Journal of Mathematical Extension
Vol. 7, No. 3, (2013), 1-14

On Topological Spaces X Determined by the Torsion Elements of $C(X)$

A. Rezaei Aliabad*
Chamran University

M. Motamedi
Chamran University

Abstract. Let $C(X)$ be the ring of real continuous functions on a Tychonoff space X and $T(X)$ be the set of all torsion elements of $C(X)$. We prove that if X and Y are two zero dimensional compact spaces, then $X \simeq Y$ if and only if the rings generated by $T(X)$ and $T(Y)$ are isomorphic.

AMS Subject Classification: 54C40; 16U60.
Keywords and Phrases: Group of units of $C(X)$, torsion subgroup, socle of a group, maximal subgroup, Frattini subgroup, $\mathcal{P}$-filter, $\mathcal{P}$-ultrafilter

1. Introduction

Throughout this paper, all topological spaces X that we consider are Tychonoff and $C(X)$ ($C^*(X)$) stands for the ring of continuous (bounded) real functions on a topological space X. Suppose $f \in C(X)$, we denote the set $f^{-1}\{0\}$ by $Z(f)$, its complement by $Coz(f)$, and the collection of all zero-sets in X by $Z(X)$. For undefined terms and notions, see [8]. We denote the group of units of the ring R by $U(R)$. Suppose that G is an abelian group, by $H \leqslant G$ we mean that H is a subgroup of G, by $T(G)$

Received: December 2012; Accepted: May 2013
*Corresponding author

we mean the torsion subgroup of G. For the sake of simplicity, $U(C(X))$ and $T(U(C(X)))$ will be denoted by $U(X)$ and $T(X)$, respectively. The set $\{f \in U(X) : f(x) > 0,\ \forall x \in X\}$ is denoted by $U^+(X)$. Suppose $f \in U(X)$, we denote the set $f^{-1}\{1\}$ by $e(f)$ and its complement by $Coe(f)$. One can easily see that $\{e(f) : f \in U(X)\} = Z(X)$. By $Max(G)$ we mean the set of all maximal subgroups of G.
In Section 2, we obtain some general facts about $U(X)$. In particular, in the same section we observe that $U(X)$ is direct product of its torsion subgroup $T(X)$ and $U^+(X)$. In Sections 3, we will focus on $T(X)$ and prove, as the main result, that if X and Y are compact zero dimensional spaces, then $X \simeq Y$ if and only if the rings generated by $T(X)$ and $T(Y)$ are isomorphic.

2. Preliminary Results

We first discuss on cardinality of $U(X)$. Let $B_{C^*}(0,1)$ be the unit ball with center 0. Define $\varphi : C(X) \longrightarrow B_{C^*}(0,1)$ by $\varphi(f) = \frac{f}{1+|f|}$, then clearly φ is one to one. Therefore, for any topological space X, we have $|C(X)| = |B_{C^*}(0,1)|$. Now, suppose that $\varphi : B_{C^*}(0,1) \longrightarrow U^+(X) \cap C^*(X)$ by $\varphi(f) = f + 2$. It is clear that φ is well defined, one-one and thus $|C(X)| = |B_{C^*}(0,1)| \leqslant |U^+(X) \cap C^*(X)| \leqslant |U^+(X) \leqslant |U(X)| \leqslant |C(X)|$. Therefore $|U^+(X)| = |U(X)| = |C(X)| = |C^*(X)|$.

Proposition 2.1. *The following statements hold.*
(a) $T(X) = \{f \in U(X) :\ f^2 = 1\}$ *and it is a subgroup of* $U(X)$.
(b) *The cardinality of the set of torsion free elements is the same as the cardinality of* $U(X)$.
(c)$T(X) = \{-1, 1\}$ *if and only if* X *is connected.*

Proof. (a) and (b) are clear.
(c $\Rightarrow$) Suppose that A is a clopen subset (i.e., closed open subset) of X. Put $\lambda_A = \chi_A - \chi_{A^c}$ (from now on, we use λ_A for $\chi_A - \chi_{A^c}$ where χ_A is the charactristic function on A). By hypothesis, $\lambda_A = -1$ or 1. Therefore, $A = \emptyset$ or $A = X$ and consequently X is connected.
(c $\Leftarrow$) Assume that X is connected and $f \in T(X)$. Then $f(X) \subseteq \{-1, 1\}$

and it follows that f is constant. Therefore $f = -1$ or 1. $\square$

Let $\mathcal{P}$ be the set of all clopen subsets of X. Clearly X is zero dimensional if and only if $\mathcal{P}$ is a base for open subset of X. Moreover, the map $f \xrightarrow{e} e(f)$ makes a one-to-one correspondence between $T(X)$ and $\mathcal{P}$ and hence $|T(X)| = |\mathcal{P}|$.

Proposition 2.2. *Let α be the cardinality of the set of connected component of a topological space X. Then*
(a) $|T(X)| \leqslant 2^\alpha$ *and the inequality may be strict.*
(b) *If α is finite, then $|T(X)| = 2^\alpha$.*

Proof. (a) It is enough to show that $|\mathcal{P}| \leqslant 2^\alpha$. To see this, letting $\mathcal{A}$ be the set of connected component of X, we define $\phi : \mathcal{P} \longrightarrow \mathbf{P}(\mathcal{A})$ with $\phi(P) = \{C \in \mathcal{A} : \ C \subseteq P\}$. We can easily see that ϕ is one-one and so we are done. Now, if we put $X = \mathbb{N}^*$ where $\mathbb{N}^*$ is the one point compactification of $\mathbb{N}$, then the cardinality of the family of clopen subsets of $\mathbb{N}^*$ is equal to $\aleph_0 = |T(X))|$.
(b) It is evident.

The socle $S(G)$ of an abelian group G consists of all $g \in G$ such that the order of g is a square free integer, see [7]. $S(G)$ is a subgroup of G; it is equal to $\{1\}$ if and only if G is torsion free and it is equal to G if and only if G is an elementary group, in the sense that every element has a square free order. It is clear that $S(G) \subseteq T(G)$. Therefore, by Proposition 2.1, we conclude that $S(U(X)) = T(X)$. $\square$

The following fact, although easy to prove, is a key result for the remainder of the paper.

Theorem 2.3. *For any topological space X, $U(X)$ is the direct product of $U^+(X)$ and $T(X)$.*

Proof. It is clear that $f = |f|Sgn(f)$ for any $f \in U(X)$ and $U^+(X) \cap T(X) = \{1\}$. $\square$

Theorem 2.4. *The following statements hold.*
(a) *If $K \in Max(U(X))$, then $U^+(X) \subseteq K$.*

(b) $K \in Max(U(X))$ *if and only if there exists* $H \in Max(T(X))$ *such that* $K = U^+(X)H$.
(c) *If* $K \leqslant U(X)$, *then* $K \in Max(U(X))$ *if and only if* $|U(X)/K| = 2$.

Proof. (a) There exists a prime number p such that $|U(X)/K| = p$, then $f^p \in K$ for any $f \in U(X)$. Now, let $f \in U^+(X)$, then $f = (f^{\frac{1}{p}})^p \in K$.
(b $\Rightarrow$) Using the part (a) and Theorem 2.3, we get $K = U^+(X)H$ where $H = K \cap T(X)$.
(b $\Leftarrow$) It is clear.
(c) By assumption, $|U(X)/K|$ is a prime number. On the other hand, by (b), $H \leqslant T(X)$ exists such that $K = U^+(X)H$, thus $U(X)/K \simeq T(X)/H$ and since every element of $T(X)$ is of order 2, $|U(X)/K| = 2$. $\square$

Recall that the Frattini subgroup of a group G is the intersection of all maximal subgroups of G, this subgroup is denoted by $\Phi(G)$, thus $\Phi(G) = \cap_{H \in Max(G)} H$.

Proposition 2.5. *For any topological space* X *we have*
(a)$\Phi(T(X)) = \{1\}$;
(b)$\Phi(U(X)) = U^+(X)$.

Proof. (a) Let $1 \neq f \in T(X)$, hence there exists $x \in X$ such that $f(x) \neq 1$. One can easily see that $H_x = \{g \in T(X) : \ x \in e(g)\} \in Max(T(X))$ and since $f \notin H_x$, we are through.
(b) It is clear that $\Phi(U(X)) = U^+(X)\Phi(T(X)) = U^+(X)$. $\square$

We conclude this section by the following remark which is useful for the next section and helps us to find an example of two zero dimensional compact spaces X and Y such that $T(X) \simeq T(Y)$ but $X \not\simeq Y$.

Remark 2.6. *The subgroup* $T(X)$ *is indeed the maximal torsion subgroup of* $U(X)$ *and is* $\mathbb{Z}_2$*-vector space (via* $(n, f) \to f^n$*). Clearly* $\varphi : T(X) \to T(Y)$ *is a group homomorphism if and only if it is a vector space homomorphism. Let* V *be a vector space over a field* F *and* S *be a base for* V*. If* $|F|$ *and* $|S|$ *are finite, then* $V \simeq F^{|S|}$ *and so* $|V| = |F|^{|S|}$*. Also, if* $|F|$ *or* $|S|$ *is infinite, then* $|V| = \max\{|F|, |S|\}$*. Therefore, if* V

and W are F-vector spaces and $|V| = |W|$, then $V \simeq W$ whenever one of the following holds.
(a) F is finite.
(b) $|F| < |V|$.

3. Zero Dimensionality is a Torsion Property

To give the main result of the paper we need to introduce and study a class of subgroups of $T(X)$ and $\mathcal{P}$-filters on X.

The next two simple facts are needed.

Proposition 3.1. *The following statements are equivalent.*
(a) $H \in Max(T(X))$.
(b) $fg \in H$ *if and only if* $f, g \in H$ *or* $f, g \notin H$.

Proof. Since $H \in Max(T(X))$ if and only if $|T(X)/H| = 2$, it is easy to prove. $\square$

Lemma 3.2. *Let $f, g \in T(X)$, then*

$$e(fg) = (e(f) \cap e(g)) \cup (Coe(f) \cap Coe(g)) \,.$$

Proof. It is evident. $\square$

Proposition 3.3. *Let X be a topological space and $p \in \beta X$, then $H^p = \{f \in T(X) : p \in cl_{\beta X} e(f)\}$ is a maximal subgroup of $T(X)$.*

Proof. Suppose that $fg \in H^p$, by Lemma 3.2

$$p \in cl_{\beta X} e(fg) = cl_{\beta X}[(e(f) \cap e(g)) \cup (Coe(f) \cap Coe(g))]$$

$$= (cl_{\beta X} e(f) \cap cl_{\beta X} e(g)) \cup (cl_{\beta X} Coe(f) \cap cl_{\beta X} Coe(g)) \,.$$

Thus, $p \in (cl_{\beta X} e(f) \cap cl_{\beta X} e(g))$ or $p \in (cl_{\beta X} Coe(f) \cap cl_{\beta X} Coe(g))$ and by Proposition 3.1, $H^p \in Max(T(X))$. $\square$

In this section, as we mentioned earlier, $\mathcal{P}(X)$ (briefly $\mathcal{P}$) stands for the set of all clopen subsets of X and by $\mathcal{P}$-filter we mean a filter whose elements are clopen subsets, see ([11] 12E). It is easy to see that if $\mathcal{F}$ is a $\mathcal{P}$-filter, then

$$e^{-1}(\mathcal{F}) = \{f : \ e(f) \in \mathcal{F}\}$$

is a subgroup of $T(X)$. On the other hand $ee^{-1}(\mathcal{F}) = \mathcal{F}$ for every $\mathcal{P}$-filter $\mathcal{F}$ on X and since $e(f) = e(g)$ implies $f = g$ for every $f, g \in T(X)$, $H = e^{-1}e(H)$ for every $H \leqslant T(X)$. But if H is a subgroup of $T(X)$, then $e(H) = \{e(f) : \ f \in H\}$ is not necessarily a $\mathcal{P}$-filter. As an example, $H = \{-1, 1\}$ is a subgroup of $T(X)$ while $e(H)$ has not even finite intersection property.

Proposition 3.4. *Let X be a topological space and $H \leqslant T(X)$, then the following statements are equivalent.*
(a) *There exists $p \in \beta X$ such that $H = H^p$.*
(b) *$e(H)$ is a $\mathcal{P}$-ultrafilter.*
(c) *The family $e(H)$ has the finite intersection property and is maximal with respect to this property.*

Proof. (a)⇒(b) Let $f_1, \cdots, f_n \in H$. By definition, $p \in \cap_{i=1}^{n} cl_{\beta X} e(f_i) = cl_{\beta X}(\cap_{i=1}^{n} e(f_i))$ and this implies $\cap_{i=1}^{n} e(f_i) \neq \emptyset$, thus $e(H)$ has the finite intersection property, and there exists a $\mathcal{P}$-ultrafilter $\mathcal{F}$ containing $e(H)$. Therefore, $H = e^{-1}e(H) \subseteq e^{-1}(\mathcal{F})$. Now, since H is maximal, $H = e^{-1}(\mathcal{F})$ and hence $e(H) = ee^{-1}(\mathcal{F}) = \mathcal{F}$.
(b)⇒(c) It is clear.
(c)⇒(a) Suppose that H satisfies the condition (c). Since βX is compact, there exists $p \in \beta X$ such that $p \in \cap_{f \in H} cl_{\beta X} e(f)$. Clearly H^p has the finite intersection property and contains H. Therefore, $H = H^p$. □

Proposition 3.5. *Let X be a topological space and $\mathcal{F}$ be a $\mathcal{P}$-filter on X, then $e^{-1}(\mathcal{F})$ is a maximal subgroup of $T(X)$ if and only if $\mathcal{F}$ is a $\mathcal{P}$-ultrafilter on X.*

Proof. ⇒) Let $e^{-1}(\mathcal{F})$ be a maximal subgroup of $T(X)$, we have to show that $\mathcal{F}$ is a $\mathcal{P}$-ultrafilter. Let $\mathcal{F} \subseteq \mathcal{G}$, then $e^{-1}(\mathcal{F}) \subseteq e^{-1}(\mathcal{G})$ and $e^{-1}(\mathcal{F}) = e^{-1}(\mathcal{G})$. We infer that $\mathcal{F} = ee^{-1}(\mathcal{F}) = ee^{-1}(\mathcal{G}) = \mathcal{G}$ and hence $\mathcal{F}$ is a $\mathcal{P}$-ultrafilter.

$\Leftarrow$) Since $ee^{-1}(\mathcal{F}) = \mathcal{F}$ is a $\mathcal{P}$-ultrafilter, by Proposition 3.4, it follows that $e^{-1}(\mathcal{F})$ is a maximal subgroup of $T(X)$. $\square$

Let X be a topological space, we will denote by $\mathcal{M}^*$ the set of all maximal subgroups of $T(X)$ which are of the form H^p. Given $f \in T(X)$, we define $\mathcal{M}^*(f) = \{M \in \mathcal{M}^* : \ f \in M\}$ and $\mathbf{M}^*_f = \bigcap \mathcal{M}^*(f)$.

Proposition 3.6. *Let X be a topological space, then the following statements are equivalent.*
(a) $e(f) \subseteq e(g)$.
(b) $g \in \bigcap \mathcal{M}^*(f)$.
(c) $\mathcal{M}^*(f) \subseteq \mathcal{M}^*(g)$.
(d) $\mathbf{M}^*_g \subseteq \mathbf{M}^*_f$.

Proof. (a $\Rightarrow$ b) Let $f \in H^p \in \mathcal{M}^*(f)$, then $p \in cl_{\beta X} e(f) \subseteq cl_{\beta X} e(g)$ and so $g \in H^p$. Hence $g \in \bigcap \mathcal{M}^*(f)$.
(b $\Rightarrow$ c) and (c $\Rightarrow$ d) are trivial.
(d $\Rightarrow$ a) Let $x \in e(f)$, then $f \in H^x$ and so $g \in \mathbf{M}^*_g \subseteq \mathbf{M}^*_f \subseteq H^x$. Therefore $x \in e(g)$. $\square$

Recall that a topological space X is called strongly zero dimensional if every two completely separated closed set can be separated by two disjoint clopen subsets of X, see ([4] 6.2).

In the following proposition, only the part (d) may not be well-known.

Proposition 3.7. *Let X be a Tychonoff space. Then the following statements are equivalent.*
(a) *X is strongly zero dimensional.*
(b) *Every two disjoint $Z_1, Z_2 \in Z(X)$ separate with two disjoint open-closed subset of X.*
(c) *If $Z \in Z(X)$, $V \in Coz(X)$ and $Z \subseteq V$, then there exists an open-closed subset U of X such that $Z \subseteq U \subseteq V$.*
(d) *The set $\{cl_{\beta X} e(f) : \ f \in T(X)\}$ is an open base for βX.*
(e) *βX is a zero dimensional space.*
(f) *βX is a strongly zero dimensional space.*

Proof. We only prove (c) $\Rightarrow$ (d), for the reminder of the proof, see ([4]

6.2). Suppose that W is an open subset of βX and $p \in W$. Clearly there exist a zero set A and a cozero set V in βX such that

$$p \in \mathrm{int}_{\beta X} A \subseteq A \subseteq V \subseteq \mathrm{cl}_{\beta X} V \subseteq W. \qquad (1)$$

By (c), there exists a clopen subset U of X such that

$$A \cap X \subseteq U \subseteq V \cap X. \qquad (2)$$

Both (1) and (2) implies that

$$p \in \mathrm{cl}_{\beta X}\mathrm{int}_{\beta X} A = \mathrm{cl}_{\beta X}(X \cap \mathrm{int}_{\beta X} A) \subseteq \mathrm{cl}_{\beta X}(X \cap A) \subseteq \mathrm{cl}_{\beta X} U \subseteq \mathrm{cl}_{\beta X} V \subseteq W.$$

Clearly $U = e(f)$ for some $f \in T(X)$ and so we are through. □

Proposition 3.8. *Let X be a topological space, then*
(a) *The set $\{\mathcal{M}^*(f) : f \in T(X)\}$ is a base for the closed sets of a topology on $\mathcal{M}^*$, which we call it the Zariski-like topology;*
(b) *$\mathcal{M}^*$ with Zariski-like topology is compact;*
(c) *If X is strongly zero dimensional, then $\mathcal{M}^*$ with this topology is Hausdorff.*

Proof. (a) It is clear that for every $f, g \in T(X)$, $e(f) \cup e(g)$ is both open and closed in X and hence there exists $h \in T(X)$ such that $e(h) = e(f) \cup e(g)$. It is enough to show that $\mathcal{M}^*(f) \bigcup \mathcal{M}^*(g) = \mathcal{M}^*(h)$. To prove this

$$\begin{aligned} H^p \in \mathcal{M}^*(f) \bigcup \mathcal{M}^*(g) \quad &\Leftrightarrow \quad p \in cl_{\beta X} e(f) \cup cl_{\beta X} e(g) = cl_{\beta X} e(h) \\ &\Leftrightarrow \quad H^p \in \mathcal{M}^*(h). \end{aligned}$$

(b) Suppose $\{\mathcal{M}^*(f_\alpha)\}_{\alpha \in A}$ is a family of basic closed subset of $\mathcal{M}^*$ with the finite intersection property. We can easily see that $\{cl_{\beta X} e(f_\alpha)\}_{\alpha \in A}$ has the finite intersection property and consequently there exists $p \in \beta X$ such that $p \in \cap_{\alpha \in A} cl_{\beta X} e(f_\alpha)$. Therefore

$$\forall \alpha \in A, \;\; f_\alpha \in H^p \quad \Leftrightarrow \quad \forall \alpha \in A, \;\; H^p \in \mathcal{M}^*(f_\alpha) \quad \Leftrightarrow \quad H^p \in \bigcap_{\alpha \in A} \mathcal{M}^*(f_\alpha)$$

$$\therefore \quad \bigcap_{\alpha \in A} \mathcal{M}^*(f_\alpha) \neq \emptyset.$$

(c) Let $H^p, H^q \in \mathcal{M}^*$ where $p \neq q$. Since X is strongly zero dimensional, by Proposition 3.7, there exists $f \in T(X)$ such that $p \in cl_{\beta X} e(f), q \notin cl_{\beta X} e(f)$ and so $p \notin cl_{\beta X} e(-f), q \notin cl_{\beta X} e(f)$. Therefore, $H^p \notin \mathcal{M}^*(-f), H^q \notin \mathcal{M}^*(f)$ and $\mathcal{M}^*(f) \bigcup \mathcal{M}^*(-f) = \mathcal{M}^*$. $\square$

Proposition 3.9. *Let X be a zero dimensional compact topological space, then $\varphi : X \longrightarrow \mathcal{M}^*$ defined by $\varphi(p) = H^p$ induces a homeomorphism between X and $\mathcal{M}^*$.*

Proof. It is clear that φ is bijection. It is sufficient to show that this function maps a base for X to a base for $\mathcal{M}^*$. To this end we write

$$\varphi(e(f)) = \{\varphi(p) \in X : \ p \in e(f)\}$$

$$= \{H^p \in \mathcal{M}^* : \ p \in e(f)\} = \{H^p \in \mathcal{M}^* : \ f \in H^p\} = \mathcal{M}^*(f)$$

which completes the proof. $\square$

Definition 3.10. *Let X and Y be topological spaces. $T(X)$ and $T(Y)$ are said to be strongly isomorphic if there exists an isomorphism from $T(X)$ to $T(Y)$ such that it maps $\mathcal{M}^*(T(X))$ onto $\mathcal{M}^*(T(Y))$.*

Proposition 3.11. *If X and Y are zero dimensional compact topological spaces, then X and Y are homeomorphic if only if $T(X)$ and $T(Y)$ are strongly isomorphic.*

Proof. $\Rightarrow$) Let $\psi : X \longrightarrow Y$ be a homeomorphism. We define $\varphi : T(Y) \longrightarrow T(X)$ by $\varphi(g) = g \circ \psi$. It can be easily shown that φ is an isomorphism. We show that φ is onto. Supposing $f \in T(X)$, we put $g = f \circ \psi^{-1}$. It is easy to show that $g \in T(Y)$ and $\varphi(g) = f$. Now, let $p \in X$ and $q = \psi(p)$, then

$$g \in H^q(Y) \ \Leftrightarrow \ g(q) = 1 \ \Leftrightarrow \ g\psi\psi^{-1}(q) = 1 \ \Leftrightarrow \ g\psi(p) = 1$$

$$\Leftrightarrow \ g \circ \psi \in H^p(X) \ \Leftrightarrow \ \varphi(g) \in H^p(X).$$

$\Leftarrow$) By Proposition 3.9 and the hypothesis, it is clear that

$$X \simeq \mathcal{M}^*(T(X)) \simeq \mathcal{M}^*(T(Y)) \simeq Y. \quad \square$$

Proposition 3.12. *If X and Y are topological spaces, then the following statements are equivalent.*
(a) $T(X) \simeq T(Y)$.
(b) $|T(X)| = |T(Y)|$.
(c) $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$.

Proof. (a)$\Leftrightarrow$ (b) is clear, by Remark 2.6. Recall that $|T(X)| = |\mathcal{P}(X)|$ for any topological space X, so (b) $\Leftrightarrow$ (c) is also clear. $\square$

Example 3.13. Let $\mathbb{Q}^*$ be the one point compactification of the $\mathbb{Q}$, ω_1 be the smallest uncountable ordinal and $W^* = \{\alpha : \alpha \text{ is an ordinal and } \alpha \leqslant \omega_1\}$. One can easily see that $|T(\mathbb{Q}^*)| = c = |T(W^*)|$ and by Proposition 3.12, $T(\mathbb{Q}^*) \simeq T(W^*)$ while $|\mathbb{Q}^*| \neq |W^*|$. Now, suppose $|A| = c$, $X = \cup_{\alpha \in A} X_\alpha$ is the disjoint union of copies of $\mathbb{Q}$, and X^* is the one point compactification of X. Then clearly $|X^*| = |W^*|$ and one can similarly show that $T(X^*) \simeq T(W^*)$ while $X^* \not\simeq W^*$. These examples show that $T(X)$ and $T(Y)$ can be isomorphic but not strongly isomorphic (even if $|X| = |Y|$). Note that if $T(X)$ and $T(Y)$ are strongly isomorphic, then it may there exists an isomorphism between them which is not strong; it is sufficient to define an isomorphism such that sends $f \neq -1$ to -1

Definition 3.14. *We say that a subgroup H of $T(X)$ is saturated if $f \in H$ and $e(f) \subseteq e(g)$ imply that $g \in H$.*

Lemma 3.15. *Let $H \leqslant T(X)$ be saturated. Then $e(H)$ is closed under finite intersection.*

Proof. Let $f, g \in H$; it is enough to show that $e(f) \cap e(g) \in e(H)$. For simplicity we let $A = e(f)$ and $B = e(g)$, then by assumption

$$D = e(fg) \cup (A \setminus B) = (B \setminus A)^c \in e(H).$$

Therefore, $\lambda_D \in H$ and hence $g\lambda_D \in H$. Thus, $A \cap B = e(g\lambda_D) \in e(H)$. $\square$

Corollary 3.16. *Let $H \leqslant T(X)$, then $e(H)$ is a $\mathcal{P}$-filter if and only if H is saturated and $-1 \notin H$.*

Corollary 3.17. *A subgroup H of $T(X)$ is saturated if and only if $h \in H$ and $1+h+f \in T(X)$ imply $-f \in H$.*

Proof. $\Rightarrow$) Suppose $h \in H$ and $1+h+f \in T(X)$. It is clear that $e(h) \cap e(f) = \emptyset$ and hence $e(h) \subseteq e(-f)$. Thus $-f \in H$.
$\Leftarrow$) Let $h \in H$ and $e(h) \subseteq e(f)$, thus $e(h) \cap e(-f) = \emptyset$. It is easy to see that $1+h-f \in T(X)$ and hence $f \in H$. $\square$

Corollary 3.18. *$e(H)$ is a $\mathcal{P}$-filter if and only if $-1 \notin H$ and if $h \in H$ and $1+h+f \in T(X)$, then $-f \in H$.*

Proof. By Corollaries 3.16 and 3.17, it is clear. $\square$

Definition 3.19. *Let X be a topological space. We will denote by $RT(X)$ the ring generated by $T(X)$.*

Proposition 3.20. *If*

$$R = \{f \in C(X),\ f(X) \text{ is finite, and } f(X) \subseteq 2\mathbb{Z} \text{ or } f(X) \subseteq 2\mathbb{Z}+1\},$$

then $RT(X) = R$.

Proof. One can easily see that R is indeed a ring that contains $T(X)$ and consequently $RT(X) \subseteq R$. Now, let $f \in R$ i.e., $f \in C(X)$ and $f(X) = \{m_1, \cdots, m_k\}$ is a subset of $2\mathbb{Z}$ or $2\mathbb{Z}+1$. We have to show $f \in RT(X)$. It is clear that $A_i = f^{-1}\{m_i\}$ is a clopen subset of X. Thus it is enough to prove that the equation $f = x_1 + \sum_{i=2}^{k} x_i \lambda_{A_i}$ has a solution for $x_1, \cdots, x_k$ in $\mathbb{Z}$. If we take $a_i \in A_i$ $(i = 1, \cdots, k)$, then we get the following equations

$$\begin{cases} x_1 - x_2 - x_3 - \cdots - x_k = m_1 \\ x_1 + x_2 - x_3 - \cdots - x_k = m_2 \\ \qquad\qquad \vdots \\ x_1 - x_2 - x_3 - \cdots + x_k = m_k \end{cases}$$

and we simply obtain the equivalent system of equations below.

$$\begin{cases} x_1 - x_2 - x_3 - \cdots - x_k = m_1 \\ \qquad x_2 = \frac{m_2 - m_1}{2} \\ \qquad\qquad \vdots \\ \qquad x_k = \frac{m_k - m_1}{2} \end{cases}$$

Now, since all the elements of the set $\{m_1, \cdots, m_k\}$ are even or all are odd, the above system has a solution in $\mathbb{Z}$. $\square$

In [5] and [6] it is defined that $C_C(X) = \{f \in C(X) : f(X) \text{ is countable}\}$ and $C^F(X) = \{f \in C(X) : |f(X)| < \infty\}$. Clearly the ring R in Proposition 3.20, is a subring of $C^F(X)$. The following result is proved in [6].

Theorem 3.21. *Let X be a topological space. Then there exists a zero dimensional space Y such that $C_C(X) \simeq C_C(Y)$ ($C^F(X) \simeq C^F(Y)$).*

The above theorem shows that, without loss of generality one may assume that X is a zero dimensional space. Therefore, considering this comment, the following fact which is our main result is in order.

Theorem 3.22. *Let X and Y be compact zero dimensional spaces. Then $RT(X) \simeq RT(Y)$ if and only if $X \simeq Y$.*

Proof. $\Rightarrow$) Suppose $\varphi : RT(X) \longrightarrow RT(Y)$ is an isomorphism. It is sufficient to show that $\varphi(T(X)) = T(Y)$ and φ maps the set $\mathcal{M}^*(X)$ onto $\mathcal{M}^*(Y)$. It is clear that $\varphi(T(X)) \subseteq T(Y)$. Now, let $g \in T(Y)$, then $f \in RT(X)$ exists such that $\varphi(f) = g$. Therefore, since φ is one-one, we can write $\varphi(f^2) = (\varphi(f))^2 = g^2 = 1$ which implies $f^2 = 1$ and therefore $f \in T(X)$. Now, we have to prove that $\varphi(\mathcal{M}^*(X)) = \mathcal{M}^*(Y)$. To this end it is sufficient to show that $e(H^p(X))$ is a $\mathcal{P}$-filter if and only if $e(\varphi(H^p(X)))$ is such. It is clear that

$$-1 \notin H^p \quad \Leftrightarrow \quad -1 = \varphi(-1) \notin \varphi(H^p).$$

On the other hand

$$f \in H^p, \quad 1 + f + g \in T(X) \quad \Leftrightarrow$$

$$\varphi(f) \in \varphi(H^p), \quad \varphi(1 + f + g) = 1 + \varphi(f) + \varphi(g) \in T(Y).$$

It is also clear that $-g \in H^p$ if and only if $-\varphi(g) \in \varphi(H^p)$. Hence, by Corollary 3.18, we are through.
$\Leftarrow$) It is obvious. $\square$

Corollary 3.23. *$T(X)$ and $T(Y)$ are strongly isomorphic if and only if $RT(X) \simeq RT(Y)$.*

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addision-wesley, Reading Mass, 1969.

[2] F. Azarpanah, Essential ideals in $C(X)$, *Period. Math. Hungar.,* 12 (1995), 105-112.

[3] F. Azarpanah and M. Motamedi, Zero divisor graph of $C(X)$, *Acta Math. Hungar.,* 108 (1-2) (2005), 25-36.

[4] R. Engelking, *General Topology*, PWN-Polishc Scientific Publishing, 1977.

[5] M. Ghademazi, *On Scattered Spaces via* $C(X)$, Ph.D. Thesis, 2009.

[6] M. Ghademazi, O. R. S. Karamzadeh, and M. Namdari, On functionally countable subalgebra of $C(X)$, *Rend. Sem. Mat. Univ. Padova,* 129 (2013), 47-69.

[7] L. Fuchs, *Infinite Abelian Groups*, Acadecic Press N. Y., 1970.

[8] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van. Nostrand Reinhold, New York, 1960.

[9] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer Verlag, 1982.

[10] R. Raphael, Rings which are generated by their units, *Journal of Algebra,* 28 (1994), 199-205.

[11] S. Willard, *General Topology*, Addison Wesly, Reading, Mass., 1970.

Ali Rezaei Aliabad
Department of Mathematics
Associate Professor of Mathematics
Chamran University
Ahvaz, Iran
E-mail: aliabady_r@scu.ac.ir

Mansoor Motamedi
Department of Mathematics
Professor of Mathematics
Chamran University
Ahvaz, Iran
E-mail: motamedi_m@scu.ac.ir