Journal of Mathematical Extension
Vol. 16, No. 12, (2022) (7)1-24
URL: https://doi.org/10.30495/JME.2022.2096
ISSN: 1735-8299
Original Research Paper

Some Types of UP-filters in UP-algebras

Z. Parvizi
Shahrekord Branch, Islamic Azad University

S. Motamed*
Bandar Abbas Branch, Islamic Azad University,

F. Khaksar Haghani
Shahrekord Branch, Islamic Azad University

Abstract. In this paper, we defined the notions of normal, prime and nodal UP-filters in UP-algebras and investigated several properties of them. Also, we stated and proved some theorems in order to determine the relationships between this notions and some types of UP-filters in a UP-algebra and by some examples we show that these notions are different.

AMS Subject Classification: 03G25; 13N15
Keywords and Phrases: UP-algebra, nodal UP-filter, normal UP-filter, prime UP-filter

1 Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [5], BCI-algebras [6], BCH-algebras [1], KU-algebras [11], SU-algebras [10] and others. They

Received: July 2021; Accepted: July 2022
*Corresponding Author

are strongly connected with logic. For example, BCI-algebras introduced by Iséki [6] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [5], [6] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Iampan [2] now introduced a new algebraic structure, called a UP-algebra and a concept of UP-ideals, congruences and UP-homomorphisms in UP-algebras, and defined a congruence relation on a UP-algebra and a quotient UP-algebra Somjanta, et al. [12] introduced the notion of UP-filters and discussed the fuzzy set theory of UP-subalgebras, UP-ideals and UP-filters. Kaijae and et al., introduced anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras concepts of UP-algebras. They also introduced the notions of Cartesian product and dot product of fuzzy sets and they discussed the relation between anti-fuzzy UP-ideals and level subsets of a fuzzy set, [9]. Jun and Iampan introduced the notions of implicative UP-filters, comparative UP-filters and shift UP-filters in a UP-algebra, ([7],[8]).

The objective of this paper is to develop and define new concepts for investigating UP-algebras. This paper motivated by the previous researches on types of UP-filters in UP-algebras, extends the new notions of UP-filters to the UP-algebras. Furthermore, several new properties for implicative UP-filters and shift UP-filters in UP-algebras are obtained. Also, new types of UP-filters in UP-algebras are introduced and several characterizations for them are found. The structure of the paper is as follows: Section 2 is a recall of some definitions and results about UP-algebras that are used in the paper. In section 3, some results in UP-algebra, are obtained. In section 4, many different characterizations and many important properties of comparative UP-filters and implicative UP-filters in UP-algebras, are proved. In section 5, a new UP-filter (normal UP-filter) in UP-algebras are introduced and some basic properties for them are provided. In section 6, a new UP-filter (prime UP-filter) in UP-algebras are introduced. In section 7, a new UP-filter (nodal UP-filter) in UP-algebras are introduced and some properties for them are investigated.

2 Preliminaries

In this section, we recall some definitions, properties and results relative to UP-algebras which will be used in the following.

Definition 2.1. [2] An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, if it satisfies following conditions, for all $, x, y, z \in X$:
(1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(2) $0 \cdot x = x$,
(3) $x \cdot 0 = 0$,
(4) if $x \cdot y = 0 = y \cdot x$ then $x = y$.
Define a binary relation $\leq$ on a UP-algebra X as follows:
(5) $x \leq y$ if and only if $x \cdot y = 0$.

Proposition 2.2. *([2],[3]) In a* UP*-algebra X, the following assertions are valid, for $a, x, y, z \in X$:*
(1) $x \cdot x = 0$,
(2) *if* $x \cdot y = 0$ *and* $y \cdot z = 0$ *then* $x \cdot z = 0$,
(3) *if* $x \cdot y = 0$ *then* $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) *if* $x \cdot y = 0$ *then* $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,
(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$,
(7) $x \cdot (y \cdot y) = 0$,
(8) $(x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0$,
(9) $(((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0$,
(10) $((x \cdot y) \cdot z) \cdot (y \cdot z) = 0$,
(11) *if* $x \cdot y = 0$ *then* $x \cdot (z \cdot y) = 0$,
(12) $((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0$,
(13) $((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0$.

Definition 2.3. Let X be a UP-algebra.
(1) A subset F of X is called a UP-filter of X, if $0 \in F$ and if $x, x \cdot y \in F$ then $y \in F$, for all $x, y \in X$, [12].
(2) A subset B of X is called a UP-ideal of X, if it satisfies in the following properties:
(i) the constant $0 \in B$, and
(ii) for any $x, y, z \in X$; $x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$, [2].

The set of all UP-filters of a UP-algebra X is denoted by $\mathrm{UF}(X)$.

Definition 2.4. [2] Let $X = (X, \cdot, 0)$ be a UP-algebra. A subset S of X is called a UP-subalgebra of X, if the constant 0 of X is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Definition 2.5. A subset F of a UP-algebra X is called

• an implicative UP-filter of X, if $0 \in F$ and for all $x, y, z \in X$, if $x \cdot (y \cdot z) \in F$ and $x \cdot y \in F$ then $x \cdot z \in F$, [7].

• a shift UP-filter of X, if $0 \in F$ and for all $x, y, z \in X$, if $x \cdot (y \cdot z) \in F$ and $x \in F$ then $((z \cdot y) \cdot y) \cdot z \in F$, [8].

• a comparative UP-filter of X, if $x \cdot ((y \cdot z) \cdot y) \in F$ and $x \in F$ then $y \in F$, for $, x, y, z \in X$, [8].

Definition 2.6. [7] Let X ba a UP-algebra.

(i) For $a \in X$, $[a) := \{x \in X \ : \ a \leq x\}$.

(ii) For any subset F of X, $[F) = \bigcap_{F \subseteq G \in \mathrm{UF}(X)} G$. Then $[F)$ is the smallest UP-filter of X containing F.

3 Some New Properties for UP-algebras

In this section, we investigate the structure of UP-algebras. Also, some new results of UP-algebras are obtained.

According to Definition 2.1 and Proposition 2.2:

Lemma 3.1. *Let X be a UP-algebra and $a, x, y, z \in X$. Then the following conditions hold:*

(1) $y \cdot z \leq (x \cdot y) \cdot (x \cdot z)$,

(2) $x \leq 0$,

(3) *if* $x \leq y$ *and* $y \leq x$ *then* $x = y$,

(4) $x \leq x$,

(5) *if* $x \leq y$ *then* $z \cdot x \leq z \cdot y$,

(6) *if* $x \leq y$ *then* $y \cdot z \leq x \cdot z$,

(7) *if* $x \leq y$ *and* $y \leq z$ *then* $x \leq z$,

(8) $x \leq y \cdot x$,

(9) $y \cdot x \leq x$ *if and only if* $x = y \cdot x$,

(10) $x \leq y \cdot y$,

(11) $x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))$,
(12) $((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z$,
(13) $(x \cdot y) \cdot z \leq y \cdot z$,
(14) *if* $x \leq y$ *then* $x \leq z \cdot y$,
(15) $(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$,
(16) $(x \cdot y) \cdot z \leq y \cdot (a \cdot z)$,
(17) *if* $0 \leq x$ *then* $x = 0$,
(18) $x \leq (y \cdot x) \cdot x$.

Lemma 3.2. *Let* X *be a* UP-*algebra and for all* $x, y, z \in X$,
(19) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.
Then the following conditions hold:
(20) $y \leq (y \cdot x) \cdot x$,
(21) $x \cdot y \leq ((y \cdot z) \cdot (x \cdot z)$.

Proof. According to condition (19) and Proposition 2.2, $y \cdot ((y \cdot x) \cdot x) = (y \cdot x) \cdot (y \cdot x) = 0$. Therefore $y \leq (y \cdot x) \cdot x$. Similary according to condition (19) and Definition 2.1,

$$(x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) = (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0.$$

Therefore $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$. □

The following example shows that the condition (19) in Lemma 3.2 is necessary.

Example 3.3. (1) Let $X = \{a, b, c, 0\}$ be a set with the binary operetion $\cdot$ which is given in the following table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	a	b
b	0	0	0	b
c	0	0	0	0

Then $(X, \cdot, 0)$ is a UP-algebra ([8]). Clearly, condition (19) is not hold, since $a \cdot (b \cdot c) \neq b \cdot (a \cdot c)$, and the condition (20) is not hold since $a \nleq (a \cdot c) \cdot c = b$.
(2) Let $X = \{a, b, c, 0\}$ be a set with the binary operation $\cdot$ which is given in the following table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	0	a	0

Then $(X, \cdot, 0)$ is a UP-algebra ([7]). Clearly, condition (19) is not hold, since $a \cdot (c \cdot b) \neq c \cdot (a \cdot b)$ and the condition (21) is not hold since $a \cdot c \nleq (c \cdot b) \cdot (a \cdot b)$.

Lemma 3.4. *Let X be a* UP*-algebra which satisfies in condition* (19)*. Then for all $x, y, z \in X$ the following statements are equivalent :*

(1) $((x \cdot y) \cdot y) \cdot x = y \cdot x$,
(2) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$,
(3) *If* $x \cdot z \leq y \cdot z$ *and* $z \leq x$ *then* $y \leq x$,
(4) *If* $x \cdot z \leq y \cdot z$ *and* $z \leq x, y$ *then* $y \leq x$,
(5) *If* $y \leq x$ *then* $(x \cdot y) \cdot y = x$.

Proof. $(1 \Leftrightarrow 2)$ Let $((x \cdot y) \cdot y) \cdot x = y \cdot x$, for all $x, y \in X$. Then using condition (19) and hypothesis, $((x \cdot y) \cdot y) \cdot ((y \cdot x) \cdot x) = (y \cdot x) \cdot (((x \cdot y) \cdot y) \cdot x) = (y \cdot x) \cdot (y \cdot x) = 0$, that is $(x \cdot y) \cdot y \leq (y \cdot x) \cdot x$. Similarly, $(y \cdot x) \cdot x \leq (x \cdot y) \cdot y$. Therefore $(y \cdot x) \cdot x = (x \cdot y) \cdot y$, for all $x, y \in X$.
$(2 \Rightarrow 3)$ Let $x, y, z \in X$, such that $x \cdot z \leq y \cdot z$ and $z \leq x$. Then using condition (19) and hypothesis, $0 = (x \cdot z) \cdot (y \cdot z) = y \cdot ((x \cdot z) \cdot z) = y \cdot ((z \cdot x) \cdot x) = y \cdot (0 \cdot x) = y \cdot x$. Therefore $y \leq x$.
$(3 \Rightarrow 4)$ It is trivial.
$(4 \Rightarrow 5)$ Let $x, y \in X$ such that $y \leq x$. Using condition (20), $x \cdot y \leq ((x \cdot y) \cdot y) \cdot y$. According to part (4), $(x \cdot y) \cdot y \leq x$, therefore based on condition (20), $(x \cdot y) \cdot y = x$, for $x, y \in X$.
$(5 \Rightarrow 2)$ As $x \leq (y \cdot x) \cdot x$, then $((y \cdot x) \cdot x) \cdot y \leq x \cdot y$. Also using condition (19), $y \leq (y \cdot x) \cdot x$. Hence according to Lemma 3.1 and Part (5), $(x \cdot y) \cdot y \leq (((y \cdot x) \cdot x) \cdot y) \cdot y = (y \cdot x) \cdot x$. Similarly, since $y \leq (x \cdot y) \cdot y$, then $((x \cdot y) \cdot y) \cdot x \leq y \cdot x$. Using $x \leq (x \cdot y) \cdot y$, Lemma 3.1 and Part (5), we get $(y \cdot x) \cdot x \leq (((x \cdot y) \cdot y) \cdot x) \cdot x = (x \cdot y) \cdot y$. Therefore $(x \cdot y) \cdot y = (y \cdot x) \cdot x$, for all $x, y \in X$. □

Proposition 3.5. *[7] Let X ba a* UP*-algebra.*

(i) In general, $[a)$ *is not a* UP*-filter of* X*.* $[a)$ *is a* UP*-filter of* X *if and only if* $\{0\}$ *is an implicative* UP*-filter of* X*.*

(ii) If X *satisfying in the condition* (19)*, for a nonempty subset* F *of* X*, then* $[F) = \{x \in X : a_1 \cdot (a_2 \cdot (\ldots (a_n \cdot x) \ldots)) = 0, \textit{for some } a_1, \ldots, a_n \in F\}$*.*

Theorem 3.6. *Let* X *be a* UP*-algebra,* F*,* G *be nonempty subsets of* X *and* $a, b \in X$*. Then*

(1) *if* X *satisfying in the condition* (19) *and* $a \leq b$ *then* $[b) \subseteq [a)$*,*
(2) *if* $[b) \subseteq [a)$ *then* $a \leq b$*,*
(3) F *is a* UP*-filter of* X *if and only if* $[F) = F$*,*
(4) *if* $F \subseteq G$ *then* $[F) \subseteq [G)$*,*
(5) *if* G *is a* UP*-filter of* X *and* $[F) \subseteq [G)$ *then* $F \subseteq G$*.*

Proof.

(1) Let $a \leq b$. Assume that $z \in [b)$, then $b \leq z$ and so $b \cdot z = 0$. Using Lemma 3.2(21), $a \cdot b \leq (b \cdot z) \cdot (a \cdot z)$ and as $a \cdot b = 0$, then $(b \cdot z) \cdot (a \cdot z) = 0$. Since $b \cdot z = 0$, thus $a \cdot z = 0$, therefore $z \in [a)$, i.e. $[b) \subseteq [a)$.

(2) Let $[b) \subseteq [a)$. As $b \in [b)$ so $b \in [a)$. Therefore $a \leq b$.

(3) It is known that $[F) = \bigcap_{F \subseteq G \in \mathrm{UF}(X)} G$. Assume that $x \in [F)$. As F is a UP-filter, then $x \in F$. Therefore $[F) = F$. The converse is clear.

(4) Let $x \in [F)$. It is known that $[F) = \bigcap_{F \subseteq H \in \mathrm{UF}(X)} H$, i.e. $x \in H$, for all $H \in \mathrm{UF}(X)$ where $F \subseteq H$. So $x \in G$. As $G \subseteq [G)$. Therefore $x \in [G)$, i.e. $[F) \subseteq [G)$.

(5) The proof is clear.

□

Definition 3.7. Let X be a UP-algebra and F be a UP-filter of X. For $x, y \in X$, we define the binary relation $\sim_F$ on X, $x \sim_F y$ if and only if $x \cdot y \in F$ and $y \cdot x \in F$.

Example 3.8. [4] Consider a UP-algebra $X = \{a, b, c, d, 0\}$ with the binary operation $\cdot$ which is given in the following table:

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Clearly, $F = \{a, c, 0\}$ is a UP-filter of X. It is easy to verify that $a \sim_F c$, since $a \cdot c = c$ and $c \cdot a = 0$. But $a \not\sim_F b$, since $a \cdot b = b \notin F$.

Proposition 3.9. *Let X be a* UP*-algebra satisfying in condition* (19) *and F be a* UP*-filter of X. A binary relation $\sim_F$ is a congruence relation of X.*

Proof. As $x \cdot x = 0 \in F$, then $x \sim_F x$. Hence we conclude that a binary relation $\sim_F$ is reflexive. Now let $x \sim_F y$, for $x, y \in X$. Then $x \cdot y \in F$ and $y \cdot x \in F$, so $y \sim_F x$. It can be concluded that $\sim_F$ is symmetric. Let $x \sim_F y$ and $y \sim_F z$, for $x, y, z \in X$. Then $x \cdot y \in F$, $y \cdot x \in F$ and $y \cdot z \in F$, $z \cdot y \in F$. As $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0 \in F$ and $y \cdot z \in F$ then $(x \cdot y) \cdot (x \cdot z) \in F$. Also $x \cdot y \in F$, it follows that $x \cdot z \in F$. Similarly, as $(z \cdot y) \cdot ((y \cdot x) \cdot (z \cdot x)) = 0 \in F$ and $z \cdot y \in F$ then $(y \cdot x) \cdot (z \cdot x) \in F$. Also $y \cdot x \in F$, we conclude that $z \cdot x \in F$. Thus $x \sim_F z$. Hence $\sim_F$ is transitive. Therefore $\sim_F$ is an equivalence relation on X. Now, assume tha $x \sim_F u$ and $y \sim_F v$, for $x, y, u, v \in X$. Then $x \cdot u \in F$, $u \cdot x \in F$ and $y \cdot v \in F$, $v \cdot y \in F$. As $(v \cdot y) \cdot ((x \cdot v) \cdot (x \cdot y)) = 0 \in F$, and $v \cdot y \in F$, then $(x \cdot v) \cdot (x \cdot y) \in F$. Similarly, we get $(y \cdot v) \cdot ((x \cdot y) \cdot (x \cdot v)) = 0 \in F$. As $y \cdot v \in F$, so $(x \cdot y) \cdot (x \cdot v) \in F$. Therefore $x \cdot y \sim_F x \cdot v$. On the other hand, $(u \cdot v) \cdot ((x \cdot u) \cdot (x \cdot v)) = 0 \in F$. Using condition (19), $(x \cdot u) \cdot ((u \cdot v) \cdot (x \cdot v)) = 0 \in F$. As $x \cdot u \in F$, then $(u \cdot v) \cdot (x \cdot v) \in F$. Similarly, $(x \cdot v) \cdot ((u \cdot x) \cdot (u \cdot v)) = 0 \in F$. Using condition (19), $(u \cdot x) \cdot ((x \cdot v) \cdot (u \cdot v)) = 0 \in F$. As $u \cdot x \in F$ then $(x \cdot v) \cdot (u \cdot v) \in F$. Thus $x \cdot v \sim_F u \cdot v$. According to transitivity of $\sim_F$, $x \cdot y \sim_F u \cdot v$. Therefore $\sim_F$ is a congruence relation on X. □

Note that, in Proposition 3.9, the condition (19) was not necessary to prove the equivalence relation $\sim_F$ and it was necessary to prove the congruence relation $\sim_F$. The following example shows that the condition (19) is necessary for congruence relation.

Example 3.10. Let $X = \{a, b, c, 0\}$ be a set with the binary operetion $\cdot$ which is given in the following table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	b	b
b	0	0	0	b
c	0	0	0	0

Then $(X, \cdot, 0)$ is a UP-algebra ([8]). Clearly, $F = \{a, 0\}$ is a UP-filter and condition (19) is not hold, since $a \cdot (b \cdot c) \neq b \cdot (a \cdot c)$. It is easy to verify that $a \sim_F 0$, $c \sim_F c$ while $a \cdot c \not\sim_F 0 \cdot c$.

Let X be a UP-algebra and ρ be a congruence relation on X. The ρ-class of $x \in X$ is the $(x)_\rho = \{y \in X \ : \ y \ \rho \ x\}$. The quotient set of X by ρ, is denoted by $X/\rho = \{(x)_\rho \ : \ x \in X\}$.

Theorem 3.11. *Let X be a* UP*-algebra satisfying in condition* (19) *and ρ be a congruence relation on X. Then the following statements hold:*
(1) *A ρ-class $(0)_\rho$ is a* UP*-filter and a* UP*-subalgebra of X,*
(2) *A ρ-class $(x)_\rho$ is a* UP*-filter of X if and only if $x \ \rho \ 0$,*
(3) *A ρ-class $(x)_\rho$ is a* UP*-subalgebra of X if and only if $x \ \rho \ 0$.*

Proof. According to Definition 2.4, the proof is clear. □

Theorem 3.12. *Let X be a* UP*-algebra satisfying in condition* (19) *and F be a* UP*-filter of X. Then the following statements hold:*
(1) *$(0)_{\sim_F}$ is a* UP*-filter and a* UP*-subalgebra of X contained in F,*
(2) *$(x)_{\sim_F}$ is a* UP*-filter of X if and only if $x \in F$,*
(3) *$(x)_{\sim_F}$ is a* UP*-subalgebra of X if and only if $x \in F$,*
(4) *$(X/\sim_F, *, (0)_{\sim_F})$ is a* UP*-algebra under a binary relation $*$ defined by $(x)_{\sim_F} * (y)_{\sim_F} = (x \cdot y)_{\sim_F}$, for all $x, y \in X$. $X/\sim_F$ is called a quotient* UP*-algebra of X induced by a congruence relation $\sim_F$.*

Proof. Based on Proposition 3.9 and Theorem 3.11, the proofs are obvious. □

Proposition 3.13. *Let X be a* UP*-algebra satisfying in condition* (19) *and F be a* UP*-filter of X. Then*

(1) $(x)_F = (0)_F$ *if and only if* $x \in F$,
(2) $(x)_F \leq (y)_F$ *if and only if* $x \cdot y \in F$,
(3) $(x)_F = (y)_F$ *if and only if* $x \cdot y \in F$ *and* $y \cdot x \in F$ *if and only if* $x_{\sim_F} y$.

Proof. According to Definition 2.1 and Theorem 3.12, the proof is easy. □

4 Comparative UP-filters and Implicative UP-filters

In this section we have established new characterizations and connections between comparative UP-filters, implicative UP-filters and maximal UP-filters in UP-algebras.

Theorem 4.1. *Let* X *be a* UP-*algebra satisfying in condition* (19) *and* F *be an implicative* UP-*filter of* X. *Then* F *is a comparative* UP-*filter if and only if* $(x \cdot y) \cdot y \in F$ *implies* $(y \cdot x) \cdot x \in F$, *for* $x, y \in X$.

Proof. Let F be a comparative UP-filter and $(x \cdot y) \cdot y \in F$. From $x \leq (y \cdot x) \cdot x$, and based on Lemma 3.1(7), $((y \cdot x) \cdot x) \cdot y \leq x \cdot y$. According to Lemma 3.1, $(x \cdot y) \cdot y \leq (y \cdot x) \cdot ((x \cdot y) \cdot x)$. Using condition (19), $(x \cdot y) \cdot y \leq (x \cdot y) \cdot ((y \cdot x) \cdot x)$ based on Lemma 3.1, $(x \cdot y) \cdot ((y \cdot x) \cdot x) \leq (((y \cdot x) \cdot x) \cdot y) \cdot ((y \cdot x) \cdot x)$. Hence $(((y \cdot x) \cdot x) \cdot y) \cdot ((y \cdot x) \cdot x) \in F$. As F is a comparative UP-filter, then $(y \cdot x) \cdot x \in F$. Conversely, let $z \cdot ((x \cdot y) \cdot x) \in F$ and $z \in F$. By Theorem 1([7]), we conclude that F is a UP-filter, so $(x \cdot y) \cdot x \in F$. By condition (20), it can be concluded that $x \leq (x \cdot y) \cdot y$. Then using Lemma 3.1, $(x \cdot y) \cdot x \leq (x \cdot y) \cdot ((x \cdot y) \cdot y)$, and we have $(x \cdot y) \cdot ((x \cdot y) \cdot y) \in F$. As F is an implicative UP-filter and $(x \cdot y) \cdot (x \cdot y) = 0 \in F$, then $(x \cdot y) \cdot y \in F$. Therefore considering to the hypothesis $(y \cdot x) \cdot x \in F$. Since $y \leq x \cdot y$, we get that $(x \cdot y) \cdot x \leq y \cdot x$. Based on Lemma 3.1, $y \cdot x \leq z \cdot (y \cdot x)$. As a result we have $(x \cdot y) \cdot x \leq z \cdot (y \cdot x)$, and so $z \cdot (y \cdot x) \in F$. As $z \in F$ and F is a UP-filter, then $y \cdot x \in F$. Hence $x \in F$. Thus the proof is completed.
□

Proposition 4.2. *Let* X *be a* UP-*algebra which satisfies in condition* (19) *and* F *be a comparative* UP-*filter of* X. *Then* F *is an implicative* UP-*filter, but the converse is not true.*

Proof. Let F be a comparative UP-filter and $x \cdot (y \cdot z) \in F$, $x \cdot y \in F$, for $x, y, z \in X$. We have $y \cdot (x \cdot z) \leq (x \cdot y) \cdot (x \cdot (x \cdot z))$. Using condition (19), $x \cdot (y \cdot z) \leq (x \cdot y) \cdot (x \cdot (x \cdot z))$. Hence $(x \cdot y) \cdot (x \cdot (x \cdot z)) \in F$. Since F is a UP-filter and $x \cdot y \in F$ then $x \cdot (x \cdot z) \in F$. In other hand we have $x \cdot (x \cdot z) \leq ((x \cdot z) \cdot z) \cdot (x \cdot z)$. So $((x \cdot z) \cdot z) \cdot (x \cdot z) \in F$. As F is a comparative UP-filter, then $x \cdot z \in F$. Therefore F is an implicative UP-filter. □

For the converse consider:

Example 4.3. Let $X = \{a, b, c, 0\}$ be a set with the binary operation $\cdot$ which is given in the following table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $(X, \cdot, 0)$ is a UP-algebra ([4]). Clearly, $F = \{b, 0\}$ is an implicative UP-filter, while it is not a comparative. Since $(c \cdot a) \cdot c = 0 \in F$, but $c \notin F$.

Theorem 4.4. *Let X be a* UP*-algebra which satisfies in condition* (19) *and F be a comparative* UP*-filter of X. Every* UP*-filter G containing F is also a comparative* UP*-filter.*

Proof. Let F be a comparative UP-filter and G be a UP-filter such that $F \subseteq G$. Then F is an implicative UP-filter. Using Theorem 13([7]), G is an implicative UP-filter. Suppose that $(y \cdot x) \cdot x \in G$, for $x, y \in X$. Denote $u = (y \cdot x) \cdot x$. Using $u \cdot ((y \cdot x) \cdot x) = 0 \in F$, F is an implicative UP-filter and Theorem 10([7]), $(y \cdot (u \cdot x)) \cdot (u \cdot x) = (u \cdot (y \cdot x)) \cdot (u \cdot x) \in F$. So according to Theorem 4.1, $((u \cdot x) \cdot y) \cdot y \in F \subseteq G$. Based on Lemma 3.1 consider

$$\begin{aligned}(y \cdot x) \cdot x &\leq (((y \cdot x) \cdot x) \cdot x) \cdot x = (u \cdot x) \cdot x \\ &\leq (x \cdot y) \cdot ((u \cdot x) \cdot y) \leq (((u \cdot x) \cdot y) \cdot y) \cdot ((x \cdot y) \cdot y).\end{aligned}$$

So $(((u \cdot x) \cdot y) \cdot y) \cdot ((x \cdot y) \cdot y) \in G$. Using $((u \cdot x) \cdot y) \cdot y \in G$, we get that $(x \cdot y) \cdot y \in G$ and so the proof is completed. □

Theorem 4.5. *Let X be a* UP-*algebra satisfying in condition* (19)*. The following condition are equivalent:*

(1) $\{0\}$ *is a comparative* UP-*filter,*
(2) *every* UP-*filter of X is a comparative* UP-*filter,*
(3) $[a) = \{x \in X \ : \ a \leq x\}$ *is a comparative* UP-*filter, for all $a \in X$,*
(4) $(x \cdot y) \cdot x = x$*, for all $x, y \in X$.*

Proof. $(1 \Leftrightarrow 2)$ According to Theorem 4.4 the proof is clear.
$(2 \Rightarrow 3)$ According to hypothesis $\{0\}$ is a comparative UP-filter, and $\{0\}$ is an implicative UP-filter. As $a \leq 0$ then $0 \in [a)$. Assume that $x, x \cdot y \in [a)$, for $x, y \in X$. Then $a \cdot x = 0$ and $a \cdot (x \cdot y) = 0$. Then $a \cdot y = 0$, since $\{0\}$ is an implicative UP-filter. And so $a \leq y$. Therefore $y \in [a)$, i.e. $[a)$ is a UP-filter. Using (2), $[a)$ is a comparative UP-filter.
$(3 \Rightarrow 4)$ We have $(x \cdot y) \cdot x \in [(x \cdot y) \cdot x)$. According to hypothesis, $[(x \cdot y) \cdot x)$ is a comparative UP-filter. Then based on Lemma $2 \cdot 6$([8]), $x \in [(x \cdot y) \cdot x)$ and so $(x \cdot y) \cdot x \leq x$. Hence $x = (x \cdot y) \cdot x$, for all $x, y \in X$.
$(4 \Rightarrow 1)$ The proof is easy, based on Lemma $2 \cdot 6$([8]). □

Definition 4.6. A UP-filter M of a UP-algebra X is called maximal, if it is not properly contained in any other UP-filter of X.

Example 4.7. In Example 4.3, $F = \{b, 0\}$ is not a maximal UP-filter and $G = \{b, c, 0\}$ is a maximal UP-filter.

Recall that $F_y = \{x \in X \ : \ y \cdot x \in F\}$, [7].

Lemma 4.8. *Let X be a* UP-*algebra satisfying in condition* (19) *and F be a* UP-*filter of X. Then the following conditions are equivalent:*

(1) *F is a maximal and comparative* UP-*filter,*
(2) *F is a maximal and implicative* UP-*filter,*
(3) *if $x, y \notin F$ then $x \cdot y \in F$ and $y \cdot x \in F$, for all $x, y \in X$.*

Proof. $(1 \Rightarrow 2)$ By Proposition 4.2, the proof is clear.
$(2 \Rightarrow 3)$ Let $x, y \notin F$, for $x, y \in X$. According to Theorem 11([7]), F_y is a UP-filter. Assume that $t \in F$, we have $t \leq y \cdot t$. So $y \cdot t \in F$, i.e. $t \in F_y$. Hence $F \subseteq F_y \subseteq X$. According to hypothesis, F is a maximal UP-filter, so $F = F_y$ or $F_y = X$. As $y \cdot y = 0 \in F$, then $y \in F_y$ and $y \notin F$, so $F_y = X$. Then $x \in F_y$ and so $y \cdot x \in F$. Similarly $x \cdot y \in F$.
$(3 \Rightarrow 1)$ Assume that F is not a comparative UP-filter. Then by Lemma

$2 \cdot 6$([8]), there exist $x, y \in X$, $(x \cdot y) \cdot x \in F$ such that $x \notin F$. The following cases are considered:

(Case 1) If $y \in F$ then $x \cdot y \in F$, since $y \leq x \cdot y$. Hence $x \in F$. Since $(x \cdot y) \cdot x \in F$, which is a contradiction.

(Case 2) If $y \notin F$, then according to part (3), $x \cdot y \in F$. Since $(x \cdot y) \cdot x \in F$, then $x \in F$, that is a contradiction.

Hence F is a comparative UP-filter. Now let G be a UP-filter of X such that $F \subsetneqq G \subseteq X$ and $t \in G - F$. We need to show that $G = X$. Now let $a \notin F$. As $a \cdot a = 0 \in F$, then $a \in F_a$. Assume that $b \in F$. As $b \leq a \cdot b$, so $b \in F_a$. Then $F \subseteq F_a$. Therefore $F \cup \{a\} \subseteq F_a$. Now let H be a UP-filter of X such that $F \cup \{a\} \subseteq H$ and assume that $x \in F_a$. Then $a \cdot x \in F$ and since $F \subseteq H$, $a \cdot x \in H$. Hence $x \in H$. Therefore F_a is the least UP-filter containing F and a. Take $u \in X$. The following cases are considered:

(Case 1) If $u \in F$, then $u \in F_a$. So $X \subseteq F_a$, i.e. $X = F_a$.

(Case 2) If $u \notin F$, then based on part (3) and $a \notin F$, $a \cdot u \in F$. So $u \in F_a$. Therefore $X \subseteq F_a$, i.e. $X = F_a$.

As F_t is the least UP-filter containing F and t, so $F \subseteq F_t \subseteq G \subseteq X$. As $t \notin F$, then $F_t = X$ and so $G = X$. Therefore F is a maximal UP-filter. □

Theorem 4.9. *Let X be a* UP*-algebra satisfying in condition* (19) *and F be a* UP*-filter of X. F is a comparative* UP*-filter of X if and only if every* UP*-filter of a quotient algebra X/F is a comparative* UP*-filter.*

Proof. Let F be a comparative UP-filter of X and $x, y \in X$ such that $((x)_F * (y)_F)) * (x)_F = (0)_F$. Then $((x \cdot y) \cdot x)_F = (0)_F$ so $(x \cdot y) \cdot x \in F$. Using Lemma $2 \cdot 6$([8]), $x \in F$. So $(x)_F = (0)_F$ which proves $\{(0)_F\}$ is a comparative UP-filter. By Theorem 4.5, every UP-filter of X/F, is a comparative UP-filter. Conversely, let $(x \cdot y) \cdot x \in F$, for $x, y \in X$. Then $((x)_F * (y)_F)) * (x)_F = ((x \cdot y) \cdot x)_F = (0)_F$. Since $\{(0)_F\}$ is a comparative of X/F, then $(x)_F = (0)_F$, i.e. $x \in F$. Hence F is a comparative UP-filter of X.

□

5 Normal UP-filter

In this section we introduce a class of new UP-filters that called normal UP-filters and we give some related results.

Definition 5.1. UP-filter F of X is called a normal UP-filter if for $x, y, z \in X$, $z \cdot ((y \cdot x) \cdot x) \in F$ and $z \in F$ then $(x \cdot y) \cdot y \in F$.

Example 5.2. Let $X = \{a, b, c, 0\}$ be a set with the binary operation $\cdot$ which is given in the following table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	c
b	0	a	0	c
c	0	a	b	0

Then $(X, \cdot, 0)$ is a UP-algebra ([8]). Clearly, $F = \{b, 0\}$ is a normal UP-filter and $G = \{c, 0\}$ is not a normal, since $c \cdot ((b \cdot a) \cdot a) \in G$ and $c \in G$, but $(a \cdot b) \cdot b = b \notin G$.

According to Definition 5.1:

Theorem 5.3. *Let F be a* UP*-filter of a* UP*-algebra X. F is a normal* UP*-filter if and only if $(y \cdot x) \cdot x \in F$ implies $(x \cdot y) \cdot y \in F$, for all $x, y \in X$.*

Proof. Let F be a normal UP-filter of X and $(y \cdot x) \cdot x \in F$, for any $x, y \in X$. Since $0 \cdot ((y \cdot x) \cdot x) \in F$ and $0 \in F$, then by using hypothesis we get that $(x \cdot y) \cdot y \in F$. Conversely, let $z \cdot ((y \cdot x) \cdot x) \in F$ and $z \in F$, for any $x, y, z \in X$. As F is a UP-filter of X, $(y \cdot x) \cdot x \in F$. According to hypothesis, we conclude that $(x \cdot y) \cdot y \in F$. Therefore, F is a normal UP-filter of X.

□

Proposition 5.4. *Let X be a* UP*-algebra which satisfies in condition* (19) *and F be an implicative* UP*-filter of X. Then the following conditions are equivalent:*

(1) *F is a normal* UP*-filter of X,*
(2) *F is a comparative* UP*-filter of X,*
(3) *$(x \cdot y) \cdot x \in F$ implies $x \in F$, for all $x, y \in X$.*

Proof. According to Theorem 4.1, Theorem 5.3 and Lemma $2 \cdot 6$([8]), the proof is clear. □

Proposition 5.5. *Let X be a* UP*-algebra which satisfies in condition* (19) *and F be a comparative* UP*-filter of X. Then F is a normal* UP*-filter, but the converse is not true.*

Proof. Based on Theorem 4.1, the proof is easy. □

Example 5.6. Consider Example 5.2. It is clear that $F = \{0\}$ is a normal UP-filter, while it is not a comparative UP-filter. Since $(b \cdot a) \cdot b = 0 \in F$ but $b \notin F$.

The following example shows that every implicative UP-filter is not a normal UP-filter.

Example 5.7. In Example 4.3, $F = \{b, 0\}$ is an implicative UP-filter, while it is not a normal UP-filter, since $(c \cdot a) \cdot a = 0 \in F$ but $(a \cdot c) \cdot c = c \notin F$.

The following example shows that a normal UP-filter is not an implicative UP-filter.

Example 5.8. Consider Example 3.3(1). It is easy to check that $F = \{0\}$ is a normal UP-filter, while it is not an implicative UP-filter. Since $b \cdot (a \cdot c) = 0 \in F$ and $b \cdot a = 0 \in F$ and $b \cdot c = b \notin F$.

The follwoing example shows that the conditions in Proposition 5.4 are necessary.

Example 5.9. In Example 3.3(1), $F = \{0\}$ is not an implicative UP-filter and F is a normal UP-filter, while F is not a comparative UP-filter, since $(a \cdot b) \cdot a = 0 \in F$ and $a \notin F$.

Lemma 5.10. *Let X be a* UP*-algebra which satisfies in condition* (19). *The following conditions are equivalent:*

(1) $\{0\}$ *is a shift* UP*-filter of X,*
(2) *Every* UP*-filter of X is a shift* UP*-filter,*
(3) $((x \cdot y) \cdot y) \cdot x = y \cdot x$*, for all $x, y \in X$.*

Proof. $(1 \Leftrightarrow 2)$ Based on Corollary $4 \cdot 15$([8]), the proof is trivial.
$(1 \Rightarrow 3)$ Assume that $\{0\}$ is a shift UP-filter and $a = (y \cdot x) \cdot x$, for $x, y \in X$. Then $y \cdot a = y \cdot ((y \cdot x) \cdot x) = (y \cdot x) \cdot (y \cdot x) = 0 \in \{0\}$. Hence according to definition of a shift UP-filter, $((a \cdot y) \cdot y) \cdot a = 0$, i.e., $(a \cdot y) \cdot y = a$. As $x \leq (y \cdot x) \cdot x = a$, then $a \cdot y \leq x \cdot y$ and $(x \cdot y) \cdot y \leq (a \cdot y) \cdot y$. And also $0 = ((a \cdot y) \cdot y) \cdot a \leq ((x \cdot y) \cdot y) \cdot a$. Then $0 = ((x \cdot y) \cdot y) \cdot a = ((x \cdot y) \cdot y) \cdot ((y \cdot x) \cdot x)$, thus $(x \cdot y) \cdot y \leq (y \cdot x) \cdot x$ and similarly, $(y \cdot x) \cdot x \leq (x \cdot y) \cdot y$. Therefore $(x \cdot y) \cdot y = (y \cdot x) \cdot x$, for all $x, y \in X$. Using Lemma 3.4, $((x \cdot y) \cdot y) \cdot x = y \cdot x$, for all $x, y \in X$.
$(3 \Rightarrow 1)$ The proof is easy. □

Theorem 5.11. *Let X be a* UP*-algebra which satisfies in condition* (19) *and F be a* UP*-filter of X. Then F is a shift* UP*-filter if and only if every* UP*-filter of the quotient* UP*-algebra X/F is a shift* UP*-filter.*

Proof. Let F be a shift UP-filter of X and $x, y \in X$ such that $(x)_F * (y)_F = (0)_F$. Then $x \cdot y \in F$, and so $((y \cdot x) \cdot x) \cdot y \in F$. Hence $(((y)_F * (x)_F) * (x)_F) * (y)_F = (((y \cdot x) \cdot x) \cdot y)_F = (0)_F$, which proves that $\{(0)_F\}$ is a shift UP-filter of X/F. Based on Lemma 5.10, every UP-filter of X/F is a shift UP-filter. Conversely, suppose that every UP-filter of X/F is a shift UP-filter and $y \cdot x \in F$, for $x, y \in X$. Then $(y)_F * (x)_F = (y \cdot x)_F = (0)_F$. Since $\{(0)_F\}$ is a shift UP-filter of X/F, then $(((x \cdot y) \cdot y) \cdot x)_F = (((x)_F * (y)_F) * (y)_F) * (x)_F = (0)_F$, i.e., $((x \cdot y) \cdot y) \cdot x \in F$. Hence according to Theorem $4 \cdot 8$([8]), F is a shift UP-filter of X. □

6 Prime UP-filter

In this section, we introduce the notion of prime UP-filter in a UP-algebra. Also, we investigate some characterizations of this UP-filter and we prove that the quotient algebra induced by a prime UP-filter in a UP-algebra is a linearly ordered UP-algebra.

Definition 6.1. A UP-filter F of a UP-algebra X is called a prime UP-filter of X, if for any $x, y \in X$, $x \cdot y \in F$ or $y \cdot x \in F$.

Example 6.2. Consider the Example 3.8. Clearly, $F = \{a, 0\}$ is not a prime UP-filter, since $b \cdot c = c \notin F$ and $c \cdot b = b \notin F$. It is clear $G = \{a, b, 0\}$ is a prime UP-filter of X.

Theorem 6.3. *Let X be a* UP*-algebra satisfying in condition* (19)*. Then F is a prime* UP*-filter if and only if X/F is a linearly ordered* UP*-algebra.*

Proof. Let F be a prime UP-filter and $(x)_F$, $(y)_F \in X/F$. Then $x \cdot y \in F$ or $y \cdot x \in F$. Thus $(x)_F \leq (y)_F$ or $(y)_F \leq (x)_F$, so X/F is a chain. Conversely, let X/F be a chain. Then for all $x, y \in X$, either $(x)_F \leq (y)_F$ or $(y)_F \leq (x)_F$. Whence either $x \cdot y \in F$ or $y \cdot x \in F$, for all $x, y \in X$. Thus F is a prime UP-filter of X. □

According to Definition 6.1:

Corollary 6.4. *Let X be a* UP*-algebra and F be a prime* UP*-filter of X. Then every* UP*-filter G containing F is also a prime* UP*-filter.*

Theorem 6.5. *Let X be a* UP*-algebra. The following conditions are equivalent:*

(1) *X is a linear* UP*-algebra,*
(2) $\{0\}$ *is a prime* UP*-filter of X,*
(3) *Every* UP*-filter of X is a prime.*

Proof. $(1 \Rightarrow 2)$ Let X be a linear UP-algebra and $x, y \in X$. Hence $x \leq y$ or $y \leq x$. Then $x \cdot y = 0$ or $y \cdot x = 0$, for all $x, y \in X$. Therefore $\{0\}$ is a prime UP-filter.
$(2 \Rightarrow 1)$ Let $\{0\}$ be a prime UP-filter. Then $x \cdot y = 0$ or $y \cdot x = 0$, for all $x, y \in X$. So $x \leq y$ or $y \leq x$, for all $x, y \in X$. Therefore X is a linear UP-algebra.
$(2 \Leftrightarrow 3)$ According to Corollary 6.4, the proof is clear. □

Corollary 6.6. *Let X be a* UP*-algebra satisfying in condition* (19) *and F be a* UP*-filter of X. Then the following conditions are equivalent:*

(1) *X/F is a linearly ordered* UP*-algebra,*
(2) *F is a prime* UP*-filter of X,*
(3) *Any* UP*-filter of X/F is a prime* UP*-filter.*

Proof. $(1 \Leftrightarrow 3)$ According to Theorem 6.5, the proof is clear.
$(2 \Rightarrow 3)$ Let F be a prime UP-filter of X. We need to show that $\{(0)_F\}$ is a prime UP-filter of X/F. Assume that $(x)_F, (y)_F \in X/F$. As F is a prime UP-filter, $x \cdot y \in F$ or $y \cdot x \in F$. And so $(x \cdot y)_F = (0)_F$ or $(y \cdot x)_F = (0)_F$, hence $(x)_F \cdot (y)_F = (0)_F$ or $(y)_F \cdot (x)_F = (0)_F$. Therefore,

$\{(0)_F\}$ is a prime UP-filter of X/F.
$(3 \Rightarrow 2)$ Let any UP-filter of X/F be a prime. Then $\{(0)_F\}$ is a prime UP-filter of X/F. So $(x)_F \cdot (y)_F = (0)_F$ or $(y)_F \cdot (x)_F = (0)_F$, for all $(x)_F, (y)_F \in X/F$. Hence $x \cdot y \in F$ or $y \cdot x \in F$, for all $x, y \in X$. Therefore F is a prime UP-filter of X. □

Remark 6.7. Let X be a UP-algebra satisfying condition (19) and F, G be two UP-filters of X, such that $F \subseteq G$. Then G is a prime UP-filter of X if and only if G/F is a prime UP-filter of a UP-algebra X/F.

Lemma 6.8. *Let F be a normal and prime* UP*-filter of a* UP*-algebra X. Then $(x \cdot y) \cdot y \in F$ implies $x \in F$ or $y \in F$, for $x, y \in X$.*

Proof. Let $x, y \in X$ and $(x \cdot y) \cdot y \in F$. Then $(y \cdot x) \cdot x \in F$, since F is a normal UP-filter. As F is a prime UP-filter, so $x \cdot y \in F$ or $y \cdot x \in F$. The following cases are considered:

(Case 1) If $x \cdot y \in F$, then $y \in F$.

(Case 2) If $y \cdot x \in F$. Then using $(y \cdot x) \cdot x \in F$, $x \in F$.

Therefore, the proof is completed. □

Lemma 6.9. *The set of* UP*-filters including a given prime* UP*-filter F of X, linearly ordered with respect to the set theoretical inclusion.*

Proof. Let F be a prime UP-filter and G, H be UP-filters containing F such that $G \nsubseteq H$ and $H \nsubseteq G$. Then there exists $a \in X$ such that $a \in G - H$ and there exists $b \in X$ such that $b \in H - G$. As F is a prime UP-filter and $a, b \in X$, then $a \cdot b \in F$ or $b \cdot a \in F$. If $a \cdot b \in F$, then $a \cdot b \in G$. So $b \in G$, which is a contradiction. If $b \cdot a \in F$, then $b \cdot a \in H$. So $a \in H$, which is a contradiction. Therefore, the proof is completed. □

Theorem 6.10. *Let F be a prime* UP*-filter of a* UP*-algebra X and G be a* UP*-filter of X such that $G \subseteq F$. Then the set of all prime* UP*-filter F' of X such that $G \subseteq F' \subseteq F$, contains a minimal element.*

Proof. Take $\sum = \{F' \ : \ F' \text{ is a prime UP} - \text{filter of } X \text{ such that } \ G \subseteq F' \subseteq F\}$. Clearly, $F \in \sum$, so $\sum$ is not void. The relation $\leq$ on $\sum$ is defined $F' \leq G'$ if and only if $G' \subseteq F'$, for all $F', G' \in \sum$. Clearly, the relation $\leq$ is a partially ordered on $\sum$. Now let T be a chain on

$\sum$. Take $d = \bigcap_{F' \in T} F'$. It is clear d is a UP-filter and for all $F' \in T$, $d \subseteq F'$. Then $F' \leq d$, for all $F' \in T$. We need to prove that d is a prime UP-filter. Now let for $x, y \in X$, $x \cdot y \notin d$ and $y \cdot x \notin d$. So there exist $F', G' \in T$ such that $x \cdot y \notin F'$ and $y \cdot x \notin G'$. Since T is a chain, so $G' \subseteq F'$ or $F' \subseteq G'$. If $F' \subseteq G'$, then $y \cdot x \notin F'$. So $x \cdot y \in F'$, since F' is a prime UP-filter. That is a contradiction. Therefore d is a prime UP-filter. Also, if $G' \subseteq F'$, the process is similarly. So d is an upper bound for T. Then by Zorn's Lemma, $\sum$ contains a maximal element, i.e. it contains a minimal element. □

Proposition 6.11. *Let X be a linear* UP*-algebra and F be a* UP*-filter of X. Then for all $x, y \in X - F$, there exists $z \in X - F$ such that $x \leq z$ and $y \leq z$.*

Proof. Let F be a UP-filter of the linear UP-algebra X. According to Theorem 6.5, F is a prime UP-filter of X. Also, assume that there exist $x, y \in X - F$ such that for all $z \in X - F$, $z < x$ or $z < y$. As $x \leq (y \cdot x) \cdot x$ and $y \leq (x \cdot y) \cdot y$, so $(y \cdot x) \cdot x \in F$ and $(x \cdot y) \cdot y \in F$. Since F is a prime UP-filter, $x \cdot y \in F$ or $y \cdot x \in F$, for all $x, y \in X$. The following are cases considered:

(Case 1) If $x \cdot y \in F$, then $y \in F$, which is a contradiction.

(Case 2) If $y \cdot x \in F$, then $x \in F$, which is a contradiction.

Therefore, the proof is completed. □

Proposition 6.12. *Let X be a linear* UP*-algebra satisfying in condition* (19) *and F be a* UP*-filter of X. Then for all $x, y \in X/F$, such that $x, y \neq (0)_F$, there exists $w \in X/F$ such that $w \neq (0)_F$, $x \leq w$ and $y \leq w$.*

Proof. Let F be a UP-filter of the linear UP-algebra X. According to Theorem 6.5, F is a prime UP-filter of X. Also, assume that $x, y \in X/F$ such that $x, y \neq (0)_F$. Hence $x = (a)_F$ and $y = (b)_F$, for some $a, b \in X$, such that $(a)_F \neq (0)_F$ and $(b)_F \neq (0)_F$, i.e. $a, b \notin F$. So according to Proposition 6.11, there exists $z \in X - F$, such that $a \leq z$ and $b \leq z$. Hence $a \cdot z = 0$ and $b \cdot z = 0$, so $(a)_F \leq (z)_F$ and $(b)_F \leq (z)_F$. Therefore $x \leq w$ and $y \leq w$. □

Proposition 6.13. *Let X be a linear* UP*-algebra and F be a* UP*-filter of X. Then $[x) \cap [y) \subseteq F$ implies $x \in F$ or $y \in F$, for all $x, y \in X$.*

Proof. Let $[x) \cap [y) \subseteq F$, for $x, y \in X$, and also $x \notin F$ and $y \notin F$. According to Proposition 6.11, there exists $z \in X - F$ such that $x \leq z$ and $y \leq z$. Hence $z \in [x)$ and $z \in [y)$, i.e. $z \in [x) \cap [y) \subseteq F$, which is a contradiction. Therefore the proof is completed. □

7 Nodal UP-filter

In this section, we introduce the notion of nodal UP-filters of UP-algebras and investigate some properties of them.

Definition 7.1. A node of a UP-algebra X is an element which is comparable with every element of X. It is clear that 0 is a node in any UP-algebra.

Note. An element $x \in X$ is a node if and only if for every $y \in X$, either $x \cdot y = 0$ or $y \cdot x = 0$.

We denote , the set of all node elements of a UP-algebra X, by $\mathrm{nod}(X)$.

Example 7.2. Consider Example 4.3. Clearly, $\mathrm{nod}(X) = \{a, 0\}$.

Definition 7.3. A UP-filter F of a UP-algebra X, will be called a nodal UP-filter of X, if F is a node of $\mathrm{UF}(X)$.

Example 7.4. In the Example 4.3, $\{0\}$ and $\{b, c, 0\}$ are all of nodal UP-filters of X and $\{b, 0\}$ and $\{c, 0\}$ are not nodal UP-filters.

We denote by $\mathrm{nod}(\mathrm{UF}(X))$ the set of all nodal UP-filters of a UP-algebra X.

Theorem 7.5. *Let F be a* UP*-filter of a* UP*-algebra X. If for all $x \in X$ and for all $y \notin F$, the relation $y < x$ is satisfied, then F is a nodal* UP*-filter of X.*

Proof. Let us suppose that there exists a UP-filter G incomparable with F. Then there are elements $x, y \in X$ such that $x \in F - G$, $y \in G - F$ and $y \nless x$. Thus it is contrary, so every UP-filter G of X is comparable with F, i.e. F is a nodal UP-filter of X. □

Recall that, according to Proposition 3.5, in general, $[a)$ is not a UP-filter of X. $[a)$ is a UP-filter of X if and only if $\{0\}$ is an implicative UP-filter of X.

Theorem 7.6. *Let F be a nodal* UP*-filter of X and $\{0\}$ be an implicative* UP*-filter of X. If $x \in F$ and $y \notin F$ then $y < x$, for every $x, y \in X$.*

Proof. Let F be a nodal UP-filter of X. Hence according to Theorem 4([7]), for all $x, y \in X$, $[x) \subseteq F$ and $F \subseteq [y)$. Thus $[x) \subseteq F \subseteq [y)$, so $x \in [y)$ i.e. $y < x$. □

Corollary 7.7. *Let X be a* UP*-algebra and $\{0\}$ be an implicative* UP*-filter of X.* $\mathrm{nod}(\mathrm{UF}(X)) = \mathrm{UF}(X)$ *if and only if X is a chain.*

Proof. Let $\mathrm{nod}(\mathrm{UF}(X)) = \mathrm{UF}(X)$. According to Theorem 4([7]), $[x) \subseteq [y)$ or $[y) \subseteq [x)$, for all $x, y \in X$. Therefore $x \in [y)$ or $y \in [x)$. So $y < x$ or $x < y$. Conversely, Let X be a chain and F be a UP-filter of X. Also assume that $x \in F$ and $y \notin F$. So $x < y$ or $y < x$. If $x < y$ then $y \in F$, which is a contradiction. Hence $y < x$. So based on Theorem 7.5, F is a nodal UP-filter of X. □

Proposition 7.8. *Let X be a* UP*-algebra and $\{0\}$ be an implicative* UP*-filter of X. Then $x \in \mathrm{nod}(X)$ if and only if $[x)$ is a nodal* UP*-filter of X.*

Proof. Let $x \in \mathrm{nod}(X)$ and F be a UP-filter of X. If $x \in F$ then $[x) \subseteq F$. Now let $x \notin F$. If $F \nsubseteq [x)$, then there exists $y \in F$ such that $y \notin [x)$. So $x \nless y$ and since x is a node, then $y < x$. So $x \in F$, it is contrary. Hence if $x \notin F$ then $F \subseteq [x)$ i.e. $[x)$ is a nodal UP-filter. Conversely, let $[x)$ be a nodal UP-filter of X and $y \in X$. Then $[x) \subseteq [y)$ or $[y) \subseteq [x)$. If $[x) \subseteq [y)$ then $x \in [y)$. Therefore $y < x$. If $[y) \subseteq [x)$ then $x < y$. Therefore x is a node of X. □

Theorem 7.9. *Let X be a* UP*-algebra satisfying in condition* (19) *and F is an implicative and nodal* UP*-filter of X and x is a node of X. Then $[F \cup \{x\})$ is a nodal* UP*-filter of X.*

Proof. Let $a \in [F \cup \{x\})$. Then based on Theorem 6([7]), $x \cdot a \in F$. If $x \in F$ then $a \in F$. Therefore $[F \cup \{x\}) = F$, i.e. then $[F \cup \{x\})$ is a nodal UP-filter of X. Let $x \notin F$. According to Proposition 7.8, $[x)$ is a nodal UP-filter of X. Using Theorem $3 \cdot 6(2)$, as $F \cup \{x\} \subseteq F \cup [x)$, so $[F \cup \{x\}) \subseteq [F \cup [x))$. Now let $a \in [F \cup [x))$. So for $g \in [x)$, $g \cdot a \in F$. Since $x \leq g$ so $g \cdot a \leq x \cdot a$ and so $x \cdot a \in F$. Hence $a \in [F \cup \{x\})$, thus

$[F \cup \{x\}) = [F \cup [x)) = F \cup [x)$. It is known that union of two nodal UP-filters, is a nodal UP-filter, thus $[F \cup \{x\})$ is a nodal UP-filter of X. □

Proposition 7.10. *Let X be a* UP*-algebra. If X has n node elements and $\{0\}$ be an implicative* UP*-filter of X, then it has at least n nodal* UP*-filters.*

Proof. Let x be a node of X, then $[x)$ is a nodal UP-filter. Now assume that x and y be two node elements of X. If $[x) = [y)$ then $x \in [y)$ and $y \in [x)$. So $x \geq y$ and $y \geq x$. Thus $x = y$. Therefore, if X has n node elements, then it has at least n nodal UP-filters. □

8 Conclusion

In this paper, we investigated the properties of UP-algebras. In addition, due to the great importance of filters in logical algebras, we introduced other types of UP-filters in these algebras and studied their properties. We defined normal, prime and nodal UP-filters in UP-algebra and examined their properties. We have also proved or disproved the relationships between the types of UP-filters in these algebras with theorems or examples.

In the continuation of this article, we can define and study other types of UP-filters and study UP-algebra in more detail. In our future work, we are going to consider the notion of the radical of UP-filters and try to define other types of UP-filters in UP-algebras. We hope this work would serve as a foundation for further studies on the structure of UP-algebras and develop corresponding many-valued logical systems.

Acknowledgements
The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

References

[1] Q. P. Hu and X. Li, On BCH-algebras, *Math. Semin. Notes, Kobe Univ*, 11 (1983), 313-320.

[2] A. Iampan, A new branch of the logical algebra: UP-algebras, *Journal of Algebra and Related Topics*, 5(1) (2017), 35-54.

[3] A. Iampan, Introducing fully UP-semigroups, *Discussiones Mathematicae-General Algebra and Applications*, 38(2) (2018), 297-306.

[4] A. Iampan, The UP-isomorphism theorems for UP-algebras, *Discussiones Mathematicae General Algebra and Applications*, 39 (2019), 113-123.

[5] Y. Imai and K. Iséki, On axiom system of propositional calculi, *XIV, Proc. Japan. Acad,* 42(1) (1966), 19-22.

[6] K. Iséki, An algebra related with a propositional calculus, *Proc. Japan Acad,* 42(1) (1966), 26-29.

[7] Y. Jun and A. Iampan, Implicative UP-filters, *Afrika Matematika*, 30 (2019), 1093-1101.

[8] Y. Jun and A. Iampan, Shift UP-filters and decompositions of UP-filters in UP-algebras, *Missouri Journal of Mathematical Sciences*, 31(1) (2019), 36-45.

[9] W. Kaijae, P. Poungsumpao, S. Arayarangsi and A. Iampan, UP-algebras charactrized by their anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras, *Italian Journal of Pure and Applied Mathematics*, 36 (2016), 667692.

[10] S. Keawrahun and U. Leerawat, On isomorphisms of SU-algebras, *Sci. Magna*, 7(2) (2011), 39-44.

[11] C. Prabpayak and U. Leerawat, On ideals and congruences in KU-algebra, *Scientia Magna*, 5(1) (2009), 54-57.

[12] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw and A. Iampan, Fuzzy sets in UP-algebra, *Annals of Fuzzy Mathematics and Informatics*, 12 (2016), 739-756.

Zahra Parvizi
Department of Mathematics
PhD student
Shahrekord Branch, Islamic Azad University
Shahrekord, Iran.
E-mail: zparvizi73@yahoo.com

Somayeh Motamed
Department of Mathematics
Assistant Professor of Mathematics
Bandar Abbas Branch, Islamic Azad University
Bandar Abbas, Iran.
E-mail: s.motamed63@yahoo.com

Farhad Khaksar Haghani
Department of Mathematics
Associate Professor of Mathematics
Shahrekord Branch, Islamic Azad University
Shahrekord, Iran.
E-mail: haghani1351@yahoo.com