Journal of Mathematical Extension
Vol. 16, No. 7, (2022) (3)1-19
URL: https://doi.org/10.30495/JME.2022.2101
ISSN: 1735-8299
Original Research Paper

Monotonic Solutions of Second Order Nonlinear Difference Equations

L. Wang*
University of Central Missouri

A. C. Willett
University of Central Missouri

Abstract. Classification, boundedness, and existence of solutions of a second order nonlinear difference equation are investigated. First, it is proved that all solutions are eventually monotone. Then, the necessary and sufficient conditions for the boundedness of all solutions are established. Finally, the existence of different types of monotonic solutions are presented. The obtained results have extended and improved some existing ones.

AMS Subject Classification: 39A05; 39A22.
Keywords and Phrases: Nonlinear difference equations, classification, bounded solutions, monotonic solutions.

1 Introduction

The aim of the paper is to consider the second order nonlinear difference equation

$$\Delta(a_n f(\Delta x_n)) = b_n g(x_{n+1}), \ n \geq 1, \tag{1}$$

Received: July 2021; Accepted: October 2021
*Corresponding Author

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $\{a_n\}$ and $\{b_n\}$ are positive real sequences for $n \geq 1$, f, $g : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying $rf(r) > 0$ and $rg(r) > 0$ for $r \neq 0$, and f is strictly increasing on $\mathbb{R}$.

Some special cases of equation (1) are widely studied in the literature, for example, the discrete half-linear equation

$$\Delta(a_n|\Delta x_n|^{\alpha}\operatorname{sgn}\Delta x_n) + b_n|x_{n+1}|^{\alpha}\operatorname{sgn} x_{n+1} = 0,$$

the discrete Emden-Fowler equation

$$\Delta(a_n|\Delta x_n|^{\alpha}\operatorname{sgn}\Delta x_n) + b_n|x_{n+1}|^{\beta}\operatorname{sgn} x_{n+1} = 0,$$

and nonlinear difference equation with p-Laplacian

$$\Delta(a_n\Phi_p(\Delta x_n)) = b_n g(x_{n+1}), \tag{2}$$

where $\Phi_p(u) = |u|^{p-2}u$ with $p > 1$ is called p-Laplacian; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein for details.

The discussion of two-dimensional difference systems can be found in the literature as well; see [12, 13, 14] and other publications.

As usual, by a solution of (1), we mean a real sequence $\{x_n\}$ that satisfies (1) and is not trivial. If $\{x_n\}$ is a monotone sequence, we say that $\{x_n\}$ is a monotone solution. If $\{x_n\}$ is an eventually monotone sequence, we say that $\{x_n\}$ is an eventually monotone solution.

The following assumptions are imposed for later discussions:

(H1) There exists a real number $M > 0$ such that

$$|f^{-1}(uv)| \leq M|f^{-1}(u)||f^{-1}(v)|, \quad \forall u, v \in \mathbb{R}.$$

(H2) Function g is increasing on $\mathbb{R}$.

(H3) There exists a real number $r_0 > 0$ such that

$$\int_{\pm r_0}^{\pm\infty} \frac{dr}{f^{-1}(g(r))} = \infty.$$

Remark 1.1. Laplacian $f(r) = \Phi_p(r)$ satisfies (H1), but (H1) is more general than p-Laplacian. For example, for any odd natural number q, the function

$$f(r) = \begin{cases} r, & |r| \le 1 \\ \sqrt[q]{r}, & |r| > 1. \end{cases}$$

satisfies (H1), but f is not p-Laplacian.

By Remark 1.1, the results in this paper have extended and improved many existing results when $f(r) = \Phi_p(r)$.

We will show that the bounded and asymptotic properties of solutions can be characterized by the convergence or divergence of two series:

$$S_1 := \sum_{k=2}^{\infty} f^{-1}\Big(\frac{1}{a_k}\sum_{j=1}^{k-1} b_j\Big),$$
$$S_2 := \sum_{k=1}^{\infty} f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j\Big).$$

The paper is organized in the follows: Section 1 is the introduction that briefs the background and the motivation of the paper. Monotonicity and classification of solutions are discussed in Section 2. In Section 3, necessary and sufficient conditions for boundedness of all solutions are established. After that, the existence of solutions in different classes are provided in the last two sections.

2 Monotonicity and Classification of Solutions

In this section, we prove that all solutions of (1) are eventually monotone and can be classified into two classes:

$$A = \{\{x_n\} \text{ solutions of } (1) : \exists n_0 \ge 1 : x_n \Delta x_n > 0, \forall n \ge n_0\},$$
$$B = \{\{x_n\} \text{ solutions of } (1) : x_n \Delta x_n < 0, \forall n \ge 1\}.$$

Further classification of solutions is addressed at the end of this section.

Theorem 2.1. *Any solution $\{x_n\}$ of (1) is eventually strongly monotone and belongs to either class A or class B.*

Proof. The proof is similar to Lemma 1 in [2] with minor adjustment. Let $\{x_n\}$ be a solution of (1) and consider the sequence $\{F_n\}$ defined as

$$F_n = a_n f(\Delta x_n) x_n.$$

By the product rule of difference we have

$$\begin{aligned}\Delta F_n &= \Delta(a_n f(\Delta x_n))x_{n+1} + a_n f(\Delta x_n)\Delta x_n \\ &= b_n g(x_{n+1})x_{n+1} + a_n f(\Delta x_n)\Delta x_n \geq 0.\end{aligned}$$

Then $\{F_n\}$ is an increasing sequence. Since $\{x_n\}$ is not eventually constant, we have either $F_n > 0$ for all $n \geq n_0$ with a natural number $n_0 > 1$ or $F_n < 0$ for all $n \geq 1$. Obviously, $\{x_n\}$ is eventually strongly monotone and $\{x_n\} \in A$ in the first case. We will show that $\{x_n\}$ is strongly monotone and $\{x_n\} \in B$ in the second case. Indeed, assume $x_1 > 0$, then $F_n < 0$ implies $\Delta x_1 < 0$, that is $x_2 < x_1$. We claim that $x_2 > 0$. Otherwise, we have $\Delta x_2 > 0$, but from (1), $a_2 f(\Delta x_2) = a_1 f(\Delta x_1) + b_1 g(x_2) < 0$. This is a contradiction. Following the same arguments we can show that $\{x_n\}$ is a positive strongly decreasing sequence for all $n \geq 1$. Similarly, we can prove that $\{x_n\}$ is a negative strongly increasing sequence for all $n \geq 1$ if $x_1 < 0$. Therefore, $\{x_n\} \in B$. $\square$

Note that a class A solution could be bounded or unbounded, and a class B solution could converge to 0 or a nonzero limit. It makes sense to further classify the solutions of (1) into four mutually disjoint subclasses:

$$\begin{aligned}A_b &:= \Big\{\{x_n\} \in A : \lim_{n\to\infty} |x_n| = l < \infty\Big\}, \\ A_\infty &:= \Big\{\{x_n\} \in A : \lim_{n\to\infty} |x_n| = \infty\Big\}, \\ B_b &:= \Big\{\{x_n\} \in B : \lim_{n\to\infty} x_n = l \neq 0\Big\}, \\ B_0 &:= \Big\{\{x_n\} \in B : \lim_{n\to\infty} x_n = 0\Big\}.\end{aligned}$$

We will discuss the existence of these four subclass solutions in the last two sections.

3 Boundedness of Solutions

We now explore the boundedness of all solutions of equation (1).

Theorem 3.1. *Let (H1), (H2), and (H3) hold. Then all solutions of (1) are bounded if and only if* $S_1 < \infty$.

Proof. Necessity. Since all class B solutions are bounded, we focus on class A solutions. Let $\{x_n\}$ be a bounded class A solution of (1). Without loss of generality, assume $x_n > 0$ and $\Delta x_n > 0$ for $n \geq n_0 \geq 1$ and $\lim_{n\to\infty} x_n = l < \infty$. Then

$$\Delta(a_n f(\Delta x_n)) \geq L b_n,$$

where

$$L = \min_{x_{n_0} \leq r \leq l} g(r) > 0.$$

Summarizing both sides of the inequality from n_0 to $n-1$, we have

$$a_n f(\Delta x_n) \geq a_{n_0} f(\Delta x_{n_0}) + L \sum_{j=n_0}^{n-1} b_j \geq L \sum_{j=n_0}^{n-1} b_j.$$

Then

$$\frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j \leq \frac{1}{L} f(\Delta x_n).$$

It follows from (H1) that

$$f^{-1}\Big(\frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j\Big) \leq M f^{-1}\Big(\frac{1}{L}\Big) \Delta x_n.$$

Hence

$$\sum_{k=n_0+1}^{\infty} f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_0}^{k-1} b_j\Big) \leq M f^{-1}\Big(\frac{1}{L}\Big)(l - x_{n_0+1})$$

and $S_1 < \infty$ follows.

Sufficiency. Let $\{x_n\}$ be an unbounded class A solution. Without loss of generality, we assume $x_n > 0$ and $\Delta x_n > 0$ for $n \geq n_0 \geq 1$. By (H2) we have

$$\begin{aligned} a_n f(\Delta x_n) &= a_{n_0} f(\Delta x_{n_0}) + \sum_{j=n_0}^{n-1} b_j g(x_{j+1}) \\ &\leq a_{n_0} f(\Delta x_{n_0}) + g(x_n) \sum_{j=n_0}^{n-1} b_j \\ &= g(x_n)\Big(\frac{a_{n_0} f(\Delta x_{n_0})}{g(x_n)} + \sum_{j=n_0}^{n-1} b_j\Big) \\ &\leq g(x_n)\Big(\frac{a_{n_0} f(\Delta x_{n_0})}{g(x_{n_0})} + \sum_{j=n_0}^{n-1} b_j\Big). \end{aligned}$$

Selecting a constant $Q > 1$ such that

$$\frac{a_{n_0} f(\Delta x_{n_0})}{g(x_{n_0})} + \sum_{j=n_0}^{n-1} b_j \leq Q \sum_{j=n_0}^{n-1} b_j,$$

we obtain

$$f(\Delta x_n) \leq Q g(x_n) \frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j.$$

(H1) implies that

$$\begin{aligned} \Delta x_n &\leq M f^{-1}(Q g(x_n)) f^{-1}\Big(\frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j\Big) \\ &\leq M^2 f^{-1}(Q) f^{-1}(g(x_n)) f^{-1}\Big(\frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j\Big). \end{aligned}$$

Then

$$\frac{\Delta x_n}{f^{-1}(g(x_n))} \leq M^2 f^{-1}(Q) f^{-1}\Big(\frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j\Big).$$

Note that the sequence $f^{-1}(g(x_n))$ is increasing

$$\int_{x_n}^{x_{n+1}} \frac{dr}{f^{-1}(g(r))} \le \frac{\Delta x_n}{f^{-1}(g(x_n))} \le M^2 f^{-1}(Q) f^{-1}\Big(\frac{1}{a_n}\sum_{j=n_0}^{n-1} b_j\Big).$$

Summarizing both sides of the inequality from n_0+1 to ∞ and noting that $\lim_{n\to\infty} x_n = \infty$ we have

$$\int_{x_{n_0+1}}^{\infty} \frac{dr}{f^{-1}(g(r))} \le M^2 f^{-1}(Q) \sum_{n=n_0+1}^{\infty} f^{-1}\Big(\frac{1}{a_n}\sum_{j=n_0}^{n-1} b_j\Big).$$

$S_1 < \infty$ yields

$$\int_{x_{n_0+1}}^{\infty} \frac{dr}{f^{-1}(g(r))} < \infty,$$

which contradicts to (H3). So, all solutions are bounded. □

Remark 3.2. Define a function g as

$$g(r) = \begin{cases} \Phi_p(r \ln |r|), & |r| > e, \\ e^{p-2} r, & |r| \le e, \end{cases}$$

where $p > 1$. It is easy to check that g is continuous and increasing on $(-\infty, \infty)$. Moreover, the major condition (3) of Theorem 1 [3] and the condition (14) of Theorem 4 [2] are not satisfied since

$$\limsup_{|r|\to\infty} \frac{g(r)}{\Phi_p(r)} = \infty.$$

However, (H3) is valid because

$$\int_{\pm e}^{\pm\infty} \frac{1}{f^{-1}(g(r))} dr = \int_{\pm e}^{\pm\infty} \frac{dr}{r \ln r} = \infty.$$

Therefore, Theorem 4 [2] Theorem 1 [3] are not applicable to these types of difference equations, but Theorem 3.1 works.

Remark 3.3. If (H3) is not satisfied, then Theorem 3.1 may fail. Consider the following difference equation

$$\Delta((n-1)\Delta x_n) = \frac{4}{n(n+1)^2} g(x_{n+1}),\ n > 2, \tag{3}$$

where $g(r) = r^2 \mathrm{sgn} r$. Clearly, $\{x_n\}$ with $x_n = (n-1)n$ is an unbounded solution of (3), but (H3) is invalid since

$$\int_{\pm 1}^{\pm\infty} \frac{1}{f^{-1}(g(r))} dr = \int_{\pm 1}^{\pm\infty} \frac{dr}{r^2} < \infty.$$

If we drop conditions (H2) and (H3) but require the boundedness of $g(r)$, we still have the boundedness result.

Theorem 3.4. *Let (H1) hold. Assume that there exists a constant $K > 0$ such that $|g(r)| \le K$ for all $r \in \mathbb{R}$. Then all solutions of (1) are bounded if and only if $S_1 < \infty$.*

Proof. The proof of necessity is similar to Theorem 3.1.

Sufficiency. Let $\{x_n\}$ be a unbounded class A solution. Without loss of generality, we assume $x_n > 0$ and $\Delta x_n > 0$ for $n \ge n_0 \ge 1$. Note that

$$\begin{aligned} a_n f(\Delta x_n) &= a_{n_0} f(\Delta x_{n_0}) + \sum_{j=n_0}^{n-1} b_j g(x_{j+1}) \\ &\le a_{n_0} f(\Delta x_{n_0}) + K \sum_{j=n_0}^{n-1} b_j. \end{aligned}$$

Select a constant $Q > 1$ such that

$$a_{n_0} f(\Delta x_{n_0}) + K \sum_{j=n_0}^{n-1} b_j \le Q \sum_{j=n_0}^{n-1} b_j,$$

then

$$f(\Delta x_n) \le Q \frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j.$$

By (H1),

$$\Delta x_n \le M f^{-1}(Q) f^{-1}\Big(\frac{1}{a_n}\sum_{j=n_0}^{n-1} b_j\Big).$$

Summarizing both sides from n_0+1 to n we have

$$x_{n+1} - x_{n_0+1} \le M f^{-1}(Q) \sum_{i=n_0+1}^{n} f^{-1}\Big(\frac{1}{a_i}\sum_{j=n_0}^{i-1} b_j\Big).$$

The limit $\lim_{n\to\infty} x_n = \infty$ implies that $S_1 = \infty$, which contradicts to the assumption. So, all solutions of (1) are bounded. □

4 Class A Solutions

In this section, we focus on class A solutions of (1) and provide the existence of different class A solutions.

Theorem 4.1. *Equation* (1) *has solutions in class A.*

Proof. Let $\{x_n\}$ be a solution of (1) with initial conditions $x_1 > 0$, $x_2 > x_1$. By Theorem 2.1, we have $F_n = a_n f(\Delta x_n) x_n > 0$ for all $n \ge 1$. Hence, $x_n \Delta x_n > 0$ for all $n \ge 1$ and $\{x_n\}$ is a positive class A solution. Similarly, Let $\{x_n\}$ be a solution of (1) with initial conditions $x_1 < 0$, $x_2 < x_1$. Then $x_n \Delta x_n > 0$ for all $n \ge 1$ and $\{x_n\}$ is a negative class A solution. □

The following corollaries are directly from Theorem 3.1 and Theorem 3.4.

Corollary 4.2. *Let (H1), (H2), and (H3) hold. If* (1) *has a bounded class A solution, then all class A solutions are bounded. On the other hand, if* (1) *has a unbounded class A solution, then all class A solutions are unbounded.*

Corollary 4.3. *Let (H1) hold and the function g be bounded in* $\mathbb{R}$*. If* (1) *has a bounded class A solution, then all class A solutions are bounded. On the other hand, if* (1) *has a unbounded class A solution, then all class A solutions are unbounded.*

Next, we deal with the existence of different types of class A solutions. The first one is the existence of subclass A_b solutions.

Theorem 4.4. *Let (H1) hold. Then equation (1) has solutions in the subclass A_b if and only if $S_1 < \infty$.*

Proof. Necessity. Suppose that $\{x_n\}$ is a solution of (1) in the subclass A_b. Assume $x_n > 0$ and $\Delta x_n > 0$ for $n \geq n_0 \geq 1$ without loss of generality. Let $\lim_{n\to\infty} x_n = l$ and define

$$L = \min_{x_{n_0} \leq r \leq l} g(r) > 0.$$

Note that

$$a_n f(\Delta x_n) = a_{n_0} f(\Delta x_{n_0}) + \sum_{j=n_0}^{n-1} b_j g(x_{j+1}) \geq L \sum_{j=n_0}^{n-1} b_j.$$

Then

$$\frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j \leq \frac{1}{L} f(\Delta x_n).$$

Applying (H1) we have

$$f^{-1}\Big(\frac{1}{a_n} \sum_{j=n_0}^{n-1} b_j\Big) \leq M f^{-1}\Big(\frac{1}{L}\Big) \Delta x_n.$$

Summarizing both sides from $n_0 + 1$ to ∞ implies

$$\sum_{k=n_0+1}^{\infty} f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_0}^{k-1} b_j\Big) \leq M f^{-1}\Big(\frac{1}{L}\Big)(l - x_{n_0+1}) < \infty.$$

Hence, we have $S_1 < \infty$.

Sufficiency. Let $M_1 = \max_{1 \leq r \leq 2} g(r)$. From $S_1 < \infty$ we can choose $n_1 > 2$ such that

$$\sum_{k=n_1}^{\infty} f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_1-1}^{k-1} b_j\Big) \leq \frac{1}{M f^{-1}(M_1)} \tag{4}$$

Let X be the Banach space of all bounded sequences $\{x_n\}$ defined for all $n \geq n_1$ with supremum norm $\sup_{n\geq n_1} |x_n|$. Consider the subset Ω of X defined by

$$\Omega = \big\{x = \{x_n\} \in X : 1 \leq x_n \leq 2, n \geq n_1\big\}.$$

Obviously, Ω is a bounded, convex, and closed subset of X. Define an operator $T : \Omega \to X$ by

$$(Tx)_n = \begin{cases} x_{n_1}, \ n = n_1, \\ 1 + \sum_{k=n_1}^{n-1} f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1} b_j g(x_{j+1})\Big), \ n > n_1. \end{cases} \tag{5}$$

T has several desirable properties for applying Schauder's fixed-point theorem as we will show in the following.

First of all, T maps Ω into Ω. Indeed, if $x = \{x_n\} \in \Omega$, then by (4), (5), and (H1),

$$1 \leq (Tx)_n \leq 1 + Mf^{-1}(M_1) \sum_{k=n_1}^{\infty} f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_1-1}^{k-1} b_j\Big) \leq 2.$$

Secondly, T is continuous. Let $x^m = \{x_n^m\} \in \Omega$ and

$$\lim_{m\to\infty} \|x^m - x\| = 0.$$

Then $x \in \Omega$ because Ω is closed. We claim $\|Tx^m - Tx\| \to 0$ as $m \to \infty$. Indeed, note that

$$\|Tx^m - Tx\| = \sup_{n\geq n_1} |(Tx^m)_n - (Tx)_n|$$

$$= \sup_{n\geq n_1} \Big| \sum_{k=n_1}^{n-1} \Big[f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_1-1}^{k-1} b_j g(x_{j+1}^m)\Big) - f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_1-1}^{k-1} b_j g(x_{j+1})\Big)\Big]\Big|$$

$$\leq \sum_{k=n_1}^{\infty} \Big|f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_1-1}^{k-1} b_j g(x_{j+1}^m)\Big) - f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_1-1}^{k-1} b_j g(x_{j+1})\Big)\Big|.$$

For each fixed k, as $m \to \infty$, we have

$$f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_1-1}^{k-1} b_j g(x_{j+1}^m)\Big) - f^{-1}\Big(\frac{1}{a_k} \sum_{j=n_1-1}^{k-1} b_j g(x_{j+1})\Big) \to 0.$$

Observe that

$$\sum_{k=n_1}^{\infty}\Big|f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j g(x_{j+1}^m)\Big)-f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j g(x_{j+1})\Big)\Big|$$
$$\le 2Mf^{-1}(M_1)\sum_{k=n_1}^{\infty}f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j\Big)<\infty.$$

By Lebesgue's Dominate Convergence theorem, $\|Tx^m - Tx\| \to 0$ as $m \to \infty$.

Finally, $T\Omega$ is precompact. Clearly, $T\Omega$ is uniformly bounded. For any $\epsilon > 0$, there exists $n^* > n_1$ such that for any $x \in \Omega$ and $m \ge n^*$ we have

$$2Mf^{-1}(M_1)\sum_{k=m}^{\infty}f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j\Big)<\epsilon.$$

Note that $n > m$

$$|(Tx)_n-(Tx)_m|$$
$$=\Big|\sum_{k=n_1}^{n-1}f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j g(x_{j+1})\Big)-\sum_{k=n_1}^{m-1}f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j g(x_{j+1})\Big)\Big|$$
$$=\sum_{k=m}^{n-1}\Big|f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j g(x_{j+1})\Big)\Big|$$
$$\le 2Mf^{-1}(M_1)\sum_{k=m}^{\infty}f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j\Big)<\epsilon.$$

This shows that $T\Omega$ is equicontinuous and hence $T\Omega$ is precompact by Ascoli-Arzela Theorem.

Since all the conditions of Schauder's fixed-point theorem are satisfied, we conclude that there exists $\bar{x} = \{\bar{x}_n\} \in \Omega$ such that $\bar{x} = T\bar{x}$, or

$$\bar{x}_n = 1+\sum_{k=n_1}^{n-1}f^{-1}\Big(\frac{1}{a_k}\sum_{j=n_1-1}^{k-1}b_j g(\bar{x}_{j+1})\Big).$$

It is easy to verify that $\bar{x} \in A_b$. □

Remark 4.5. Theorem 4.4 generalizes Proposition 1 and Theorem 2 [3].

The following results follow from Corollary 4.2, Corollary 4.3, and Theorem 4.4.

Corollary 4.6. *Let (H1),(H2) and (H3) hold. Then*

1. $A = A_b$ *if and only if* $S_1 < \infty$.
2. $A = A_\infty$ *if and only if* $S_1 = \infty$.

Corollary 4.7. *Let (H1) hold. Assume that the function g is bounded in $\mathbb{R}$. Then*

1. $A = A_b$ *if and only if* $S_1 < \infty$.
2. $A = A_\infty$ *if and only if* $S_1 = \infty$.

5 Class B Solutions

In this section, we discuss the existence of different types of class B solutions.

Theorem 5.1. *Let (H1) hold. Then (1) has class B solutions.*

Proof. The proof is similar to Theorem 1 in [2] with minor changes. For real numbers $\mu > 0$ and α, let $x = \{x_n\}$ be the solution of (1) with $x_1 = \mu$ and $\Delta x_1 = \alpha$. Consider the set Γ given by

$$\Gamma = \{\alpha < 0 : \exists n_1 \geq 1 \text{ such that } x_{n_1} x_{n_1+1} \leq 0\}.$$

Then $\Gamma \neq \emptyset$. Indeed, if $x_1 = \mu$ and $\Delta x_1 = \alpha_1 < -\mu$, then $\alpha_1 \in \Gamma$ by noting that $x_1 x_2 = \mu(\mu + \alpha_1) < 0$. Define

$$\bar{\alpha} = \sup \Gamma. \tag{6}$$

Then $\bar{\alpha} \leq 0$. We claim that the solution $\bar{x} = \{\bar{x}_n\}$ of (1) such that $\bar{x}_1 = \mu$ and $\Delta \bar{x}_1 = \bar{\alpha}$ is a class B solution. Indeed, assume, instead, that $\bar{x} \in A$. Then either there exists an integer $n_1 > 1$ such that $\bar{x}_n < 0$

and $\Delta\bar{x}_n < 0$ for all $n > n_1$, or there exists an integer $n_1 \geq 1$ such that $\Delta\bar{x}_n > 0$ and $\bar{x}_n > 0$ for all $n > n_1$.

In the first case, take $\beta > \bar{\alpha}$ such that $\beta - \bar{\alpha}$ is sufficiently small. Consider the solution $\{y_n\}$ of (1) given by the initial conditions $y_1 = \mu$ and $\Delta y_1 = \beta$. We know that there is some $1 \leq n^* \leq n_1$ such that $\bar{x}_{n^*} > 0$ while $\bar{x}_{n^*+1} \leq 0$. If $\bar{x}_{n^*+1} < 0$, then $\bar{x}_{n^*}\bar{x}_{n^*+1} < 0$. It follows from the continuous dependence on initial conditions we also have that $y_{n^*}y_{n^*+1} < 0$. If $\bar{x}_{n^*+1} = 0$, then $\bar{x}_{n^*+2} < 0$. Again, by the continuous dependence on initial conditions, we have $y_{n^*} > 0$ and $y_{n^*+2} < 0$. Note that the sign of y_{n^*+1} could be greater than, less than, or equal to 0. For any case, we either have $y_{n^*}y_{n^*+1} \leq 0$ or $y_{n^*+1}y_{n^*+2} \leq 0$. Therefore, we have $\beta \in \Gamma$ which is a contradiction to (6)

In the second case, take $\beta < \bar{\alpha}$ such that $\bar{\alpha} - \beta$ is sufficiently small. Again, we consider a solution $\{y_n\}$ of (1) given by the initial conditions $y_1 = \mu$ and $\Delta y_1 = \beta$. Since $\Delta\bar{x}_n > 0$ and $\bar{x}_n > 0$ for all $n > n_1$, the continuous dependence on initial conditions implies that y_n is positive for all $n \geq 1$. Thus, $\beta \notin \Gamma$ for all $\beta < \bar{\alpha}$ that is sufficiently close to $\bar{\alpha}$ which is a contradiction to (6). □

Remark 5.2. Theorem 5.1 generalizes Theorem 1 [2].

Theorem 5.3. *Let (H1) hold. Then* (1) *has solutions in the subclass* B_b *if and only if* $S_2 < \infty$.

Proof. Necessity. Let $\{x_n\}$ be a solution of (1) in the subclass B_b. Without loss of generality we assume $x_n > 0$ and $\Delta x_n < 0$ for $n \geq 1$. Let $\lim_{n\to\infty} x_n = l$. Then $0 < l < \infty$.

Note that $a_n f(\Delta x_n) < 0$ and $\Delta(a_n f(\Delta x_n)) = b_n g(x_{n+1}) > 0$. The sequence $\{a_n f(\Delta x_n)\}$ is increasing and bounded above, so

$$\lim_{n\to\infty} a_n f(\Delta x_n) = H \in (-\infty, 0].$$

Let $L = \min_{l \leq r \leq x_1} g(r)$. Then $L > 0$. Summarizing (1) from k to infinity

$$H - a_k f(\Delta x_k) = \sum_{j=k}^{\infty} b_j g(x_{j+1}) \geq L \sum_{j=k}^{\infty} b_j.$$

Thus

$$\frac{1}{a_k}\sum_{j=k}^{\infty} b_j \le -\frac{1}{L} f(\Delta x_k).$$

By (H1) we have

$$f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j\Big) \le M f^{-1}\Big(-\frac{1}{L}\Big)\Delta x_k.$$

Then

$$\sum_{k=1}^{\infty} f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j\Big) \le M f^{-1}\Big(-\frac{1}{L}\Big)(l - x_1)$$

and $S_2 < \infty$.

Sufficiency. Let $M_1 = \max_{1\le r\le 2} g(r) > 0$. Since $S_2 < \infty$, we can choose $n_1 > 1$ such that

$$\sum_{k=n_1}^{\infty} f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j\Big) \le \frac{1}{-M^2 f^{-1}(-1) f^{-1}(M_1)}.$$

Consider the same Banach space X and subset Ω of X as in Theorem 4.4. Define an operator $T : \Omega \to X$ by

$$(Tx)_n = 2 + \sum_{k=n}^{\infty} f^{-1}\Big(-\frac{1}{a_k}\sum_{j=k}^{\infty} b_j g(x_{j+1})\Big),\ n \ge n_1.$$

Obviously, $(Tx)_n \le 2$ for all $n \ge n_1$. By (H1) we have

$$\begin{aligned} & f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j g(x_{j+1})\Big) \\ & \le f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} M_1 b_j\Big) \\ & \le M f^{-1}(M_1) f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j\Big). \end{aligned}$$

By (H1) again

$$\begin{aligned} &f^{-1}\Big(-\frac{1}{a_k}\sum_{j=k}^{\infty} b_j g(x_{j+1})\Big) \\ &\geq M f^{-1}(-1) f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j g(x_{j+1})\Big) \\ &\geq M^2 f^{-1}(-1) f^{-1}(M_1) f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j\Big). \end{aligned}$$

Then

$$(Tx)_n \geq 2 + M^2 f^{-1}(-1) f^{-1}(M_1) \sum_{n=n_1}^{\infty} f^{-1}\Big(\frac{1}{a_k}\sum_{j=k}^{\infty} b_j\Big) \geq 1.$$

Therefore, T maps Ω into Ω. Following the same discussions as Theorem 4.4, we can show that T is continuous and $T\Omega$ is precompact. So, all conditions of Schauder's fixed-point theorem are satisfied, we conclude that there exists $\bar{x} = \{\bar{x}_n\} \in \Omega$ such that $\bar{x} = T\bar{x}$, or

$$\bar{x}_n = 2 + \sum_{k=n}^{\infty} f^{-1}\Big(-\frac{1}{a_k}\sum_{j=k}^{\infty} b_j g(\bar{x}_{j+1})\Big).$$

It is easy to verify that $\bar{x} \in B_b$. □

Remark 5.4. Theorem 5.3 improves Theorem 2 [2] by providing necessary and sufficient conditions.

From Theorem 5.3 we have the result that the subclass B_b is empty.

Corollary 5.5. *Let (H1) hold. Then $B = B_0$ if and only if $S_2 = \infty$.*

6 Conclusion

The classification, existence, boundedness, and monotony of solutions of a second order nonlinear difference equation (1) are studied in this paper. It is proved that all the solutions are eventually monotonous in

Chapter 2. The necessary and sufficient conditions for the boundedness of all solutions are established in Chapter 3. As it is pointed out in Remark 3.2, the major conditions (3) of Theorem 1 [3] and (14) of Theorem 4 [2] are not satisfied for some types of difference equations, but our assumption (H3) is valid. Therefore, Theorem 3.1 improves Theorem 4 [2] and Theorem 1 [3]. The existence results of different types of monotonous solutions are presented in Chapters 4 and 5. In particular, Theorem 4.4 generalizes Theorem 2 [3], Theorem 5.1 generalizes Theorem 1 [2], and Theorem 5.3 improves Theorem 2 [2] by providing necessary and sufficient conditions.

References

[1] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publishing Corporation, New York, 2005.

[2] M. Cecchi, Z. Došlá and M. Marini, Positive decreasing solutions of quasi-linear difference equations, *Comput. Math. Appl.*, 42 (2001) 1401–1410.

[3] M. Cecchi, Z. Došlá and M. Marini, Unbounded solutions of quasi-linear difference equations, *Comput. Math. Appl.*, 45 (2003) 1113–1123.

[4] M. Cecchi, Z. Došlá and M. Marini, Nonoscillatory half-linear difference equations and recessive solutions, *Adv. Difference Equ.*, 2 (2005) 193–204.

[5] M. Cecchi, Z. Došlá and M. Marini, On the growth of nonoscillatory solutions for difference equations with deviating argument, *Adv. Difference Equ.*, 2008, Article ID 505324.

[6] M. Cecchi, Z. Došlá and M. Marini, On oscillation and nonoscillation properties of Emden-Fowler difference equations, *Cent. Eur. J. Math.*, 7 (2009) 322–334.

[7] M. Cecchi, Z. Došlá and M. Marini, Intermediate solutions for nonlinear difference equations with p-Laplacian, *Adv. Stud. Pure Math.*, 53 (2009) 33–40.

[8] M. Cecchi, Z. Došlá, M. Marini and I. Vrkoč, Summation inequalities and half-linear difference equations, *J. Math. Anal. Appl.*, 302 (2005) 1–13.

[9] M. Cecchi, Z. Došlá, M. Marini and I. Vrkoč, Asymptotic properties for half-linear difference equations, *Math. Bohem.*, 131 (2006) 347363.

[10] Z. Došlá, M. Marini and S. Matucci, A fixed-point approach for decaying solutions of difference equations, *Philos Trans A Math Phys Eng Sci*, 379 (2021) 1-13.

[11] X. He, Oscillatory and asymptotic behaviour of second order nonlinear difference equations, *J. Math. Anal. Appl.*, 175 (1993) 482–498.

[12] H. F. Huo and W. T. Li, Oscillation of certain two-dimensional nonlinear difference systems, *Comput. Math. Appl.*, 45 (2003) 1221–1226.

[13] J. Jiang and X. Li, Oscillation and nonoscillation of two-dimensional difference systems, *J. Comput. Appl. Math.*, 188 (2006) 77–88.

[14] W. T. Li, Classification schemes for nonoscillatory solutions of two-dimensional nonlinear difference systems, *Comput. Math. Appl.*, 42 (2001) 341–355.

Lianwen Wang
Professor of Statistics and Mathematics
School of Computer Science and Mathematics
University of Central Missouri
Warrensburg, MO 64093, USA
E-mail: lwang@ucmo.edu

Anthony C. Willett
MSc of Mathematics

School of Computer Science and Mathematics
University of Central Missouri
Warrensburg, MO 64093, USA
E-mail: awillett@ucmo.edu