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Wolfe Type Duality for Nonsmooth Optimization Problems with Vanishing Constraints

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Abstract. In this paper, we formulate and study the duality problems in Wolfe type for optimization problems with vanishing constraints in nonsmooth case, whereas Mishra *et al.* (Ann Oper Res 243(1):249–272, 2016) investigated them in smooth case. Also, we derive the weak, strong and strict converse duality results for the problems with Lipschitzian functions utilizing the Clarke subdifferential.

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1 Introduction

The different types dual problems attributed to an optimization problem and the existing relationships between their solutions are one of the most important topics that are studied in optimization theory, which leads to finding algorithms and methods for solving the optimization problem (see [3]). In general, for a given function $f : \mathbb{R}^n \to \mathbb{R}$ and a set $D \subseteq \mathbb{R}^n$, the dual problem of the minimization problem $\min_{x\in D} f(x)$ is in the form of the maximization problem $\max_{z\in E} \varphi(z)$, where the set $E \subseteq \mathbb{R}^m$ and the function $\varphi : \mathbb{R}^m \to \mathbb{R}$ are defined in terms of f and D.

The purpose of the duality theory is that if $\hat{x}$ and $\hat{z}$ are optimal solutions of the the primal and dual problems, respectively, first it finds the relationship between $f(\hat{x})$ and $\varphi(\hat{z})$, introduces conditions that are equal, and then finds the relationship between $\hat{x}$ and $\hat{z}$; see, e.g., [5].

As an important case, for the classical optimization problem (OP, in brief), the feasible set D is defined as

$$D := \{x \in \mathbb{R}^n \mid \; g_j(x) \leq 0, \; j = 1, \ldots, r\},$$

where the functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are given. The most famous duals for OP are in the Lagrange type, the Fenchel type, the Wolfe type, and the Mond-Weir type. To observe the duality theory for OP, we can refer to [2, 12, 13, 14, 17] and their references.

A difficult category of optimization problems introduced by Kansow in [1, 6] is mathematical programs with vanishing constraints (MPVC). The general form of this problem is as follows:

$$(P): \qquad \min f(x) \quad s.t. \quad x \in S, \tag{1}$$

in which

$$S := \{x \in \mathbb{R}^n | \; H_i(x) \geq 0, \; G_i(x)H_i(x) \leq 0, \; i = 1, \ldots, m\}, \tag{2}$$

where, for each $i \in \{1, \ldots, m\}$, the functions f, H_i and G_i are defined from $\mathbb{R}^n$ to $\mathbb{R}$. The optimality conditions for MPVCs, as a category of problems containing multiplicative constraints, have been considered by many researchers; for example, one can refer to [7, 8, 9] for the continuously differentiable case, to [10, 11, 16, 18] for the locally Lipschitz case, and to [19, 20] for the convex case.

The Wolfe type dual problem to the MPVCs with continuously data were considered by Mishra *et al* in [15]. The purpose of this paper is to investigate the Wolfe type dual in the case of that the functions f, G_i and H_i are locally Lipschitz and non-differential. It seems natural that a suitable subdifferential is used instead of a differential in this process, which due to condition of being locally Lipschitz for functions, the Clarke subdifferential seems more suitable than the other subdifferentials.

The remaining sections of this article are arranged as follows; in Section 2, we will introduce definitions and theorems of nonsmooth analysis that will be used in the preparation of the main provisions. The main results of the paper are presented in Section 3. As we shall see, we consider two kinds of Wolf type dual problems for (P) in nonsmooth cases.

2 Preliminaries

This section contains some material on nonsmooth analysis widely used in what follows. We refer the reader to the books by Clarke [4] for details and examples.

Definition 2.1. Let $B \subseteq \mathbb{R}^n$. The polar cone of B is defined by

$$B^- := \{x \in \mathbb{R}^n | \, \langle x, d\rangle \leq 0, \quad \forall d \in B\}.$$

Definition 2.2. The Bouligand tangent cone (cotingent cone) $\Gamma(A, \hat{x})$ of $\emptyset \neq A \subseteq \mathbb{R}^n$ at $\hat{x} \in A$ is defined by

$$\Gamma(A, \hat{x}) := \{v \in \mathbb{R}^n \mid \exists t_r \downarrow 0, \, \exists v_r \to v \textit{ such that } \hat{x} + t_r v_r \in A, \, \forall r \in \mathbb{N}\}.$$

Definition 2.3. The function $f : \mathbb{R}^n \to \mathbb{R}$ is called locally Lipschitz if for every $z \in \mathbb{R}^n$ there exist a neighbourhood U of z and a positive constant L_U such that

$$|f(x) - f(y)| \leq L_U \, \|x - y\|, \qquad \forall x, y \in U.$$

Definition 2.4. For a given locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, the generalized Clarke directional derivative and the Clarke subdifferential of f at $\hat{x} \in \mathbb{R}^n$ are defined as

$$f^0(\hat{x}; d) := \limsup_{y \to \hat{x}, \; t \downarrow 0} \frac{f(y + td) - f(y)}{t},$$

$$\partial_c f(\hat{x}) := \left\{ \zeta \in \mathbb{R}^n \mid f^0(\hat{x}; d) \geq \langle \zeta, d \rangle, \quad \forall d \in \mathbb{R}^n \right\}.$$

Theorem 2.5. *Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions. Then, the following relations hold:*

$$\partial_c(f+g)(x) \subseteq \partial_c f(x) + \partial_c g(x), \quad \forall x \in \mathbb{R}^n,$$

$$\partial_c(\alpha f)(x) = \alpha \partial_c f(x), \quad \forall \alpha \in \mathbb{R}, \ \forall x \in \mathbb{R}^n.$$

Definition 2.6. [12] For a given function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be

(i): η-invex at $\hat{x} \in \mathbb{R}^n$ if for each $x \in \mathbb{R}^n$ one has:

$$f(x) - f(\hat{x}) \geq \langle \zeta, \eta(x, \hat{x}) \rangle, \quad \forall \zeta \in \partial_c f(\hat{x}).$$

(ii): strict η-invex at $\hat{x} \in \mathbb{R}^n$ if for each $x \in \mathbb{R}^n$ one has:

$$f(x) - f(\hat{x}) > \langle \zeta, \eta(x, \hat{x}) \rangle, \quad \forall \zeta \in \partial_c f(\hat{x}).$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is η-invex or strict η-invex at $\hat{x}$, then η is named the kernel of f at $\hat{x}$.

3 Main Results

In this section, we consider the problem (P), defined in (1), with the feasible set $S \neq \emptyset$, defined in (2), in which the emerging functions f, H_i, and G_i are locally Lipschitz for each $i = 1, \ldots, m$. At starting point, for each feasible point $x \in S$, we define the following index sets:

$$\begin{aligned}
&I_{+0}(x) := \{i \in \{1, \ldots, m\} : H_i(x) > 0, \ G_i(x) = 0\}, \\
&I_{+-}(x) := \{i \in \{1, \ldots, m\} : H_i(x) > 0, \ G_i(x) < 0\}, \\
&I_{0+}(x) := \{i \in \{1, \ldots, m\} : H_i(x) = 0, \ G_i(x) > 0\}, \\
&I_{0-}(x) := \{i \in \{1, \ldots, m\} : H_i(x) = 0, \ G_i(x) < 0\}, \\
&I_{00}(x) := \{i \in \{1, \ldots, m\} : H_i(x) = 0, \ G_i(x) = 0\}, \\
&I_{+}(x) := I_{+0}(x) \cup I_{+-}(x), \\
&I_{0}(x) := I_{0+}(x) \cup I_{0-}(x) \cup I_{00}(x).
\end{aligned}$$

It is easy to see that for each $x \in S$ we have $\{1, \ldots, m\} = I_+(x) \cup I_0(x)$.

As mentioned in Introduction, for each feasible point $x \in S$, we consider the following two kinds of dual problems (WD_1^x) and (WD_2^x) in Wolfe type for (P):

$$(WD_1^x): \qquad \max f(y) + \sum_{i=1}^m \Big(-\alpha_i H_i(y) + \beta_i G_i(y)\Big),$$
$$s.t. \begin{cases} 0 \in \partial_c \Big(f(\cdot) + \sum_{i=1}^m \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot)\Big)\Big)(y), \\ \alpha_i \geq 0, \quad i \in I_+(x), \\ \beta_i \leq 0, \quad i \in I_{0+}(x), \\ \beta_i \geq 0, \quad i \in I_{0-}(x) \cup I_{+-}(x), \end{cases}$$

$$(WD_2^x): \qquad \max f(y) + \sum_{i=1}^m \Big(-\alpha_i H_i(y) + \beta_i G_i(y)\Big),$$
$$s.t. \begin{cases} 0 \in \partial_c f(y) - \sum_{i=1}^m \alpha_i \partial_c H_i(y) + \sum_{i=1}^m \beta_i \partial_c G_i(y), \\ \alpha_i \geq 0, \quad i \in I_+(x), \\ \beta_i \leq 0, \quad i \in I_{0+}(x), \\ \beta_i \geq 0, \quad i \in I_{0-}(x) \cup I_{+-}(x). \end{cases}$$

The feasible sets of (WD_1^x) and (WD_2^x) are respectively denoted by S_1^x and S_2^x. In this paper we take $\alpha := (\alpha_1, \ldots, \alpha_m)$ and $\beta := (\beta_1, \ldots, \beta_m)$.

Remark 3.1. Taking into account the well-known inclusion

$$\partial_c \Big(f(\cdot) + \sum_{i=1}^m \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot)\Big)\Big)(y) \subseteq \partial_c f(y)$$
$$- \sum_{i=1}^m \alpha_i \partial_c H_i(y) + \sum_{i=1}^m \beta_i \partial_c G_i(y),$$

we obtain that $S_1^x \subseteq S_2^x$, and equality holds if the functions f, G_i, and $-H_i$ are convex or differentiable at y for all $i \in \{1, \ldots, m\}$ (see, e.g. [4]). Thus, (WD_1^x) and (WD_2^x) are both generalizations of the dual problem VC-WD(x) that is defined in [15].

We can state the weak duality result for (WD_1^x) as follows.

Theorem 3.2. ***(First Weak Duality)*** *Suppose that the feasible points $x \in S$ and $(y, \alpha, \beta) \in S_1^x$ are given for (P) and (WD_1^x), respectively. If $f(\cdot) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot)\Big)$ is η-invex at y, then,*

$$f(x) \geq f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y)\Big).$$

Proof. Assume on the contrary that

$$f(x) < f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y)\Big). \tag{3}$$

Owing to $x \in S$ and $(y, \alpha, \beta) \in S_1^x$, we can obtain that,

$$\begin{cases} -H_i(x) < 0, & \alpha_i \geq 0, \quad \forall i \in I_+(x), \\ -H_i(x) = 0, & \alpha_i \in \mathbb{R}, \quad \forall i \in I_0(x), \\ G_i(x) > 0, & \beta_i \leq 0, \quad \forall i \in I_{0+}(x), \\ G_i(x) = 0, & \beta_i \in \mathbb{R}, \quad \forall i \in I_{00}(x) \cup I_{+0}(x), \\ G_i(x) < 0, & \beta_i \geq 0, \quad \forall i \in I_{0-}(x) \cup I_{+-}(x), \end{cases}$$

and hence,

$$\sum_{i=1}^{m} \Big(-\alpha_i H_i(x) + \beta_i G_i(x)\Big) \leq 0. \tag{4}$$

Adding both sides of (3) and (4), we conclude that:

$$f(x) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(x) + \beta_i G_i(x)\Big) < f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y)\Big).$$

By the above inequality and the η-invexity of $f(\cdot) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot)\Big)$ at y, for each $\zeta \in \partial_c\Bigg(f(\cdot) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot)\Big)\Bigg)(y)$ one has:

$$\langle \zeta, \eta(x, y) \rangle \leq \Bigg(f(x) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(x) + \beta_i G_i(x)\Big)\Bigg) -$$

$$\left(f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y) \Big) \right) < 0. \tag{5}$$

Now, since $0 \in \partial_c \Bigg(f(\cdot) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot) \Big) \Bigg)(y)$ by feasibility of (y, α, β), owing to (5), we get,

$$0 = \langle 0, \eta(x, y) \rangle < 0.$$

This contradiction proves the result. □

In order to analyze the (WD_2^x), we will need the following index sets in the future. Giving $(y, \alpha, \beta) \in S_2^x$, put:

$$\begin{aligned}
I_+^+(x) &:= \{i \in I_+(x) : \alpha_i > 0\}, \\
I_0^+(x) &:= \{i \in I_0(x) : \alpha_i > 0\}, \\
I_0^-(x) &:= \{i \in I_0(x) : \alpha_i < 0\}, \\
I_{0+}^-(x) &:= \{i \in I_{0+}(x) : \beta_i < 0\}, \\
I_{00}^-(x) &:= \{i \in I_{00}(x) : \beta_i < 0\}, \\
I_{+0}^-(x) &:= \{i \in I_{+0}(x) : \beta_i < 0\}, \\
I_{00}^+(x) &:= \{i \in I_{00}(x) : \beta_i > 0\}, \\
I_{+0}^+(x) &:= \{i \in I_{+0}(x) : \beta_i > 0\}, \\
I_{0-}^+(x) &:= \{i \in I_{0-}(x) : \beta_i > 0\}, \\
I_{+-}^+(x) &:= \{i \in I_{+-}(x) : \beta_i > 0\}.
\end{aligned}$$

For simplicity, consider the following set, which includes the objective function f and a number of constraint functions:

$$\begin{aligned}
\Omega(x) \quad := \quad & \Big\{ f, \quad H_i \ for \ i \in I_0^-(x), \quad G_i \ for \ i \in I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{+0}^+(x) \cup I_{+-}^+(x), \\
R \qquad & -H_i \ for \ i \in I_+^+(x) \cup I_0^+(x), \quad -G_i \ for \ i \in I_{0+}^-(x) \cup I_{00}^-(x) \cup I_{+0}^-(x) \Big\}.
\end{aligned}$$

Following theorem presents the weak duality result for (WD_2^x).

Theorem 3.3. ***(Second Weak Duality)*** *Suppose that $x \in S$ and $(y, \alpha, \beta) \in S_2^x$ are feasible points for the problems (P) and (WD_2^x), respectively. If all the functions in $\Omega(x)$ are η-invex at y by a common* *kernel η, then*

$$f(x) \geq f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y)\Big).$$

Proof. According to $0 \in \partial_c f(y) - \sum_{i=1}^{m} \alpha_i \partial_c H_i(y) + \sum_{i=1}^{m} \beta_i \partial_c G_i(y)$, we can find some $\zeta^f \in \partial_c f(y)$, $\zeta_i^H \in \partial_c H_i(y)$, and $\zeta_i^G \in \partial_c G_i(y)$, for $i = 1, \ldots, m$, satisfying

$$\zeta^f - \sum_{i=1}^{m} \alpha_i \zeta_i^H + \sum_{i=1}^{m} \beta_i \zeta_i^G = 0. \tag{6}$$

Let $i \in I_+(x)$. If $i \in I_+^+(x)$, the η-invexity of $-H_i$, the inequality $\alpha_i > 0$, and the fact that $-\zeta_i^H \in -\partial_c H_i(y) = \partial_c\big(-H_i\big)(y)$ imply

$$-\alpha_i H_i(x) - \big(-\alpha_i H_i(y)\big) \geq \big\langle -\alpha_i \zeta_i^H, \eta(x,y)\big\rangle. \tag{7}$$

If $i \in I_+ \setminus I_+^+(x)$, by $(y, \alpha, \beta) \in S_2^x$ and definition of $I_+^+(x)$ we get $\alpha_i = 0$, and so, (7) is automatically correct. Thus, we have:

$$\sum_{i \in I_+} -\alpha_i H_i(x) - \sum_{i \in I_+} -\alpha_i H_i(y) \geq \Big\langle \sum_{i \in I_+} -\alpha_i \zeta_i^H, \eta(x,y)\Big\rangle.$$

Similarly, by the η-invexity of f, $-H_i$ for $i \in I_+^+(x) \cup I_0^+(x)$, H_i for $i \in I_0^-(x)$, $-G_i$ for $i \in I_{0+}^-(x) \cup I_{00}^-(x) \cup I_{+0}^-(x)$, and G_i for $i \in I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{+0}^+(x) \cup I_{+-}^+(x)$ at y, we conclude that:

$$\begin{cases} f(x) - f(y) \geq \big\langle \zeta^f, \eta(x,y)\big\rangle, \\ -\sum_{i=1}^{m} \alpha_i H_i(x) + \sum_{i=1}^{m} \alpha_i H_i(y) \geq \Big\langle \sum_{i=1}^{m} -\alpha_i \zeta_i^H, \eta(x,y)\Big\rangle, \\ \sum_{i=1}^{m} \beta_i G_i(x) - \sum_{i=1}^{m} \beta_i G_i(y) \geq \Big\langle \sum_{i=1}^{m} \beta_i \zeta_i^G, \eta(x,y)\Big\rangle. \end{cases} \tag{8}$$

Adding the above three inequalities, and owing to (6), we obtain that

$$f(x) - f(y) - \sum_{i=1}^{m} \alpha_i H_i(x) + \sum_{i=1}^{m} \alpha_i H_i(y) + \sum_{i=1}^{m} \beta_i G_i(x) - \sum_{i=1}^{m} \beta_i G_i(y)$$
$$\geq \Big\langle \zeta^f - \sum_{i=1}^{m} \alpha_i \zeta_i^H + \sum_{i=1}^{m} \beta_i \zeta_i^G, \eta(x,y) \Big\rangle = 0.$$

Therefore,

$$f(y)+\sum_{i=1}^{m} \Big(-\alpha_i H_i(y)+\beta_i G_i(y)\Big) \leq f(x)+\sum_{i=1}^{m} \Big(-\alpha_i H_i(x)+\beta_i G_i(x)\Big) \leq f(x),$$

where the last inequality satisfies by (4). □

To compare two weak duality Theorems 3.2 and 3.3, we must note that condition

$$0 \in \partial_c \Bigg(f(\cdot) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot)\Big)\Bigg)(\hat{x}),$$

in Theorem 3.2 is stronger than condition

$$0 \in \partial_c f(\hat{x}) + \sum_{i=1}^{m} (-\alpha_i H_i(\hat{x}) + \beta_i G_i(\hat{x})),$$

in Theorem 3.3, but, the condition of η-invexity for functions within $\Omega(\hat{x})$ in Theorem 3.3 is stronger than the condition of η-invexity for

$$f(\cdot) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot)\Big).$$

So, none of these two theorems is stronger than another one, and each of them is used in cases where the other is not used.

It should be noted that Theorems 3.2 and 3.3 are generalizations of [15, Theorem 3] to nonsmooth MPVCs. Moreover, their results state that:

$$\min_{x\in S} f(x) \geq \max_{(y,\alpha,\beta)\in S_1^x} f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y)\Big), \quad (9)$$

$$\min_{x\in S} f(x) \geq \max_{(y,\alpha,\beta)\in S_2^x} f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y)\Big). \quad (10)$$

We say that the strong duality holds if the inequality increases to equality in (9) and\or (10). As we know from the classical optimization theory, the Karush-Kahn-Tucker (KKT) necessary condition is required for satisfying the strong duality equalities. Since all presented KKT conditions for MPVCs are in the form of

$$0 \in \partial_c f(\hat{x}) + \sum_{i=1}^{m} \Big(-\alpha_i \partial_c H_i(\hat{x}) + \beta_i \partial_c G_i(\hat{x})\Big),$$

in the following we will prove the strong duality theorem only for (WD_2^x). It is clear that to prove the strong duality theorem for (WD_1^x), we need to prove some new forms of necessary KKT conditions as

$$0 \in \partial_c \Bigg(f(\cdot) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(\cdot) + \beta_i G_i(\cdot)\Big)\Bigg)(\hat{x}),$$

which are more precise than the existed ones, and require a separate research.

For stating the strong duality result for (WD_2^x), the following definition and theorem are required from [16].

Definition 3.4. We say that (P) satisfies the "generalized VC_4-Abadie constraint qualification" ($GVC_4 - ACQ$, in short) at $\hat{x} \in S$, if $\mathcal{A}_4{}^{-} \subseteq \Gamma(S, \hat{x})$ and $cone\big(\mathcal{A}_4\big)$ is a closed subset of $\mathbb{R}^n$, where $\mathcal{A}_4$ is defined as:

$$\mathcal{A}_4 := \Big(\bigcup_{i \in I_0} \partial_c H_i(\hat{x})\Big) \cup \Big(\bigcup_{i \in I_{0+}} -\partial_c H_i(\hat{x})\Big) \cup \Big(\bigcup_{i \in I_{+0} \cup I_{00}} \partial_c G_i(\hat{x})\Big),$$

and $cone\big(\mathcal{A}_4\big)$ denotes the convex cone of $\mathcal{A}_4$.

Theorem 3.5. *[16, Theorem 4(iii)] Suppose that $\hat{x}$ is a local solution of (P) and $GVC_4 - ACQ$ holds at $\hat{x}$. Then, there exist real coefficients α_i and β_i (for each $i \in \{1, \ldots, m\}$) such that*

$$\begin{cases} 0 \in \partial_c f(\hat{x}) + \sum\limits_{i=1}^{m} \Big(- \alpha_i \partial_c H_i(\hat{x}) + \beta_i \partial_c G_i(\hat{x})\Big), \\ \alpha_i = 0 \quad \textit{for } i \in I_{+}; \quad \alpha_i \textit{ free for } i \in I_{0+}; \quad \alpha_i \geq 0 \quad \textit{for } i \in I_{0-} \cup I_{00}, \\ \beta_i = 0 \quad \textit{for } i \in I_{+-} \cup I_{0+} \cup I_{0-}; \quad \beta_i \geq 0 \quad \textit{for } i \in I_{+0} \cup I_{00}. \end{cases} \tag{11}$$

For study the properties of $GVC_4 - ACQ$ and its relationship with other constraint qualifications, we can refer to [16].

Theorem 3.6. ***(Strong Duality)*** *Assume that $\hat{x} \in S$ is a local solution for (P) and $GVC_4 - ACQ$ holds at $\hat{x}$. Then, there exist some vectors $\hat{\alpha} \in \mathbb{R}^m$ and $\hat{\beta} \in \mathbb{R}^m$ such that $(\hat{x}, \hat{\alpha}, \hat{\beta}) \in S_2^{\hat{x}}$.*

Furthermore, if all the functions in $\Omega(\hat{x})$ are η-invex by a common kernel η, then $(\hat{x}, \hat{\alpha}, \hat{\beta})$ is a global solution of the problem (WD_2^x), and

$$f(\hat{x}) = f(\hat{x}) + \sum_{i=1}^{m} \Big(-\hat{\alpha}_i H_i(\hat{x}) + \hat{\beta}_i G_i(\hat{x}) \Big).$$

Proof. Employing Theorem 3.5, there exist some multipliers $\hat{\alpha}_i$ and $\hat{\beta}_i$, for $i \in \{1, \ldots, m\}$, such that (11) holds. Clearly, $(\hat{x}, \hat{\alpha}, \hat{\beta})$ satisfies in constraints of problem (WD_2^x), i.e., $(\hat{x}, \hat{\alpha}, \hat{\beta}) \in S_2^{\hat{x}}$. Suppose that an index $\hat{i} \in \{1, \ldots, m\}$ is given. If $\hat{i} \in I_+$, then $\hat{\alpha}_{\hat{i}} H_{\hat{i}}(\hat{x}) = 0$ by (11); and if $\hat{i} \in I_0$, then $\hat{\alpha}_{\hat{i}} H_{\hat{i}}(\hat{x}) = 0$ by the definition of index set I_0. Similarly, we can see $\hat{\beta}_{\hat{i}} G_{\hat{i}}(\hat{x}) = 0$, and so

$$\sum_{i=1}^{m} \Big(-\hat{\alpha}_i H_i(\hat{x}) + \hat{\beta}_i G_i(\hat{x}) \Big) = 0. \tag{12}$$

Now, if $(y, \alpha, \beta) \in S_2^{\hat{x}}$ is given, by the second weak duality Theorem 3.3 we have

$$f(\hat{x}) \geq f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y) \Big).$$

The above inequality and (12) imply that

$$f(\hat{x}) + \sum_{i=1}^{m} \Big(-\hat{\alpha}_i H_i(\hat{x}) + \hat{\beta}_i G_i(\hat{x}) \Big) \geq f(y) + \sum_{i=1}^{m} \Big(-\alpha_i H_i(y) + \beta_i G_i(y) \Big),$$

for all $(y, \alpha, \beta) \in S_2^{\hat{x}}$, that is, $(\hat{x}, \hat{\alpha}, \hat{\beta})$ is a global maximum of the (WD_2^x), as required. Finally, (12) concludes

$$f(\hat{x}) = f(\hat{x}) + \sum_{i=1}^{m} \Big(-\hat{\alpha}_i H_i(\hat{x}) + \hat{\beta}_i G_i(\hat{x}) \Big),$$

and the proof is completed. $\square$

Remark 3.7. The following points are noteworthy about strong duality Theorem 3.6.

i) We can replace $GVC_4 - ACQ$ with each stronger constraint qualifications; see a complete list of the various constraint qualifications for nonsmooth MPVCs in [10, 11, 16, 18]. It should be noted that, as shown in [10], $GVC_4 - ACQ$ cannot be replaced by some weaker constraint qualifcations, which are called Guignard-type constraint qualifications.

ii) Since the coefficients of functions H_i for $i \in I_+$ and G_i for $i \in I_{+-} \cup I_{0+} \cup I_{0-}$ are zero in (11), the assumption of η-invexity of the functions within $\Omega(\hat{x})$ can be changed to the weaker assumption of η-invexity of functions within $\Omega_1(\hat{x})$, in which

$$\Omega_1(\hat{x}) = \Omega(\hat{x}) \setminus \left\{ -H_i \left(i \in I_+^+(\hat{x}) \right), G_i \left(i \in I_{0-}^+(\hat{x}) \cup I_{+-}^+(\hat{x}) \right) \right\}.$$

iii) As seen in the proof of Theorem 3.6, $GVC_4 - ACQ$ leads to (11) which is stronger than the required inclusion $(\hat{x}, \hat{\alpha}, \hat{\beta}) \in S_2^{\hat{x}}$. So, it seems possible to define a weaker constraint qualification that leads to $(\hat{x}, \hat{\alpha}, \hat{\beta}) \in S_2^{\hat{x}}$ that is exactly equivalent to $(\hat{x}, \hat{\alpha}, \hat{\beta}) \in S_2^{\hat{x}}$. Of course, fiding such a constraint qualification can be proposed as topic of a independent and difficult research for interested researchers.

Theorem 3.8. ***(Strict Converse Duality)*** *Suppose that $\hat{x} \in S$ is a local minimum for the (P), and $GVC_4 - ACQ$ holds at $\hat{x}$. Furthermore, assume that $(\hat{y}, \bar{\alpha}, \bar{\beta}) \in S_2^{\hat{x}}$ is a global maximum for the $(WD_2^{\hat{x}})$, all the functions in $\Omega(\hat{x}) \setminus \{f\}$ are η-invex at $\hat{y}$, and f is strict η-invex, with a common kernel η, then*

$$\hat{x} = \hat{y}.$$

Proof. Considering the assumption that $(\hat{y}, \bar{\alpha}, \bar{\beta}) \in S_2^{\hat{x}}$, we can find some $\zeta^f \in \partial_c f(\hat{y})$, $\zeta_i^H \in \partial_c H_i(\hat{y})$, and $\zeta_i^G \in \partial_c G_i(\hat{y})$ such that

$$\zeta^f + \sum_{i=1}^{m} -\bar{\alpha}_i {\zeta_i}^H + \sum_{i=1}^{m} \bar{\beta}_i {\zeta_i}^G = 0. \tag{13}$$

On the contrary, suppose that $\hat{x} \neq \hat{y}$. According to the strict η-invexity of f at $\hat{y}$, we have

$$f(\hat{x}) - f(\hat{y}) > \left\langle \zeta^f, \eta(\hat{x}, \hat{y}) \right\rangle. \tag{14}$$

By the η-invexity of functions in $\Omega(\hat{x})\backslash\{f\}$ at $\hat{y}$, following (8), we obtain that

$$\begin{cases} -\sum_{i=1}^{m} \bar{\alpha}_i H_i(\hat{x}) + \sum_{i=1}^{m} \bar{\alpha}_i H_i(\hat{y}) \geq \Big\langle \sum_{i=1}^{m} -\bar{\alpha}_i \zeta_i^H, \eta(\hat{x}, \hat{y}) \Big\rangle, \\ \\ \sum_{i=1}^{m} \bar{\beta}_i G_i(\hat{x}) - \sum_{i=1}^{m} \bar{\beta}_i G_i(\hat{y}) \geq \Big\langle \sum_{i=1}^{m} \bar{\beta}_i \zeta_i^G, \eta(\hat{x}, \hat{y}) \Big\rangle. \end{cases}$$

Adding the both sides of (14) and above two inequalities, and considering (13), we conclude that

$$\begin{aligned} & f(\hat{x}) + \sum_{i=1}^{m} -\bar{\alpha}_i H_i(\hat{x}) + \sum_{i=1}^{m} \bar{\beta}_i G_i(\hat{x}) > \\ & f(\hat{y}) + \sum_{i=1}^{m} -\bar{\alpha}_i H_i(\hat{y}) + \sum_{i=1}^{m} \bar{\beta}_i G_i(\hat{y}) \\ & + \left\langle \overbrace{\zeta^f + \sum_{i=1}^{m} -\bar{\alpha}_i {\zeta_i}^H + \sum_{i=1}^{m} \bar{\beta}_i {\zeta_i}^G}^{=0}, \eta(\hat{x}, \hat{y}) \right\rangle. \end{aligned} \tag{15}$$

Owing to the $\hat{x} \in S$ and $(\hat{y}, \bar{\alpha}, \bar{\beta}) \in S_2^{\hat{x}}$, following (4), we can obtain that,

$$0 \geq \sum_{i=1}^{m} -\bar{\alpha}_i H_i(\hat{x}) + \sum_{i=1}^{m} \bar{\beta}_i G_i(\hat{x}).$$

Combining the last inequality and (15), we get

$$f(\hat{x}) > f(\hat{y}) + \sum_{i=1}^{m} -\bar{\alpha}_i H_i(\hat{y}) + \sum_{i=1}^{m} \bar{\beta}_i G_i(\hat{y}). \tag{16}$$

On the other hand, employing the strong duality Theorem 3.6, there exist some vectors $\tilde{\alpha} \in \mathbb{R}^m$ and $\tilde{\beta} \in \mathbb{R}^m$ such that $(\hat{x}, \tilde{\alpha}, \tilde{\beta}) \in S_2^{\hat{x}}$ is a global solution of the problem $(WD_2^{\hat{x}})$, and

$$f(\hat{x}) = f(\hat{x}) + \sum_{i=1}^{m} \Big(-\tilde{\alpha}_i H_i(\hat{x}) + \tilde{\beta}_i G_i(\hat{x})\Big).$$

The last relation and (16) imply that

$$f(\hat{x}) + \sum_{i=1}^{m} \Big(-\tilde{\alpha}_i H_i(\hat{x}) + \tilde{\beta}_i G_i(\hat{x})\Big) > f(\hat{y}) + \sum_{i=1}^{m} -\bar{\alpha}_i H_i(\hat{y}) + \sum_{i=1}^{m} \bar{\beta}_i G_i(\hat{y}),$$

which is a contradiction, since it states that the objective function of $(WD_2^{\hat{x}})$ has two different values at its two global solutions $(\hat{x}, \tilde{\alpha}, \tilde{\beta}) \in S_2^{\hat{x}}$ and $(\hat{y}, \bar{\alpha}, \bar{\beta}) \in S_2^{\hat{x}}$. The proof is complete. □

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