., \delta(\alpha_n(t), x_n)) = (1, ..., 1)$, then $D^{\boldsymbol{\alpha}(t)}f(\boldsymbol{a}) = 0$.*

Proof. Suppose f have a local maximum at $\boldsymbol{a} \in U$. Therefore,

$$\exists\delta > 0 \ \ \forall y \in U \subseteq \mathbb{R}^{n^+} \ \ \|\boldsymbol{a} - y\| < \delta \ \ \Rightarrow \ \ f(y) \leq f(\boldsymbol{a}).$$

Let $L = D^{\boldsymbol{\alpha}(t)}f(x)$. Therefore for each $h = (h_1, ..., h_n)$

$$f(a_1 + h_1 a_1^{1-\alpha_1(t)}, ..., a_n + h_n a_n^{1-\alpha_n(t)}) - f(a_1, ..., a_n) = L(h) + r(h).$$

Since $f : U \subseteq \mathbb{R}^{n^+} \longrightarrow \mathbb{R}$ is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a}$ then $\lim_{h\to 0} \dfrac{\|r(h)\|}{\|h\|} = 0$.
Let $v \in \mathbb{R}^n$ and $\varepsilon > 0$ small enough such that $h = tv$, then $L(tv)+r(tv) \leq 0$ because f have a local maximum at $\boldsymbol{a}$. Therefore,

$$\lim_{t\to 0} \frac{L(tv) + r(tv)}{t} = \lim_{t\to 0} L(v) + \frac{r(tv)}{t} \leq 0 \ \ \Rightarrow \ \ L(v) \leq 0.$$

Since L is linear transformation then with the displacement v to $-v$, we have $L(v) \geq 0$. Therefore $L(v) = 0$. Since v is arbitrary then $L(v) = D^{\boldsymbol{\alpha}(t)}f(a)(v) = 0$. When f have a local minimum, the proof is similar. □

3 The n-ary variable-order partial $\boldsymbol{\alpha}(t)$-derivatives ($(n, \boldsymbol{\alpha}(t)) - VOP$)

In this section we introduce the definition of $(n, \boldsymbol{\alpha}(t)) - VOP$ of a multi-variable real-valued functions with n variables.

Definition 3.1. Let $f : \mathbb{R}^{n^+} \to \mathbb{R}^m$ be a multi-variable vector-valued function of n variables. Suppose that $\boldsymbol{a} \in \mathbb{R}^{n^+}$. If the limit

$$\lim_{h \to 0} \frac{f(a_1, ..., a_i + h a_i^{1-\alpha_i(t)}, ..., a_n) - f(a_1, ..., a_i, ..., a_n)}{h},$$

there existed, such that $\alpha_i(t) \in \mathbb{T}$, then denoted by $\dfrac{\partial^{\alpha_i(t)} f}{\partial x_i}(\boldsymbol{a})$ and called the i^{th} partial $\boldsymbol{\alpha}(t)$-derivative of f of order $\alpha_i(t)$ at $\boldsymbol{a}$.

Theorem 3.2. *Suppose f be multi-variable vector-valued function of n variables. If f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a} \in \mathbb{R}^{n^+}$ where $\boldsymbol{\alpha}(t) \in \mathbb{T}^n$, then $\dfrac{\partial^{\alpha_j(t)} f_i}{\partial x_j}(\boldsymbol{a})$ exists for $i = 1, ..., m$; $j = 1, ..., n$ and the $\boldsymbol{\alpha}(t)$-derivative fractional Jacobian of f at $\boldsymbol{a}$ is the $m \times n$ matrix $\left(\dfrac{\partial^{\alpha_j(t)} f_i}{\partial x_j}(\boldsymbol{a})\right)_{m \times n}$.*

Proof. Define $f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$. We first give the proof only for the case $m = 1$, so $f(x_1, ..., x_n) \in \mathbb{R}$. Define $P : \mathbb{R} \longrightarrow \mathbb{R}^n$ by

$$P(y) = (P_1(y), ..., P_n(y)) = (a_1, ..., a_{j-1}, y, a_{j+1}, ..., a_n).$$

Then $\dfrac{\partial^{\alpha_j(t)} f_i}{\partial x_j}(a_j) = D^{\boldsymbol{\gamma}(t)}(f \circ P)(a_j)$, where $\boldsymbol{\gamma}(t)$ is obtained from equation (7). Thus, by Corollary 2.12 the matrix $J^{\boldsymbol{\gamma}(t)}_{f \circ P}(a_j)$ becomes $J^{\boldsymbol{\alpha}(t)}_f(P(a_j)) \times$

$$\begin{pmatrix} (P_1(a_j))^{\alpha_1(t)-1} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & (P_j(a_j))^{\alpha_j(t)-1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & (P_n(a))^{\alpha_n(t)-1} \end{pmatrix} J^{\alpha_j(t)}_P(a_j),$$

which is equal to $J_f^{\boldsymbol{\alpha}(t)}(a)\times$

$$\begin{pmatrix} (a_1)^{\alpha_1(t)-1} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & (a_j)^{\alpha_j(t)-1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & (a_n)^{\alpha_n(t)-1} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ (a_j)^{1-\alpha_j(t)} \\ \vdots \\ 0 \end{pmatrix},$$

finally equal to

$$J_f^{\boldsymbol{\alpha}(t)}(a) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $J_{f\circ P}^{\gamma(t)}(a_j)$ has the single entry $\dfrac{\partial^{\alpha_j(t)} f_i}{\partial x_j}(\boldsymbol{a})$. This shows that $\dfrac{\partial^{\alpha_j(t)} f_i}{\partial x_j}(\boldsymbol{a})$ exists and is the jth entry of the $1 \times n$ matrix $J_f^{\boldsymbol{\alpha}(t)}(\boldsymbol{a})$. The theorem now follows for arbitrary m since, according to Theorem 2.13 each f_i is $\boldsymbol{\alpha(t)}$-differentiable. □

Proposition 3.3. *Let $f : \mathbb{R}^{n^+} \to \mathbb{R}^m$ be a multi-variable vector-valued function. If f is $\boldsymbol{\alpha}(t)$-differentiable at $\boldsymbol{a} \in \mathbb{R}^{n^+}$ where $\boldsymbol{\alpha}(t) \in \mathbb{T}^n$, then* $\dfrac{\partial^{\alpha_j(t)} f_i}{\partial x_j}(\boldsymbol{a}) = \boldsymbol{a}^{\alpha_j(t)-1} \dfrac{\partial f_i}{\partial x_j}(\boldsymbol{a})$.

Proof. Let $\varepsilon = h\boldsymbol{a}^{1-\alpha_i(t)}$ in above definition and then $h = \boldsymbol{a}^{\alpha_i(t)-1}\varepsilon$. Therefore, we can obtain

$$\frac{\partial^{\alpha_i(t)} f}{\partial x_i}(\boldsymbol{a}) = \lim_{h\to 0} \frac{f(a_1, ..., a_i + h a_i^{1-\alpha_i(t)}, ..., a_n) - f(a_1, ..., a_i, ..., a_n)}{h},$$

$$= \lim_{\varepsilon\to 0} \frac{f(a_1, ..., a_i + \varepsilon, ..., a_n) - f(a_1, ..., a_i, ..., a_n)}{\varepsilon \boldsymbol{a}^{\alpha_j(t)-1}}$$

$$= \boldsymbol{a}^{1-\alpha_j(t)} \lim_{\varepsilon\to 0} \frac{f(a_1, ..., a_i + \varepsilon, ..., a_n) - f(a_1, ..., a_i, ..., a_n)}{\varepsilon} = \boldsymbol{a}^{\alpha_j(t)-1} \frac{\partial f_i}{\partial x_j}(\boldsymbol{a}).$$

□

4 Conclusions

The FC generalizes the concept of derivative $D^{\boldsymbol{\alpha(t)}}[f(x)]$ to non-integer orders and the physical applications of FC to describe complex media and processes are considered by many mathematicians, physicists, and engineers in recent decades. The VO-FDE's with time t and space x has been successfully applied to investigate time, space-dependent dynamics. The purpose of this article is to make the basic premise of generalizing this theory for application in science.

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