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Quasi-multipliers on Group Algebras Related to a Locally Compact Group

M. J. Mehdipour
Shiraz University of Technology

GH. R. Moghimi*
Payame Noor University

Abstract. In this paper, we first characterize quasi-multipliers of $(M(\mathcal{G})_0^*)^*$ and show that the Banach algebra of all quasi-multipliers of $(M(\mathcal{G})_0^*)^*$ is isometrically isomorphic to $(M(\mathcal{G})_0^*)^*$. We also establish that quasi-multipliers of $(M(\mathcal{G})_0^*)^*$ are separately continuous. Then, we investigate the existence of weakly compact quasi-multipliers of $(M(\mathcal{G})_0^*)^*$. Finally, we prove that the Banach algebra of quasi-multipliers of $(M(\mathcal{G})_0^*)^*$ is commutative if and only if $\mathcal{G}$ is abelian and discrete.

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1 Introduction

Throughout this paper, $\mathcal{G}$ denotes a locally compact group with a fixed left Haar measure m and the identity element e. Let $L^1(\mathcal{G})$ be the space of all integrable functions on $\mathcal{G}$ and $L^\infty(\mathcal{G})$ be the Banach algebra as

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*Corresponding Author

defined in [4]. We denote by $L_0^\infty(\mathcal{G})$ the space of all functions $f \in L^\infty(\mathcal{G})$ that vanish at infinity; i. e, for each $\varepsilon > 0$ there exists a compact subset K of $\mathcal{G}$ such that

$$|\int_{\mathcal{G}} f(x)\phi(x)\ d\ m(x)| < \varepsilon$$

for all $\phi \in L^1(\mathcal{G})$ with $\|\phi\|_1 = 1$ and $|\phi|\ |_K = 0$. It is well-known from [8] that the dual space of $L_0^\infty(\mathcal{G})$ is a Banach algebra with the first Arens product.

Let $M(\mathcal{G})$ be the Banach space of all complex regular Borel measures on $\mathcal{G}$ with the total variation norm. Note that $M(\mathcal{G})$ is the dual space of $C_0(\mathcal{G})$, the Banach space of all complex-valued continuous functions on $\mathcal{G}$ vanishing at infinity. By the convolution multiplication

$$\mu * \nu(f) = \int_{\mathcal{G}}\int_{\mathcal{G}} f(xy)\ d\mu(x)\ d\nu(y) \quad (f \in C_0(\mathcal{G}), \mu, \nu \in M(\mathcal{G})),$$

$M(\mathcal{G})$ becomes a Banach algebra with the identity element δ_e, the Dirac measure at e.

Let us recall from [9] that a functional $\lambda \in M(\mathcal{G})^*$ *vanishes at infinity* if for every $\varepsilon > 0$, there exists a compact subset K of $\mathcal{G}$, for which $|\langle\lambda, \mu\rangle| < \varepsilon$, where $\mu \in M(\mathcal{G})$ with $|\mu|(K) = 0$ and $\|\mu\| = 1$. This space is denoted by $M(\mathcal{G})_0^*$ and is proved that it is a left introverted subspace of $M(\mathcal{G})^*$. Hence if $F \in (M(\mathcal{G})_0^*)^*$ and $\lambda \in M(\mathcal{G})_0^*$, we may define the functional $F\lambda$ in $M(\mathcal{G}, \omega)^*$ by

$$\langle F\lambda, \mu\rangle = \langle F, \lambda\mu\rangle, \quad \text{in which} \quad \langle\lambda\mu, \nu\rangle = \langle\lambda, \mu * \nu\rangle \quad (\mu, \nu \in M(\mathcal{G})).$$

This fact let us to define the first Arens product "·" by

$$\langle F \cdot H, \lambda\rangle := \langle F, H\lambda\rangle$$

for all $F, H \in (M(\mathcal{G})_0^*)^*$, $\lambda \in M(\mathcal{G})_0^*$. Then, with this product $(M(\mathcal{G})_0^*)^*$ becomes a unital Banach algebra with the identity element δ_e [9]. One can prove that $L_0^\infty(\mathcal{G})^*$ and $L^1(\mathcal{G})$ are ideals in $(M(\mathcal{G})_0^*)^*$. Also, when $\mathcal{G}$ is discrete, we have

$$(M(\mathcal{G})_0^*)^* = L_0^\infty(\mathcal{G})^* = L^1(\mathcal{G}).$$

For a Banach algebra A, a bilinear mapping $\mathfrak{m} : A \times A \to A$ is called a *quasi-multiplier* if

$$\mathfrak{m}(ab, cd) = a\mathfrak{m}(b, c)d,$$

for all $a, b, c, d \in A$. We write $QM(A)$ for the set of all quasi-multipliers from $A \times A$ into A. It is easy to see that $QM((A)$ is a normed space with the following norm.

$$\|\mathfrak{m}\| = \sup\{\|\mathfrak{m}(a, b)\| : a, b \in B_1\},$$

where $B_1 = \{a \in A : \|a\| = 1\}$.

Quasi-multipliers have been studied by several authors. For example, McKennon [10] studied quasi-multipliers of Banach algebra with minimal approximate identities and proved that $QM(L^1(\mathcal{G}))$ is isometrically isomorphic to $M(\mathcal{G})$. Vasudevan and Goel [13] studied the question of embedding $QM(A)$ in the second dual A^{**} of A. They constructed a Banach algebra B_0 and gave conditions under which $QM(A)$ can be embedded isometrically isomorphic into B_0^*; see also [2, 3, 5, 6, 7, 14, 15].

In this paper, we investigate quasi-multipliers of $(M(\mathcal{G})_0^*)^*$ and characterize them. We prove that the elements of $QM((M(\mathcal{G})_0^*)^*)$ are separately continuous and $QM((M(\mathcal{G})_0^*)^*)$ is isometrically isomorphic to $(M(\mathcal{G})_0^*)^*$ and so it is isometrically isomorphic to $QM(L^1(\mathcal{G}))$ if and only if $\mathcal{G}$ is discrete. Under certain condition, we also show that if $QM((M(\mathcal{G})_0^*)^*)$ has a nonzero weakly compact element, then $\mathcal{G}$ is compact. Finally, we prove that $QM((M(\mathcal{G})_0^*)^*)$ is commutative if and only if $\mathcal{G}$ is abelian and discrete.

2 Quasi-Multipliers of $(M(\mathcal{G})_0^*)^*$

We commence this section with the following result.

Theorem 2.1. *Let $\mathcal{G}$ be a locally compact group and $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$. Then there is an $\eta \in (M(\mathcal{G})_0^*)^*$ such that $\mathfrak{m}(F, H) = F \cdot \eta \cdot H$, for all $F, H \in (M(\mathcal{G})_0^*)^*$. Moreover, $\|\mathfrak{m}\| = \|\eta\|$.*

Proof. Let $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$ and $F, H \in (M(\mathcal{G})_0^*)^*$. Then

$$\begin{aligned} \mathfrak{m}(F, H) &= \mathfrak{m}(F \cdot \delta_e, \delta_e \cdot H) \\ &= F \cdot \mathfrak{m}(\delta_e, \delta_e) \cdot H \\ &= F \cdot \eta \cdot H, \end{aligned}$$

where $\eta = \mathfrak{m}(\delta_e, \delta_e)$. Now we have

$$\begin{aligned}\|\mathfrak{m}\| &= \sup\{\|\mathfrak{m}(F,H)\| : F, H \in B_1\}\\ &= \sup\{\|F\cdot\eta\cdot H\| : F, H \in B_1\}\\ &\leq \|\eta\| = \|\mathfrak{m}(\delta_e,\delta_e)\|\\ &\leq \|\mathfrak{m}\|.\end{aligned}$$

Therefore $\|\mathfrak{m}\| = \|\eta\|$. □

A bounded linear operator $T : A \to A$ is called a *left* (*right*) *multiplier* if for every $a, b \in A$

$$T(ab) = T(a)b \qquad (T(ab) = aT(b)).$$

The following result which is an immediate consequence of Theorem (2.1) shows that quasi-multipliers of $(M(\mathcal{G})_0^*)^*$ are an extension of left and right multipliers on $(M(\mathcal{G})_0^*)^*$.

Corollary 2.2. *Let $\mathcal{G}$ be a locally compact group and $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$. Then $\mathfrak{m}$ is separately continuous and*

$$\mathfrak{m}(F_1\cdot F_2, F_3) = F_1\cdot\mathfrak{m}(F_2,F_3) \quad \textit{and} \quad \mathfrak{m}(F_1, F_2\cdot F_3) = \mathfrak{m}(F_1,F_2)\cdot F_3$$

for all $F_1, F_2, F_3 \in (M(\mathcal{G})_0^*)^*$.

For $\mathfrak{m}_1, \mathfrak{m}_2 \in QM((M(\mathcal{G})_0^*)^*)$, the quasi-multiplier $\mathfrak{m}_1 \odot \mathfrak{m}_2 \in QM((M(\mathcal{G})_0^*)^*)$ is defined by

$$(\mathfrak{m}_1 \odot \mathfrak{m}_2)(F,H) = F\cdot\eta_1\cdot\eta_2\cdot H,$$

where $\eta_i = \mathfrak{m}_i(\delta_e, \delta_e)$ for $i = 1, 2$.

Proposition 2.3. *Let $\mathcal{G}$ be a locally compact group. Then $(QM((M(\mathcal{G})_0^*)^*), \odot, \|.\|)$ is a unital Banach algebra.*

Proof. Let $(\mathfrak{m}_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $QM((M(\mathcal{G})_0^*)^*)$ and $\eta_i = \mathfrak{m}_i(\delta_e, \delta_e)$ for $i \in \mathbb{N}$. Then

$$\|\eta_i - \eta_j\| = \|\mathfrak{m}_i - \mathfrak{m}_j\| \to 0.$$

This shows that $(\eta_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $(M(\mathcal{G})_0^*)^*$ and hence converges to an element $\eta \in (M(\mathcal{G})_0^*)^*$. Now we define $\mathfrak{m} : (M(\mathcal{G})_0^*)^* \times (M(\mathcal{G})_0^*)^* \to (M(\mathcal{G})_0^*)^*$ by $\mathfrak{m}(F,H) = F \cdot \eta \cdot H$. Then

$$\|\mathfrak{m}_i - \mathfrak{m}\| = \|\eta_i - \eta\| \to 0.$$

Therefore, $QM((M(\mathcal{G})_0^*)^*)$ is complete. From this and

$$\begin{aligned}
\|\mathfrak{m}_1 \odot \mathfrak{m}_2\| &= \|(\mathfrak{m}_1 \odot \mathfrak{m}_2)(\delta_e, \delta_e)\| \\
&= \|\delta_e \cdot \eta_1 \cdot \eta_2 \cdot \delta_e\| \\
&= \|\eta_1 \cdot \eta_2\| \\
&\leq \|\eta_1\|\|\eta_2\| \\
&= \|\mathfrak{m}_1\|\|\mathfrak{m}_2\|
\end{aligned}$$

we infer that $QM((M(\mathcal{G})_0^*)^*)$ is a Banach algebra. Note that the quasi-multiplier $\mathfrak{m} : (M(\mathcal{G})_0^*)^* \times (M(\mathcal{G})_0^*)^* \to (M(\mathcal{G})_0^*)^*$ defined by $\mathfrak{m}(F,H) = F \cdot H$ is the identity of $QM((M(\mathcal{G})_0^*)^*)$. □

Let $LM((M(\mathcal{G})_0^*)^*)$ be the space of left multipliers on $(M(\mathcal{G})_0^*)^*$.

Lemma 2.4. *The Banach algebra $QM((M(\mathcal{G})_0^*)^*)$ is isometrically isomorphic to $LM((M(\mathcal{G})_0^*)^*)$.*

Proof. Define the mapping $\Theta : QM((M(\mathcal{G})_0^*)^*) \to LM((M(\mathcal{G})_0^*)^*)$ by

$$(\Theta(\mathfrak{m}))(H) = \eta \cdot H,$$

where $\eta = \mathfrak{m}(\delta_e, \delta_e)$ and $H \in (M(\mathcal{G})_0^*)^*$. It is easy to see that Θ is linear. Now, let $\mathfrak{m}_1, \mathfrak{m}_2 \in QM((M(\mathcal{G})_0^*)^*)$ and $\eta_i = \mathfrak{m}_i(\delta_e, \delta_e)$, for $i = 1, 2$. Then for every $H \in (M(\mathcal{G})_0^*)^*$

$$\begin{aligned}
\Theta(\mathfrak{m}_1 \odot \mathfrak{m}_2)(H) &= (\eta_1 \cdot \eta_2) \cdot H \\
&= \eta_1 \cdot (\eta_2 \cdot H) \\
&= \Theta(\mathfrak{m}_1)(\eta_2 \cdot H) \\
&= \Theta(\mathfrak{m}_1)(\Theta(\mathfrak{m}_2)(H)) \\
&= \Theta(\mathfrak{m}_1) \circ \Theta(\mathfrak{m}_2)(H).
\end{aligned}$$

So Θ is an algebra homomorphism. If $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$, then

$$\begin{aligned}\|\Theta(\mathfrak{m})\| &= \sup\{\|\Theta(\mathfrak{m})(H)\| : H \in B_1\}\\ &= \sup\{\|\eta \cdot H\| : H \in B_1\}\\ &= \|\eta\| = \|\mathfrak{m}\|.\end{aligned}$$

This shows that Θ is an isometry. To complete the proof, let $T \in LM((M(\mathcal{G})_0^*)^*)$. Then

$$T(H) = T(\delta_e \cdot H) = T(\delta_e) \cdot H$$

for all $H \in (M(\mathcal{G})_0^*)^*$. Define

$$\mathfrak{m} : (M(\mathcal{G})_0^*)^* \times (M(\mathcal{G})_0^*)^* \to (M(\mathcal{G})_0^*)^*$$

by $\mathfrak{m}(F, H) = F \cdot T(\delta_e) \cdot H$. Then

$$\begin{aligned}\Theta(\mathfrak{m})(H) &= \mathfrak{m}(\delta_e, \delta_e) \cdot H\\ &= (\delta_e \cdot T(\delta_e) \cdot \delta_e) \cdot H\\ &= T(\delta_e) \cdot H\\ &= T(H).\end{aligned}$$

Thus $\Theta(\mathfrak{m}) = T$ and so Θ is onto. Therefore, $QM((M(\mathcal{G})_0^*)^*)$ is isometrically isomorphic to $LM((M(\mathcal{G})_0^*)^*)$. $\square$

Corollary 2.5. *The Banach algebra $QM((M(\mathcal{G})_0^*)^*)$ is isometrically isomorphic to $(M(\mathcal{G})_0^*)^*$.*

Proof. This follows from Lemma (2.4) and the fact that $LM((M(\mathcal{G})_0^*)^*)$ is isometrically isomorphic to $(M(\mathcal{G})_0^*)^*$. $\square$

Let $\Lambda(L_0^\infty(\mathcal{G})^*)$ be the set of all weak*-cluster points of an approximate identity in $L^1(\mathcal{G})$ bounded by one. Then $n \cdot u = n$ and $u \cdot \phi = \phi$ for all $n \in L_0^\infty(\mathcal{G})^*$, $\phi \in L^1(\mathcal{G})$ and $u \in \Lambda(L_0^\infty(\mathcal{G})^*)$, for more details see [8].

Proposition 2.6. *Let $\mathcal{R} : QM((M(\mathcal{G})_0^*)^*) \to QM(L^1(\mathcal{G}))$ be the restriction map. Then the following statements hold.*

(i) *$\mathcal{R}$ is an epimorphism.*

(ii) *$\mathcal{R}$ is a monomorphism if and only if $\mathcal{G}$ is discrete.*

Proof. First we note that if $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$, then $\mathcal{R}(\mathfrak{m}) \in QM(L^1(\mathcal{G}))$, because for every $\phi, \psi \in L^1(\mathcal{G})$

$$\begin{aligned} \mathcal{R}(\mathfrak{m})(\phi, \psi) &= \mathfrak{m}(\phi, \psi) \\ &= \phi \cdot \mathfrak{m}(\delta_e, \delta_e) \cdot \psi \in L^1(\mathcal{G}). \end{aligned}$$

(i) Let $\tilde{\mathfrak{m}} \in QM(L^1(\mathcal{G}))$. Then there is a $\mu \in M(\mathcal{G})$ such that

$$\tilde{\mathfrak{m}}(\phi, \psi) = \phi * \mu * \psi$$

for all $\phi, \psi \in L^1(\mathcal{G})$. We define

$$\mathfrak{m}(F, H) = F \cdot \mu \cdot H.$$

It is obvious that $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$ and $\mathcal{R}(\mathfrak{m}) = \tilde{\mathfrak{m}}$. Therefore, $\mathcal{R}$ is an epimorphism.

(ii) Assume that $\mathcal{R}$ is a monomorphism. So $QM((M(\mathcal{G})_0^*)^*)$ is isomorphic to $QM(L^1(\mathcal{G}))$. Hence $(M(\mathcal{G})_0^*)^*$ is isomorphic to $M(\mathcal{G})$ and so $L_0^\infty(\mathcal{G})^*$ is contained in $M(\mathcal{G})$. From this and the fact that $M(\mathcal{G})$ is isometrically isomorphic to $u \cdot L_0^\infty(\mathcal{G})^*$ we infer that $L_0^\infty(\mathcal{G})^* = u \cdot L_0^\infty(\mathcal{G})^*$, where $u \in \Lambda(L_0^\infty(\mathcal{G})^*)$; see [8]. Therefore

$$L^1(\mathcal{G}) = \cap_{u \in \Lambda(L_0^\infty(\mathcal{G})^*)} u \cdot L_0^\infty(\mathcal{G})^* = L_0^\infty(\mathcal{G})^*.$$

It follows from Proposition 3.1 of [11] that $\mathcal{G}$ is discrete. The converse is clear. $\square$

In the following, let $C_0(\mathcal{G})^\perp$ be the space of all $F \in (M(\mathcal{G})_0^*)^*$ such that $F|_{C_0(\mathcal{G})} = 0$. Let also η be $\mathfrak{m}(\delta_e, \delta_e)$, where $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$.

Proposition 2.7. *Let $\mathcal{G}$ be a locally compact group. If there exists a weakly compact element $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$ with $\eta \notin C_0(\mathcal{G})^\perp$, then $\mathcal{G}$ is compact.*

Proof. Let $\mathfrak{m}$ be a weakly compact element in $QM((M(\mathcal{G})_0^*)^*)$ with $\eta \notin C_0(\mathcal{G})^\perp$. Define the bounded linear operator $T : L^1(\mathcal{G}) \to L^1(\mathcal{G})$ by

$$T(\phi) = \mathfrak{m}(\delta_e, \phi).$$

It is clear that T is weakly compact left multiplier on $(M(\mathcal{G})_0^*)^*$. Note that δ_e is the identity element of $(M(\mathcal{G})_0^*)^*$. Since $\eta \notin C_0(\mathcal{G})^\perp$ and

$$L^1(\mathcal{G})M(\mathcal{G})_0^* = C_0(\mathcal{G}),$$

there exist $\phi \in L^1(\mathcal{G})$ and $\lambda \in M(\mathcal{G})_0^*$ such that $\langle \eta, \phi\lambda\rangle \neq 0$. On the hand,

$$\begin{aligned} \langle \eta, \phi\lambda\rangle &= \langle \mathfrak{m}(\delta_e, \delta_e), \phi\lambda\rangle \\ &= \langle \mathfrak{m}(\delta_e, \delta_e)\phi, \lambda\rangle \\ &= \langle \mathfrak{m}(\delta_e, \delta_e\phi), \lambda\rangle \\ &= \langle T(\phi), \lambda\rangle. \end{aligned}$$

Therefore, T is nonzero. By [12], $\mathcal{G}$ is compact. □

At this point in the paper the reader will be expect to see the converse of Proposition (2.7). This principle reason why it dose not appear is that we have been unable to prove it.

Now, we characterize weakly compact elements $QM((M(\mathcal{G})_0^*)^*)$.

Corollary 2.8. *Let $\mathcal{G}$ be a locally compact group and $\mathfrak{m}$ be a weakly compact element of $QM((M(\mathcal{G})_0^*)^*)$. Then there exists $\phi \in L^1(\mathcal{G})$ and $\Gamma \in C_0(\mathcal{G})^\perp$ such that $\eta = \phi + \Gamma$.*

Proof. Let T be as defined in the proof of Theorem (2.7). Then there is $\phi \in L^1(\mathcal{G})$ such that $T(\psi) = \phi * \psi$ for all $\psi \in L^1(\mathcal{G})$; see [1]. Hence

$$\phi \cdot \psi = \mathfrak{m}(\delta_e, \psi) = \delta_e \cdot \eta \cdot \psi = \eta \cdot \psi$$

for all $\psi \in L^1(\mathcal{G})$. So

$$(\phi - \eta) \cdot \psi = 0$$

for all $\psi \in L^1(\mathcal{G})$. This implies that $\phi - \eta \in C_0(\mathcal{G})^\perp$. Therefore, $\eta = \phi + \Gamma$, for some $\Gamma \in C_0(\mathcal{G})^\perp$. □

Theorem 2.9. *Let $\mathcal{G}$ be a locally compact group. Then the following assertions are equivalent.*

(a) *$QM((M(\mathcal{G})_0^*)^*)$ is commutative;*
(b) *$(M(\mathcal{G})_0^*)^*$ is commutative;*
(c) *$\mathcal{G}$ is abelian and discrete;*

Proof. Let $F_1, F_2 \in (M(\mathcal{G})_0^*)^*$. We define the quasi-multiplier

$$\mathfrak{m}_i : (M(\mathcal{G})_0^*)^* \times (M(\mathcal{G})_0^*)^* \to (M(\mathcal{G})_0^*)^*$$

by

$$\mathfrak{m}_i(H_1, H_2) = H_1 \cdot F_i \cdot H_2 \qquad (i = 1, 2).$$

If (a) holds, then

$$\mathfrak{m}_1 \odot \mathfrak{m}_2 = \mathfrak{m}_2 \odot \mathfrak{m}_1.$$

Hence

$$(\mathfrak{m}_1 \odot \mathfrak{m}_2)(\delta_e, \delta_e) = (\mathfrak{m}_2 \odot \mathfrak{m}_1)(\delta_e, \delta_e)$$

and so

$$\delta_e \cdot F_1 \cdot F_2 \cdot \delta_e = \delta_e \cdot F_2 \cdot F_1 \cdot \delta_e.$$

Consequently, $F_1 \cdot F_2 = F_2 \cdot F_1$. That is, $(M(\mathcal{G})_0^*)^*$ is commutative. Thus (b) holds. The implication (b)$\Rightarrow$(a) is clear. Let's show that (b)$\Leftrightarrow$(c). Let $(M(\mathcal{G})_0^*)^*$ be commutative. Then $L^1(\mathcal{G})$ is commutative. It follows that $\mathcal{G}$ is abelian. Now, if $\mathcal{G}$ is not discrete, the by Hahn-Banach theorem, there exists a nonzero functional $n \in L_0^\infty(\mathcal{G})^*$ such that $n|_{C_0(\mathcal{G})} = 0$. Choose $u \in \Lambda(L_0^\infty(\mathcal{G})^*)$. Then for every $f \in L_0^\infty(\mathcal{G})$, we have $\frac{1}{\Delta}\tilde{e_\alpha} * f \in C_0(\mathcal{G})$, where Δ is the modular function of $\mathcal{G}$ and $(e_\alpha)_\alpha$ is a net in $L^1(\mathcal{G})$ with $e_\alpha \to u$ in the weak* topology of $L_0^\infty(\mathcal{G})^*$ and $\tilde{e_\alpha}(x) = e_\alpha(x^{-1})$. It follows that

$$\begin{aligned} \langle n, f\rangle &= \langle n \cdot u, f\rangle \\ &= \langle u \cdot n, f\rangle \\ &= \lim_\alpha \langle n, \frac{1}{\Delta}\tilde{e_\alpha} * f\rangle = 0 \end{aligned}$$

for all $f \in L_0^\infty(\mathcal{G})$. Thus $n = 0$, a contradiction. So (c) holds. Conversely, assume that $\mathcal{G}$ is abelian and discrete. Since $\mathcal{G}$ is abelian, $L^1(\mathcal{G})$ is commutative. On the other hand, $\mathcal{G}$ is discrete. Thus

$$(M(\mathcal{G})_0^*)^* = L_0^\infty(\mathcal{G})^* = L^1(\mathcal{G}).$$

This implies that $(M(\mathcal{G})_0^*)^*$ is commutative. $\square$

The right annihilator $L_0^\infty(\mathcal{G})^*$ is denoted by $Ann_r(L_0^\infty(\mathcal{G})^*)$ and is defined by

$$Ann_r(L_0^\infty(\mathcal{G})^*) = \{r \in L_0^\infty(\mathcal{G})^* : L_0^\infty(\mathcal{G})^* \cdot r = \{0\}\}.$$

It is easy to see that the following remark are true and we mention them only for the readers attention.

Remark 2.10. *Let* $\mathfrak{m} \in QM((M(\mathcal{G})_0^*)^*)$. *Then the following statements hold.*

(i) $\mathfrak{m}(L^1(\mathcal{G}) \times L^1(\mathcal{G})) \subseteq L^1(\mathcal{G})$.
(ii) $\mathfrak{m}(L_0^\infty(\mathcal{G})^* \times L_0^\infty(\mathcal{G})^*) \subseteq L_0^\infty(\mathcal{G})^*$.
(iii) $\mathfrak{m}(Ann_r(L_0^\infty(\mathcal{G})^*) \times Ann_r(L_0^\infty(\mathcal{G})^*) \subseteq Ann_r(L_0^\infty(\mathcal{G})^*)$.

Let B be a Banach algebra and A be a closed subalgebra of B. Then A is called a *quasi-ideal* in B if $ABA \subseteq A$. Furthermore, if the linear mapping $\Phi : B \to QM(A)$ defined by

$$\Phi(b)(x, y) = xby \qquad (b \in B, x, y \in A),$$

is an isometry, then B is called an *intermediate algebra* for A.
If $H \in (M(\mathcal{G})_0^*)^*$. Then we have

$$\|\Phi(H)\| \;=\; \sup\{\|F_1 \cdot H \cdot F_2\| : F_1, F_2 \in B_1\} = \|H\|.$$

Therefore, Φ is an isometry. Now, if we note that every ideal is a quasi-ideal and $L^1(\mathcal{G})$ and $L_0^\infty(\mathcal{G})^*$ are quasi-ideals of $(M(\mathcal{G})_0^*)^*$, we have the following remark.

Remark 2.11. *The following statements hold.*

(i) $(M(\mathcal{G})_0^*)^*$ *is an intermediate algebra for ideals and quasi-ideals of* $(M(\mathcal{G})_0^*)^*$.
(ii) $(M(\mathcal{G})_0^*)^*$ *is an intermediate algebra for* $(M(\mathcal{G})_0^*)^*$.
(iii) $(M(\mathcal{G})_0^*)^*$ *is an intermediate algebra for* $L^1(\mathcal{G})$ *and* $L_0^\infty(\mathcal{G})^*$.

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Mohammad Javad Mehdipour
Associated Professor of Mathematics
Department of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: mehdipour@sutech.ac.ir

Gholam Reza Moghimi
Phd student of Mathematics
Department of Mathematics
Payame Noor University (PNU)
P.O. Box 19395-4697,Tehran, Iran
E-mail: moghimimath@pnu.ac.ir