Journal of Mathematical Extension
Vol. 15, SI-NTFCA, (2021) (23)1-18
URL: https://doi.org/10.30495/JME.SI.2021.2094
ISSN: 1735-8299
Original Research Paper

Certain Inequalities for s-Convex Functions via Caputo Fractional Derivative and Caputo-Fabrizio Integral Operator

G. Tınaztepe
Akdeniz University

I. Yeşilce Işık
Aksaray University

S. Kemali *
Akdeniz University

G. Adilov
Akdeniz University

Abstract. In this paper, several integral inequalities for the functions whose derivatives are s-convex functions in the fourth sense are obtained by means of Caputo fractional derivative and Caputo-Fabrizio integral operator. Also the Hermite-Hadamard type inequalities for s-convex functions and their products are stated via Caputo-Fabrizio integral operator.

AMS Subject Classification: 26A33, 26D07, 26D10, 26D15, 26A51

Keywords and Phrases: Convex functions, s-convexity, Hermite-Hadamard type inequality, Caputo fractional derivatives, Caputo-Fabrizio integral operator.

Received:July 2021; Published: September 2021

*Corresponding Author

1 Introduction

The complexity of the problems encountered in the ongoing scientific and technological progresses, which cannot be easily modelled and solved via ordinary calculus, push forward the usage of the fractional calculus, which is based on the existence of the fractional order derivative. The first track of it was seen in the studies of Abel and Lacroix in early 1800s [39]. Since the extension of the order of the derivative from integer to fraction yields to fractional differential equations, the fractional derivatives and integrals have found a lot of application areas from medicine to engineering [2, 6, 7, 9, 10, 12, 24, 40–42].

There have been defined various fractional derivative and integral operators. Some of them are the Riemann-Liouville fractional derivative and integral, the Hadamard fractional derivative and integral, Grünwald-Letnikov fractional derivative, the Caputo fractional derivative and the others (see [3] and the references therein). The Caputo-Fabrizio fractional derivative is one of the most useful among them due to the fact that it has no singular kernel [8, 11, 29–31]. Also the Caputo-Fabrizio integral operator is defined and used as a fractional integral operator in many works, although it is shown that this operator is not a fractional derivative or integral operator in the context of the classical definitions of fractional calculus, but just an auxiliary operator to simplify the problem solutions [13, 15]. Nevertheless, this misusage continues in a lot of papers.

Besides the applications in different disciplines, one favor of it is the generalization of the inequalities in mathematics [5, 16, 17, 22, 32]. The investigation of the integral inequalities of some certain functions , especially convex functions, via the fractional derivatives and operator takes part in the research areas of fractional calculus. In literature, many fractional integral inequalities are given for different classes of convex functions such as exponentially convex functions, m-convex functions, quasiconvex functions, exponentional s-convex functions, harmonically convex functions [1, 18, 35, 43]. The Hermite-Hadamard inequality is the prominent one among these integral inequalities. [14, 19–21, 25–28, 38]. This article aims at making a connection between a novel convexity type and Caputo fractional derivative and integral operator in setting Hermite-Hadamard type inequalities. We focus on the obtaining the Hermite-Hadamard type inequalities for s-convex functions in the fourth sense via the Caputo fractional derivative operator and the Caputo-Fabrizio integral operator.

s-convexity is one of the generalizations of convexity and it has some applications in fractal theory [4, 44]. Its origin is based on the studies on modular spaces and Orlicz spaces [45]. In literature, essentially, four different types of s-convex functions are introduced [5, 23, 33, 36, 46]. In this paper, we state new inequalities including the Hermite-Hadamard type inequalities for s-convex functions in the fourth senses via Caputo derivative and Caputo-Fabrizio integral operator. Also, the fractional theory has many important applications.

The organization of the paper is as follows. Section 2 is separated into two subsections. The first includes the inequalities by the Caputo-Fabrizio fractional derivative for the functions whose nth derivatives are s-convex in the fourth sense. The second is allocated to the inequalities by the Caputo-Fabrizio integral operator for the functions whose second derivatives and itself are s-convex in the fourth sense, respectively. Also some inequalities for the product of this kind of functions are presented.

2 Preliminaries

In this section, some definitions and theorems needed throughout the study are given.

Definition 2.1. [46] Let $s \in (0,1]$ and U be a convex set on the vector space X. A function $\psi : U \to \mathbb{R}$ is said to be s-convex function in the fourth sense if the following inequality holds

$$\psi(\lambda x + (1-\lambda)y) \leq \lambda^{\frac{1}{s}} \psi(x) + (1-\lambda)^{\frac{1}{s}} \psi(y)$$

for all $x, y \in U$ and $\lambda \in [0,1]$.

The class of these functions is denoted by K_s^4. Using both definitions for $\lambda = \frac{1}{2}$ and $x = y$, we easily see that if $\psi \in K_s^4$, then ψ is not positive.

Theorem 2.2. *[36] Let $\psi : \mathbb{R} \to \mathbb{R}_-$ be s-convex function in the fourth sense where $s \in (0,1]$ and integrable on $[a,b] \subseteq \mathbb{R}$. Then the following inequality holds*

$$2^{\frac{1}{s}-1} \psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \psi(x)dx \leq \frac{s\left[\psi(a)+\psi(b)\right]}{s+1}.$$

Definition 2.3. [29,30] Let $AC^n[a,b]$ be a space of the functions having nth derivatives absolutely continuous, $\psi \in AC^n[a,b]$,$\alpha \notin \{1,2,3,...\}$ and $n = [\alpha]+1$ where $[.]$ denotes floor function. The left side Caputo fractional derivative is as follows

$$\left({}^C D_{a+}^{\alpha} \psi\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\psi^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a,$$

the right side Caputo fractional derivative is as follows

$$\left({}^C D_{b-}^{\alpha} \psi\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{\psi^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < b.$$

If $\alpha = n \in \{1,2,3,...\}$ and usual derivative $\psi^{(n)}(x)$ of order n exists, then Caputo fractional $\left({}^C D_{a+}^{n} \psi\right)(x)$ coincides with $\psi^{(n)}(x)$ whereas $\left({}^C D_{b-}^{n} \psi\right)(x)$ coincides with $\psi^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular, we have

$$\left({}^C D_{a+}^{0} \psi\right)(x) = \left({}^C D_{b-}^{0} \psi\right)(x) = \psi(x)$$

where $n = 1$, $\alpha = 0$.

Definition 2.4. [31] Let $H^1(a,b)$ be the Sobolev space of order 1 given as follows

$$H^1(a,b)=\left\{u\in L^2(a,b):u'\in L^2(a,b)\right\}.$$

Let $\psi\in H^1(a,b)$, $a<b$, $\alpha\in[0,1]$, then the definition of the left derivative in the sense of Caputo-Fabrizio becomes

$$\left({}_{a}^{CFD}D^{\alpha}\psi\right)(x)=\frac{B(\alpha)}{1-\alpha}\int_{a}^{x}\psi\prime(t)e^{\frac{-\alpha(x-t)^{\alpha}}{1-\alpha}}dt,\qquad x>a,$$

and the associated integral operator is

$$\left({}_{a}^{CF}I^{\alpha}\psi\right)(x)=\frac{1-\alpha}{B(\alpha)}\psi(x)+\frac{\alpha}{B(\alpha)}\int_{a}^{x}\psi(t)dt,$$

where $B(\alpha)>0$ is a normalization function satisfying $B(0)=B(1)=1$. In the cases $\alpha=0$ and $\alpha=1$, the left derivative is defined as follows

$$\left({}_{a}^{CFD}D^{0}\psi\right)(x)=\psi'(x)\quad\text{and}\quad\left({}_{a}^{CFD}D^{1}\psi\right)(x)=\psi(x)-\psi(a).$$

For the right derivative operator, we have

$$\left({}_{b}^{CFD}D^{\alpha}\psi\right)(x)=\frac{-B(\alpha)}{1-\alpha}\int_{x}^{b}\psi'(t)e^{\frac{-\alpha(t-x)^{\alpha}}{1-\alpha}}dt,\quad x<b,$$

and the associated integral operator is

$$\left({}^{CF}I_{b}^{\alpha}\psi\right)(x)=\frac{1-\alpha}{B(\alpha)}\psi(x)+\frac{\alpha}{B(\alpha)}\int_{x}^{b}\psi(t)dt$$

where $B(\alpha)>0$ is a normalization function satisfying $B(0)=B(1)=1$.

3 Main results

This section contains two subsections. First one includes some new integral inequalities obtained for the functions whose nth derivatives are s-convex functions in the fourth sense via the Caputo fractional derivative. Second one displays the results obtained by means of Caputo-Fabrizio integral operator, which covers a generalization of Hermite-Hadamard type inequalities for the s-convex functions of fourth sense and the products of them. Also, in this subsection, owing to some integral identities, new inequalities involving Caputo-Fabrizio operators are presented for the functions whose second derivatives are s-convex functions.

3.1 Some new integral inequalities with the Caputo fractional derivative

In this subsection, some inequalities are obtained for the functions whose nth order derivative is s-convex function in the fourth sense via the Caputo fractional derivative. The first theorem sets an inequality associated with the all values of an interval over which function is integrated.

Theorem 3.1. *Let $\psi : [a, b] \subset [0, \infty) \to \mathbb{R}$ be n-times differentiable function where n is a positive integer. If $\psi^{(n)}$ is s-convex function in the fourth sense, then for $\alpha, \beta > 1$ with $n > \max\{\alpha, \beta\}$, the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned}\Gamma(n-\alpha+1)\left({}^{C}D_{a^+}^{\alpha-1}\psi\right)(x) &+ \Gamma(n-\beta+1)\left({}^{C}D_{b^-}^{\beta-1}\psi\right)(x)\\ &\leq \frac{s\left[(x-a)^{n-\alpha+1}\psi^{(n)}(a) + (-1)^n (b-x)^{n-\beta+1}\psi^{(n)}(b)\right]}{1+s}\\ &+ \psi^{(n)}(x)\frac{s\left[(x-a)^{n-\alpha+1} + (-1)^n (b-x)^{n-\beta+1}\right]}{1+s}.\end{aligned}$$

for $x \in [a, b]$.

Proof. For $t \in [a, x]$, $x \in [a, b]$, $n \in N^+$and $n > \alpha$ the following inequality holds

$$(x-t)^{n-\alpha} \leq (x-a)^{n-\alpha}. \tag{1}$$

Since $\psi^{(n)} \in K_s^4$, we have

$$\psi^{(n)}(t) \leq \left(\frac{x-t}{x-a}\right)^{\frac{1}{s}} \psi^{(n)}(a) + \left(\frac{t-a}{x-a}\right)^{\frac{1}{s}} \psi^{(n)}(x). \tag{2}$$

Multiplying (1) and (2), then integrating with respect to t over $[a, x]$ we get

$$\int_a^x (x-t)^{n-\alpha}\psi^{(n)}(t)dt \leq (x-a)^{n-\alpha-\frac{1}{s}}\left[\psi^{(n)}(a)\int_a^x (x-t)^{\frac{1}{s}}dt + \psi^{(n)}(x)\int_a^x (t-a)^{\frac{1}{s}}dt\right].$$

We can write the same inequality as follows,

$$\int_a^x \frac{\psi^{(n)}(t)}{(x-t)^{\alpha-n}}dt \leq \frac{s}{1+s}(x-a)^{n-\alpha+1}\left[\psi^{(n)}(a) + \psi^{(n)}(x)\right].$$

From here, the following inequality is obtained for Caputo fractional derivative;

$$\Gamma(n-\alpha+1)\left({}^{C}D_{a^+}^{\alpha-1}\psi\right)(x) \leq \frac{s}{1+s}(x-a)^{n-\alpha+1}\left[\psi^{(n)}(a) + \psi^{(n)}(x)\right]. \tag{3}$$

Now, we consider $t \in [x, b]$ and $n > \beta$ the following inequality holds

$$(t-x)^{n-\beta} \leq (b-x)^{n-\beta}. \tag{4}$$

Since $\psi^{(n)} \in K_s^4$, then we have

$$\psi^{(n)}(t) \leq \left(\frac{t-x}{b-x}\right)^{\frac{1}{s}} \psi^{(n)}(b) + \left(\frac{b-t}{b-x}\right)^{\frac{1}{s}} \psi^{(n)}(x). \tag{5}$$

Multiplying inequalities (4) and (5), then integrating with respect to t over $[x,b]$ we have

$$\int_x^b (t-x)^{n-\beta}\psi^{(n)}(t)dt \leq (b-x)^{n-\beta-\frac{1}{s}} \left[\psi^{(n)}(b)\int_x^b (t-x)^{\frac{1}{s}}dt + \psi^{(n)}(x)\int_x^b (b-t)^{\frac{1}{s}}\,dt\right].$$

We can write the same inequality as follows;

$$\int_x^b \frac{\psi^{(n)}(t)}{(t-x)^{\beta-n}}dt \leq \frac{s}{1+s}\,(b-x)^{n-\beta-\frac{1}{s}}\left[\psi^{(n)}(b)(b-x)^{\frac{1}{s}+1} + \psi^{(n)}(x)\,(b-x)^{\frac{1}{s}+1}\,dt\right].$$

Multiplying both sides of the inequality above with $(-1)^n$ by taking into account the oddness and evenness of n, we have

$$\int_x^b (-1)^n\frac{\psi^{(n)}(t)}{(t-x)^{\beta-n}}dt \leq (-1)^n\frac{s}{1+s}\,(b-x)^{n-\beta-\frac{1}{s}}\left[\psi^{(n)}(b)(b-x)^{\frac{1}{s}+1} + \psi^{(n)}(x)\,(b-x)^{\frac{1}{s}+1}\,dt\right].$$

From here, the following inequality is obtained for Caputo fractional derivative;

$$\Gamma\left(n-\beta+1\right)\left({}^{C}D_{b^-}^{\beta-1}\psi\right)(x) \leq (-1)^n\frac{s}{1+s}\,(b-x)^{n-\beta+1}\left[\psi^{(n)}(b)+\psi^{(n)}(x)\right]. \tag{6}$$

Adding to (3) and (6) side by side, we have the desired inequality. □

Corollary 3.2. *If we take $\alpha=\beta$ in Theorem 3.1, then we have the following fractional derivative inequality*

$$\begin{aligned}\Gamma\left(n-\alpha+1\right)&\left[\left({}^{C}D_{a^+}^{\alpha-1}\psi\right)(x)+\left({}^{C}D_{b^-}^{\alpha-1}\psi\right)(x)\right]\\ &\leq \frac{s\left[(x-a)^{n-\alpha+1}\psi^{(n)}(a)+(-1)^n\,(b-x)^{n-\alpha+1}\,\psi^{(n)}(b)\right]}{1+s}\\ &\quad+\psi^{(n)}(x)\frac{s\left[(x-a)^{n-\alpha+1}+(-1)^n\,(b-x)^{n-\alpha+1}\right]}{1+s}.\end{aligned}$$

Corollary 3.3. *If we take $\alpha=\beta$, $s=1$ in Theorem 3.1, then we have the following fractional derivative inequality given in [17]*

$$\begin{aligned}\Gamma\left(n-\alpha+1\right)&\left[\left({}^{C}D_{a^+}^{\alpha-1}\psi\right)(x)+\left({}^{C}D_{b^-}^{\alpha-1}\psi\right)(x)\right]\\ &\leq \frac{(x-a)^{n-\alpha+1}\psi^{(n)}(a)+(-1)^n\,(b-x)^{n-\alpha+1}\,\psi^{(n)}(b)}{2}\\ &\quad+\psi^{(n)}(x)\frac{(x-a)^{n-\alpha+1}+(-1)^n\,(b-x)^{n-\alpha+1}}{2}.\end{aligned}$$

Remark 3.4. When n is even number in Corollary 3.3, the inequality in Corollary 2.1 in [17] is obtained.

The following theorem sets an inequality involving Caputo fractional derivative, which is stated for only the end points of the interval over which function is integrated.

Theorem 3.5. *Let $\psi : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a n-times differentiable function where n is a positive integer. If $\psi^{(n)}$ is s-convex function in the fourth sense and integrable on $[a,b]$, then the following inequality for Caputo fractional derivatives holds*

$$\frac{2^{\frac{1}{s}}}{n-\alpha}\psi^{(n)}(\frac{a+b}{2}) \leq \frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left[\left({}^{C}D^{\alpha}_{a+}\psi\right)(b) + (-1)^n\left({}^{C}D^{\alpha}_{b-}\psi\right)(a)\right]$$
$$\leq \left(\psi^{(n)}(a)+\psi^{(n)}(b)\right)\left[\frac{s}{ns-\alpha s+1} + \frac{\Gamma(1+\frac{1}{s})\Gamma(n-\alpha)}{\Gamma\left(n-\alpha+\frac{1}{s}+1\right)}\right]$$

or

$$\frac{2^{\frac{1}{s}}}{n-\alpha}\psi^{(n)}(\frac{a+b}{2}) \leq \frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left[\left({}^{C}D^{\alpha}_{a+}\psi\right)(b) + (-1)^n\left({}^{C}D^{\alpha}_{b-}\psi\right)(a)\right]$$
$$\leq \left(\psi^{(n)}(a)+\psi^{(n)}(b)\right)\left[\frac{s}{ns-\alpha s+1} + B\left(1+\frac{1}{s}, n-\alpha\right.\right.$$

Proof. Since $\psi^{(n)} \in K^4_s$, for $x, y \in [a,b]$ we have

$$\psi^{(n)}(\frac{x+y}{2}) \leq \frac{\psi^{(n)}(x)+\psi^{(n)}(y)}{2^{\frac{1}{s}}}. \tag{7}$$

Let $x = ta + (1-t)b$, $y = (1-t)a + tb$ for $t \in [0,1]$. Then the inequality (7) gives

$$2^{\frac{1}{s}}\psi^{(n)}(\frac{a+b}{2}) \leq \psi^{(n)}(ta+(1-t)b) + \psi^{(n)}((1-t)a+tb). \tag{8}$$

Multiplying both sides of the inequality (8) by $t^{n-\alpha-1}$, then integrating with respect to t over $[0,1]$ we get

$$2^{\frac{1}{s}}\psi^{(n)}(\frac{a+b}{2})\int_0^1 t^{n-\alpha-1}dt$$
$$\leq \int_0^1 t^{n-\alpha-1}\psi^{(n)}(ta+(1-t)b)dt + \int_0^1 t^{n-\alpha-1}\psi^{(n)}((1-t)a+tb)dt.$$

Thus, we have the following inequalities;

$$\frac{2^{\frac{1}{s}}}{n-\alpha}\psi^{(n)}(\frac{a+b}{2}) \leq \frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left[\left({}^{C}D^{\alpha}_{a+}\psi\right)(b) + (-1)^n\left({}^{C}D^{\alpha}_{b-}\psi\right)(a)\right]. \tag{9}$$

On the other hand, the convexity of $\psi^{(n)}$ gives

$$\psi^{(n)}(ta+(1-t)b) + \psi^{(n)}((1-t)a+tb) \leq \left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\left(\psi^{(n)}(a)+\psi^{(n)}(b)\right). \tag{10}$$

Multiplying both sides of inequality (10) by $t^{n-\alpha-1}$, then integrating with respect to t over $[0,1]$ we get

$$\int_0^1 t^{n-\alpha-1}\psi^{(n)}(ta+(1-t)b)dt+\int_0^1 t^{n-\alpha-1}\psi^{(n)}((1-t)a+tb)dt \le \int_0^1 t^{n-\alpha-1}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\left(\psi^{(n)}(a)+\psi^{(n)}(b)\right)dt,$$

so one can have

$$\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left[\left({}^{C}D^{\alpha}_{a+}\psi\right)(b)+(-1)^n\left({}^{C}D^{\alpha}_{b-}\psi\right)(a)\right] \le \left(\psi^{(n)}(a)+\psi^{(n)}(b)\right)\left[\frac{s}{ns-\alpha s+1}+\frac{\Gamma(1+\frac{1}{s})\Gamma(n-\alpha)}{\Gamma\left(n-\alpha+\frac{1}{s}+1\right)}\right]. \tag{11}$$

Inequalities (9) and (11) give required inequality. □

3.2 Various inequalities involving the Hermite-Hadamard type inequalities via Caputo-Fabrizio integral operator

In this subsection, Hermite-Hadamard type inequalities for s-convex functions of the fourth sense and their products are obtained in terms of Caputo-Fabrizio integral operators. Also some inequalities are stated via integral identities involving Caputo-Fabrizio integral operators are given for the functions whose second derivatives are s-convex functions.

In the following theorem, the Hermite-Hadamard type inequalities and related inequalities are obtained for the s-convex functions in the fourth sense by means of Caputo-Fabrizio type integral operators.

Theorem 3.6. *Let $\psi:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be s-convex function in the fourth sense and integrable on $[a,b]$. The following double inequality holds for $\alpha\in[0,1]$,*

$$2^{\frac{1}{s}-1}\psi\left(\frac{a+b}{2}\right)\le\frac{B(\alpha)}{\alpha(b-a)}\left[\left({}^{CF}_{a}I^{\alpha}\psi\right)(x)+\left({}^{CF}I^{\alpha}_{b}\psi\right)(x)-\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\right] \le\frac{s\left(\psi(a)+\psi(b)\right)}{s+1}$$

where $x\in[a,b]$, $B(\alpha)>0$ is normalization function.

Proof. From Theorem 2.2, we have

$$2^{\frac{1}{s}-1}\psi\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b\psi(t)dt\le\frac{s\left(\psi(a)+\psi(b)\right)}{s+1}. \tag{12}$$

By multiplying both sides of (12) with $\frac{\alpha(b-a)}{B(\alpha)}$, we get

$$\frac{\alpha(b-a)}{B(\alpha)}2^{\frac{1}{s}-1}\psi\left(\frac{a+b}{2}\right)\leq\frac{\alpha}{B(\alpha)}\int\limits_a^b\psi(t)dt\leq\frac{\alpha(b-a)}{B(\alpha)}\frac{s\left(\psi(a)+\psi(b)\right)}{s+1},$$

and adding $\frac{2(1-\alpha)}{B(\alpha)}\psi(x)$, we get

$$\begin{aligned}\frac{\alpha(b-a)}{B(\alpha)}2^{\frac{1}{s}-1}\psi\left(\frac{a+b}{2}\right)+\frac{2(1-\alpha)}{B(\alpha)}\psi(x)&\leq\frac{\alpha}{B(\alpha)}\int\limits_a^b\psi(t)dt+\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\\&\leq\frac{\alpha(b-a)}{B(\alpha)}\frac{s\left(\psi(a)+\psi(b)\right)}{s+1}+\frac{2(1-\alpha)}{B(\alpha)}\psi(x).\end{aligned}\tag{13}$$

Considering the left side of inequality (13), we have

$$\begin{aligned}\frac{\alpha(b-a)}{B(\alpha)}2^{\frac{1}{s}-1}\psi\left(\frac{a+b}{2}\right)+\frac{2(1-\alpha)}{B(\alpha)}\psi(x)&\leq\frac{\alpha}{B(\alpha)}\int\limits_a^x\psi(t)dt+\frac{(1-\alpha)}{B(\alpha)}\psi(x)\\&+\frac{\alpha}{B(\alpha)}\int\limits_x^b\psi(t)dt+\frac{(1-\alpha)}{B(\alpha)}\psi(x)\\&=\left({}_a^{CF}I^{\alpha}\psi\right)(x)+\left({}^{CF}I_b^{\alpha}\psi\right)(x).\end{aligned}\tag{14}$$

Considering the right side of inequality (13), we have

$$\begin{aligned}\frac{\alpha(b-a)}{B(\alpha)}\frac{s\left(\psi(a)+\psi(b)\right)}{s+1}+\frac{2(1-\alpha)}{B(\alpha)}\psi(x)&\geq\frac{\alpha}{B(\alpha)}\int\limits_a^x\psi(t)dt+\frac{(1-\alpha)}{B(\alpha)}\psi(x)\\&+\frac{\alpha}{B(\alpha)}\int\limits_x^b\psi(t)dt+\frac{(1-\alpha)}{B(\alpha)}\psi(x)\\&=\left({}_a^{CF}I^{\alpha}\psi\right)(x)+\left({}^{CF}I_b^{\alpha}\psi\right)(x).\end{aligned}\tag{15}$$

If $\frac{2(1-\alpha)}{B(\alpha)}\psi(x)$ is subtracted from both sides of the inequalities (14) and (15), divided by both sides by $\frac{\alpha(b-a)}{B(\alpha)}$, and the two inequalities are combined, then the proof is complete. □

In the following theorems, using integral operators of Caputo-Fabrizio type, we obtain the Hermite-Hadamard type inequalities for the product of s-convex functions in the fourth sense.

Theorem 3.7. *Let $\psi:[a,b]\subseteq\mathbb{R}\to\mathbb{R}_-$ be s_1-convex function in the fourth sense and $\phi:[a,b]\subseteq\mathbb{R}\to\mathbb{R}_-$ be s_2-convex function in the fourth sense. If $\psi\phi$ integrable* *on* $[a,b]$*, then the following inequality holds for* $\alpha \in [0,1]$*,* $x \in [a,b]$*,*

$$\frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{CF}I^{\alpha}\psi\phi\right)(x)+\left({}^{CF}I_{b}^{\alpha}\psi\phi\right)(x)-\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)\right] \geq \frac{s_1 s_2}{s_1+s_1 s_2+s_2}M(a,b)+\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})N(a,b)$$

where $B(\alpha)>0$ *is a normalization function and*

$$M(a,b)=\psi(a)\phi(a)+\psi(b)\phi(b), \quad N(a,b)=\psi(a)\phi(b)+\psi(b)\phi(a).$$

Proof. Since $\psi \in K_{s_1}^4$, and $\phi \in K_{s_2}^4$, we have following inequalities

$$\psi(\lambda a+(1-\lambda)b) \leq \lambda^{\frac{1}{s_1}}\psi(a)+(1-\lambda)^{\frac{1}{s_1}}\psi(b),$$

and

$$\phi(\lambda a+(1-\lambda)b) \leq \lambda^{\frac{1}{s_2}}\phi(a)+(1-\lambda)^{\frac{1}{s_2}}\phi(b).$$

We know that ψ and ϕ are negative functions since these are s-convex functions in the fourth sense. If we multiply these inequalities side by side and integrate with λ over $[0,1]$, we have

$$\int_0^1 \psi(\lambda a+(1-\lambda)b)\phi(\lambda a+(1-\lambda)b)d\lambda$$
$$\geq \int_0^1 \left[\lambda^{\frac{1}{s_1}}\psi(a)+(1-\lambda)^{\frac{1}{s_1}}\psi(b)\right]\left[\lambda^{\frac{1}{s_2}}\phi(a)+(1-\lambda)^{\frac{1}{s_2}}\phi(b)\right]d\lambda$$
$$= \frac{s_1 s_2}{s_1+s_1 s_2+s_2}M(a,b)+\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})N(a,b).$$

If we substitute variables $t=\lambda a+(1-\lambda)b$, this inequality gives the following inequality;

$$\frac{1}{b-a}\int_a^b \psi(t)\phi(t)dt \geq \frac{s_1 s_2}{s_1+s_1 s_2+s_2}M(a,b)+\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})N(a,b). \tag{16}$$

By multiplying both sides of (16) with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)$, we have

$$\frac{\alpha(b-a)}{B(\alpha)}\left[\frac{s_1 s_2}{s_1+s_1 s_2+s_2}M(a,b)+\beta(1+\frac{1}{s_1},1+\frac{1}{s_2})N(a,b)\right]+\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)$$
$$\leq \frac{\alpha}{B(\alpha)}\left[\int_a^x \psi(t)\phi(t)dt+\int_x^b \psi(t)\phi(t)dt\right]+\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)$$
$$= \frac{\alpha}{B(\alpha)}\int_a^x \psi(t)\phi(t)dt+\frac{(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)+\frac{\alpha}{B(\alpha)}\int_x^b \psi(t)\phi(t)dt+\frac{(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)$$
$$= \left({}_{a}^{CF}I^{\alpha}\psi\phi\right)(x)+\left({}^{CF}I_{b}^{\alpha}\psi\phi\right)(x).$$

If $\frac{2(1-\alpha)}{B(\alpha)}\psi(x)$ is subtracted from both sides of this inequality and divided by $\frac{\alpha(b-a)}{B(\alpha)}$, the desired inequality is achieved. □

Corollary 3.8. *If we take $s_1 = s_2 = 1$, then inequality in Theorem 3.5 is reduced to the following inequality*

$$\frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{CF}I^{\alpha}\psi\phi\right)(x)+\left({}^{CF}I_{b}^{\alpha}\psi\phi\right)(x)-\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)\right]$$
$$\geq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b).$$

Theorem 3.9. *Let $\psi:[a,b]\subseteq\mathbb{R}\to\mathbb{R}_-$ be s_1-convex function in the fourth sense and $\phi:[a,b]\subseteq\mathbb{R}\to\mathbb{R}_-$ be s_2-convex function in the fourth sense. If $\psi\phi$ integrable on $[a,b]$, then the following inequality holds for $\alpha\in[0,1]$, $x\in[a,b]$*

$$\frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{CF}I^{\alpha}\psi\phi\right)(x)+\left({}^{CF}I_{b}^{\alpha}\psi\phi\right)(x)-\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)\right]$$
$$\leq 8\psi\left(\frac{a+b}{2}\right)\phi\left(\frac{a+b}{2}\right)-\frac{1}{6}N(a,b)-\frac{1}{3}M(a,b)$$

where $B(\alpha)>0$ is a normalization function and $M(a,b)$ and $N(a,b)$ are denoted as in Theorem 3.7.

Proof. In [37], from Corollary 2 we have the following inequality

$$\frac{1}{b-a}\int_a^b\psi(t)\phi(t)dt\leq 8\psi\left(\frac{a+b}{2}\right)\phi\left(\frac{a+b}{2}\right)-\frac{1}{3}N(a,b)-\frac{1}{6}M(a,b). \tag{17}$$

By multiplying both sides of (17) with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)$, we have

$$\frac{\alpha(b-a)}{B(\alpha)}\int_a^x\psi(x)\phi(x)dx+\frac{(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)+\frac{\alpha(b-a)}{B(\alpha)}\int_x^b\psi(x)\phi(x)dx+\frac{(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)$$
$$\leq\frac{\alpha(b-a)}{B(\alpha)}\left[8\psi\left(\frac{a+b}{2}\right)\phi\left(\frac{a+b}{2}\right)-\frac{1}{3}N(a,b)-\frac{1}{6}M(a,b)\right]+\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x).$$

□

In the following result, a generalization of Corollary 8 in [37] is presented via Caputo-Fabrizio integral operators.

Corollary 3.10. *Using Theorem 3.9 and Corollary 3.8, we have the following double inequality*

$$\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)\leq\frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{CF}I^{\alpha}\psi\phi\right)(x)+\left({}^{CF}I_{b}^{\alpha}\psi\phi\right)(x)-\frac{2(1-\alpha)}{B(\alpha)}\psi(x)\phi(x)\right]$$
$$\leq 8\psi\left(\frac{a+b}{2}\right)\phi\left(\frac{a+b}{2}\right)-\frac{1}{6}N(a,b)-\frac{1}{3}M(a,b).$$

Using the following lemma, we can obtain an inequality via the Caputo-Fabrizio integral operators for the functions whose second derivative is s-convex function in the fourth sense.

Lemma 3.11. *[16] Let $\psi : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on (a,b). If $\psi'' \in L_1[a,b]$ and $\alpha \in [0,1]$, the following equality holds*

$$\frac{\psi(a)+\psi(b)}{2} - \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{CF}I^{\alpha}\psi\right)(x) + \left({}^{CF}I_{b}^{\alpha}\psi\right)(x)\right] + \frac{2(1-\alpha)}{\alpha(b-a)}\psi(x)$$
$$= \frac{(b-a)^2}{2}\int_0^1 \lambda(1-\lambda)\psi''(\lambda a + (1-\lambda)b)d\lambda$$

where $x \in [a,b]$ and $B(\alpha) > 0$ is a normalization function.

Theorem 3.12. *Let $\psi : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{-}$ be twice differentiable on (a,b) and ψ'' be s-convex function in the fourth sense. The following inequality holds*

$$\frac{\psi(a)+\psi(b)}{2} - \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{CF}I^{\alpha}\psi\right)(x) + \left({}^{CF}I_{b}^{\alpha}\psi\right)(x)\right] + \frac{2(1-\alpha)}{\alpha(b-a)}\psi(x)$$
$$\leq \frac{(s(b-a))^2}{(1+2s)(1+3s)}\frac{\psi''(b)+\psi''(a)}{2}$$

where $x \in [a,b]$ and $\alpha \in [0,1]$, $B(\alpha) > 0$ is a normalization function.

Proof. From Lemma 3.11 and convexity of ψ'', we have

$$\frac{\psi(a)+\psi(b)}{2} - \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{CF}I^{\alpha}\psi\right)(x) + \left({}^{CF}I_{b}^{\alpha}\psi\right)(x)\right] + \frac{2(1-\alpha)}{\alpha(b-a)}\psi(x)$$
$$\leq \frac{(b-a)^2}{2}\int_0^1 \lambda(1-\lambda)\left[\lambda^{\frac{1}{s}}\psi''(a) + (1-\lambda)^{\frac{1}{s}}\psi''(b)\right]d\lambda$$
$$= \frac{(b-a)^2}{2}\int_0^1 \left[\left(\lambda^{\frac{1}{s}+1} - \lambda^{\frac{1}{s}+2}\right)\psi''(a) + \lambda(1-\lambda)^{\frac{1}{s}+1}\psi''(b)\right]d\lambda$$
$$= \frac{(b-a)^2}{2}\frac{s^2}{(1+2s)(1+3s)}\left[\psi''(a)+\psi''(b)\right].$$

□

Corollary 3.13. *If we choose $s = 1$ in Theorem 3.12, then the following inequality is obtained*

$$\frac{\psi(a)+\psi(b)}{2} - \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{CF}I^{\alpha}\psi\right)(x) + \left({}^{CF}I_{b}^{\alpha}\psi\right)(x)\right] + \frac{2(1-\alpha)}{\alpha(b-a)}\psi(x)$$
$$\leq (b-a)^2\frac{\psi''(b)+\psi''(a)}{48}.$$

4 Applications

We consider the applications of some of the results to the special means. Let us recall the following means for positive real numbers a, b.

Let a, b, p be positive number with $a \neq b$ and $p \in Z/\{-1, 0\}$,

$$A(a,b) = \frac{a+b}{2},$$

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}},$$

$$L_p(a,b) = \begin{cases} a & , \quad \text{if } a = b \\ \left(\frac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right)^{1/p} & , \quad \text{else} \end{cases}$$

are called Arithmetic mean, Power mean and Stolarsky mean (Generalized logarithmic mean) respectively.

Proposition 4.1. *Let $a, b \in \mathbb{R}_+$ with $a < b$. The inequality holds*

$$2^{1-s} A(a,b) \geq L_{\frac{1+s}{s}}(a,b) \geq \left(\frac{2s}{s+1}\right)^s M_{\frac{1}{s}}(a,b)$$

for all $s \in (0, 1]$.

Proof. The assertion follows from Theorem 3.6 applied to the s-convex function in the fourth sense $\psi(x) = -x^{\frac{1}{s}}$, $x \in [a, b]$ and $\alpha = 1$, $B(\alpha) = 1$,

$$\frac{(a+b)^{\frac{1}{s}}}{2} \geq \frac{s\left(b^{\frac{1}{s}+1} - a^{\frac{1}{s}+1}\right)}{(b-a)(s+1)} \geq \frac{s}{s+1}\left(a^{\frac{1}{s}} + b^{\frac{1}{s}}\right). \tag{18}$$

If s-th power of each side is taken in this inequality, then we get

$$\frac{a+b}{2^s} \geq \left[\frac{b^{\frac{1}{s}+1} - a^{\frac{1}{s}+1}}{\left(\frac{1}{s}+1\right)(b-a)}\right]^s \geq \left(\frac{s}{s+1}\right)^s \left(a^{\frac{1}{s}} + b^{\frac{1}{s}}\right)^s.$$

We can write this inequality as follows

$$2^{1-s}\frac{a+b}{2} \geq \left[\frac{b^{\frac{1}{s}+1} - a^{\frac{1}{s}+1}}{\left(\frac{1}{s}+1\right)(b-a)}\right]^s \geq \left(\frac{2s}{s+1}\right)^s \left(\frac{a^{\frac{1}{s}} + b^{\frac{1}{s}}}{2}\right)^s.$$

From this inequality, we get

$$2^{1-s} A(a,b) \geq S_{\frac{1+s}{s}}(a,b) \geq \left(\frac{2s}{s+1}\right)^s M_{\frac{1}{s}}(a,b).$$

□

Using the results, one can obtain some inequalities relevant to special function expressed by integral. To illustrate, we present one of them.

Proposition 4.2. *Let $x \geq 3$. Then*

$$\frac{x}{x-1} \leq \Psi(x) + \gamma \leq \frac{2^{x-1}-1}{2}$$

where $\Psi(x)$ is digamma function, i.e.

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \text{ for } x > 0$$

and γ is Euler-Mascheroni constant i.e. $\gamma \approx 0.5772156649...$

Proof. Let us suppose a, b as the same in Proposition 4.1, write $t = \frac{a}{b}$ and simplify the expression in (18). Then we have

$$\frac{(1+t)^{\frac{1}{s}}}{2} \geq \frac{s}{s+1} \frac{1-t^{\frac{1}{s}+1}}{1-t} \geq \frac{s}{s+1}\left(1+t^{\frac{1}{s}}\right).$$

Integrating the inequality with respect to t on $[0,1]$ and simplifying the expression, we have

$$2^{\frac{1}{s}} - 2^{-1} \geq \int_0^1 \frac{1-t^{\frac{1}{s}+1}}{1-t} dt \geq \frac{2s+1}{s+1}$$

By making use of the following integral representation of digamma function given in [34],

$$\Psi(r) = \int_0^1 \frac{1-t^{r-1}}{1-t} dt - \gamma$$

where $r > 0$, it is obtained that

$$2^{\frac{1}{s}} - 2^{-1} \geq \Psi(2 + \frac{1}{s}) + \gamma \geq \frac{2s+1}{s+1}.$$

The substitution $x = 2 + \frac{1}{s}$ above yields to

$$\frac{2^{x-1}-1}{2} \geq \Psi(x) + \gamma \geq \frac{x}{x-1}$$

for $x \geq 3$. □

5 Conclusion

The restatement of certain integral inequalities for the certain function classes via Caputo fractional derivative and Caputo-Fabrizio operators is one of the common topics in fractal calculus. In this paper, Hermite-Hadamard type and related inequalities via Caputo fractional derivative and Caputo-Fabrizio operators are studied for s-convex functions in the fourth sense on the real interval. Some Hermite-Hadamard type inequalities by means of Caputo fractional derivatives for the functions whose nth derivative is s-convex in the fourth sense are obtained. Also some integral inequalities via Caputo-Fabrizio operators for the functions whose second derivatives and themselves are the s-convex functions in the fourth sense are given. By using the results, some inequalities involving means and special functions such as digamma function are obtained as applications. In future works, if the generalization of s-convex functions in the fourth sense on fractal sets likewise in the second sense can be achieved, some connections between fractals and Caputo fractional may be accessed via the obtained inequalities in this paper.

References

[1] A. Ali, G. Gulshan, R. Hussain, A. Latif, M. Muddassar, Generalized Inequalities of the type of Hermite-Hadamard-Fejer with Quasi-Convex Functions by way of k-Fractional Derivatives. Journal of Computational Analysis Applications, 22(1), (2017).

[2] A. Alsaedi, D. Baleanu, S. Etemad, S. Rezapour, On coupled systems of time-fractional differential problems by using a new frac-tional derivative. *J. Funct. Spaces*, (2016), 8 pages, Article ID 4626940.

[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional diferential equations* . North-Holland Mathematics, Studies, Elsevier, New York-London, 2006.

[4] A. Kılıçman, W. Saleh, Notions of generalized s-convex functions on fractal sets. *Journal of Inequalities and Applications*, (1), (2015), 1-16.

[5] B. Meftah, B. Benssaad, M. Kaidouchi, W.S. Ghomrani, Confortable fractional Hermite-Hadamard type inequalities for product of two Harmonic s-convex functions,*Proceedings of the American Mathematical Society*, Article electronically published on February 5, (2021), https://doi.org/10.1090/proc/15396

[6] D. Baleanu, A. Jajarmi, H. Mohammadi, S. Rezapour, A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative, *Chaos, Solitons Fractals*, 134, (2020), 109705.

[7] D. Baleanu, H. Mohammadi, S. Rezapour, Analysis of the model of HIV-1 infection of CD4+ $CD4^+$ T-cell with a new approach of fractional derivative, *Advances in Difference Equations*, (1), (2020), 1-17.

[8] D. Baleanu, S.S. Sajjadi, A. Jajarmi, O. Defterli, J.H. Asad, P. Tulkarm, The fractional dynamics of a linear triatomic molecule, *Rom. Rep. Phys.* , 73(1), (2021), 105.

[9] D. Baleanu, S.S. Sajjadi, A. Jajarmi, O. Defterli, On a nonlinear dynamical system with both chaotic and nonchaotic behaviors: a new fractional analysis and control,*Advances in Difference Equations*, (1), (2021), 1-17.

[10] D. Baleanu, S.S. Sajjadi, J.H. Asad, A. Jajarmi, E. Estiri, Hyperchaotic behaviors, optimal control, and synchronization of a nonautonomous cardiac conduction system, *Advances in Difference Equations*, (1), (2021), 1-24.

[11] D. Baleanu, S. Zibaei, M. Namjoo, A Jajarmi, A nonstandard finite difference scheme for the modeling and nonidentical synchronization of a novel fractional chaotic system, *Advances in Difference Equations*, (1), (2021).

[12] D. Baleanu, A. Jajarmi, H. Mohammadi, S. Rezapour, A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. Chaos, Solitons Fractals, 134, (2020), 109705.

[13] D. Manuel, J. Ortigueira, T. Machado, A critical analysis of the Caputo-Fabrizio operators,*Communications in Nonlinear Science and Numerical Simulation*, Volume 59, (2018), 608-611.

[14] D. Bertsimas, I. Popescu, Optimal inequalities in probability theory, *A Convex Optimization Approach Sloan* WP #4025, June 1998.

[15] E.V. Tarasov, No nonlocality. No fractional derivative, *Communications in Nonlinear Science and Numerical Simulation*, Volume 62, (2018), 157-163.

[16] E.R. Nwaeze, S Kermausuor, Caputo–Fabrizio fractional Hermite–Hadamard type and associated results for strongly convex functions,*The Journal of Analysis* https://doi.org/10.1007/s41478-021-00315-8.

[17] G. Farid, On Caputo fractional derivatives via convexity, *Kragujevac Journal of Math*, 44(3), (2020), 393-399.

[18] G. Farid, A. Javed, A.U. Rehman, Fractional integral inequalities of Hadamard type for m-convex functions via Caputo k-fractional derivatives, ıJournal of Fractional Calculus and Applications, 10(1), (2019), 120-134.

[19] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*,Cambridge, UK : Cambridge University Press, Cambridge mathematical library, 1952.

[20] G. Tinaztepe, S. Kemali, S. Sezer, Z. Eken, The sharper form of Brunn-Minkowski type inequality for boxes,*Hacettepe Journal of Mathematics and Statistics* .31(47), (2020), 1-10.

[21] G. Adilov, G. Tinaztepe, R. Tinaztepe, On the global minimization of increasing positively homogeneous functions over the unit simplex, *International Journal of Computer Mathematics*, 87, (2010), 2733-2746.

[22] G. Farid, S. Naqvi, A.U. Rehman, Aversion of the Hadamard inequality for Caputo fractional derivatives and related results.*RGMIA, Research Report Collection*, 20, (2017).

[23] H. Hudzik, L. Maligranda, Some remarks on s-convex functions.*Aequ. Math.* 48(1), (1994), 100-111.

[24] H. Mohammadi, S. Kumar, S. Rezapour, S. Etemad, A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control, *Chaos, Solitons Fractals*, 144, (2021), 110668.

[25] I. Yesilce, G. Adilov, Hermite-Hadamard type inequalities for B^{-1}-convex functions involving generalized fractional integral operators, *Filomat* 32,18 (2018), 6457-6464.

[26] I. Yesilce, G. Adilov, Fractional integral inequalities for B-convex functions, *Creat. Math.* , Inform 26.3 (2017), 345-351.

[27] I. Yesilce, Some inequalities for B^{-1}- convex functions via fractional integral operator, *TWMS Journal of Applied and Engineering Mathematics* 9.3 (2019), 620-625.

[28] K. Lange, Convexity, optimization and inequalities, *Applied Probability* , (2010), 55-73.

[29] M. Caputo, Linear model of dissipation whose Q is almost frequency independent, *Geophysical Journal International*, 13(5), (1967), 529-539.

[30] M. Caputo, Elasticita e Dissipazione, *Zanichelli*, Bologna, 1969.

[31] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel,*Progr. Fract. Differ. Appl.* 1(2), (2015),73-85.

[32] M. Gurbuz, A.O. Akdemir, S. Rashid, E. Set, Hermite-Hadamard inequality for fractional integrals of Caputo-Fabrizio type and related inequalities, *J. Ineq. and Appl.* 2020, http://doi.org/10.1186/s13660-020-02438-1http://hdl.handle.net/11489/2037.

[33] M.A. Khan, Y. Chu, T.U. Khan, J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications.*Open Mathematics*, 15(1), (2017), 1414-1430, https://doi.org/10.1515/math-2017-0121

[34] S.S. Dragomir, R.P. Agarwal, N.S. Barnett, Inequalities for beta and gamma functions via some classical and new integral inequalities.*RGMIA research report collection* , 2(3), (1999).

[35] S. Butt, M. Nadeem, G. Farid , On Caputo Fractional Derivatives via Exponential s-Convex Functions, *Turkish Journal of Science*, vol. 5, no. 2, pp. 140-146, Oct. 2020

[36] S. Kemali, Hermite-Hadamard inequality for s-convex functions in the fourth sense. Submitted to Turkish Journal of Mathematics.

[37] S. Kemali, S. Sezer Evcan, I. Yesilce, G. Adilov, Hermite-Hadamard type inequalities for product of s-convex functions in the fourth sense. *International Journal of Nonlinear Analysis and Applications,* Accepted to online publish.

[38] S. Kemali, G. Tinaztepe, G. Adilov, New type inequalities for B–convex functions involving Hadamard fractional integral,*Facta Universitatis Series Mathematics and Informatics*, 33.5 (2018), 697-704.

[39] S.F. Lacroix, *Traité du Calcul Différentiel et du Calcul Intégral.* 2nd Edition, Mme, Paris, Ve Courcier, Tome Troisiéme, (1819), 409-410.

[40] S. Rezapour, S. Etemad, H. Mohammadi, A mathematical analysis of a system of Caputo–Fabrizio fractional differential equations for the anthrax disease model in animals. Advances in Difference Equations, *Advance Different Equations* 481 (1), (2020), 1-30.

[41] S. Rezapour, H. Mohammadi, A. Jajarmi, A new mathematical model for Zika virus transmission, *Advances in Difference Equations*, (1), (2020), 1-15.

[42] S.S. Ray, A. Atangana, S.C.O. Noutchie, M. Kurulay, N. Bildik, A. Kılıcman, Fractional Calculus and Its Applications in Applied Mathematics and Other Sciences, *Mathematical Problems in Engineering*, vol. 2014, Article ID 849395, (2014), 2 pages.

[43] S. Rashid, M.A. Noor, K.I. Noor, Y.M. Chu, Ostrowski type inequalities in the sense of generalized K-fractional integral operator for exponentially convex functions. AIMS Math, 5(3), (2020), 2629-2645.

[44] Y.C. Kwun, M. Tanveer, W. Nazeer, M. Abbas, S.M. Kang, Fractal generation in modified Jungck–S orbit.*IEEE Access*, 7, (2019), 35060-35071.

[45] W. Orlicz, A note on modular spaces I, *Bull. Acad. Polon. Soi., Ser. Math. Astronom Phys.*, (9), (1961), 157-162.

[46] Z. Eken, S. Sezer, G. Tinaztepe, G. Adilov, s-Convex functions in the fourth sense and some of their properties. *Konuralp Journal of Mathematics, 9(2), 260-267, 2021.*

Gültekin Tınaztepe
Vocational School of Technical Sciences
Associate Professor of Mathematics
Akdeniz University
Antalya, Turkey
E-mail: skemali@akdeniz.edu.tr

Ilknur Yeşilce Işık
Department of Mathematics
Associate Professor of Mathematics
Aksaray University
Aksaray, Turkey
E-mail: ilknuryesilce@gmail.com

Serap Kemali
Vocational School of Technical Sciences
Assistant Professor of Mathematics
Akdeniz University
Antalya, Turkey
E-mail: skemali@akdeniz.edu.tr

Gabil Adilov
Faculty of Education,
Professor of Mathematics
Akdeniz University
Antalya, Turkey
E-mail: gabil@akdeniz.edu.tr