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# The Dual Notion of the Square Submodule of a Module

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**Abstract.** The dual notion of a nil module and the square submodule of a module over a commutative ring are defined. Moreover, besides other results, some properties of these new concepts are investigated.

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## 1 Introduction

The concepts of a nil module and the square submodule of a module for any arbitrary module $M$ over a commutative ring $R$ are both inspired from their analogous concepts in the theory of abelian groups. Indeed, such notions were first introduced by A. E. Stratton and M. C.

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Webb [15] for abelian groups. Several authors have investigated problems concerning to nil groups and the square subgroup of an abelian group over the past decades in [1, 2, 3, 7, 14, 16]. For the first time, Aghdam and Najafizadeh [3] defined the notion of the square submodule over a commutative ring and discussed it over some classes of commutative domains. The square submodule of a module is the most natural generalization of the concept of a nil module. Indeed, any module over a commutative ring is a nil module exactly if its square submodule vanishes. In this paper we define a dual notion for the square submodule of a module. Moreover, besides some other results, we investigate some properties of the square submodule of a module and its dual notion.

## 2 Preliminaries and Notations

Throughout this paper, $R$ denotes a commutative ring with identity element. Let $M$ be an $R$-module. An element $m$ in a module $M$ over a ring $R$ is said to be a regular element if there exists an $R$-homomorphism $\alpha : M \to R$ such that $m = \alpha(m)m$. If every element of the module $M$ is regular, then $M$ is called a regular module. The module $M$ is said to be divisible if $rM = M$ for all non zero-divisors $r \in R$. Following [8], an $R$-module $M$ is said to be a multiplication (co-multiplication) module if for any submodule $N$ of M there exists a two sided ideal $I$ of $R$ such that $N = IM$ ($N = (0 :_M I)$). A proper submodule $P$ of $M$ is called prime provided that for $r \in R$ and $m \in M$, if $rm \in P$ then either $m \in P$ or $rM \subseteq P$. For a module $M$, $M^{(I)}$ denotes the direct sum of $|I|$ copies of $M$ in which $|I|$ is the cardinality of the index set $I$. We also say that $N$ (finitely) generates $M$ if there is an epimorphism $N^{(A)} \to M \to 0$ for some (finite) set $A$. The trace of $M$ in $N$, which is denoted by $Tr_N(M)$, is defined as $Tr_N(M) = \sum\{\mathrm{Im}(\varphi) : \varphi \in Hom_R(M, N)\}$. Moreover, the reject of $M$ in $N$, which is denoted by $Rej_N(M)$, is defined as $Rej_N(M) = \cap\{\ker(\varphi) : \varphi \in Hom_R(N, M)\}$.

Let $M$ be an $R$-module over a commutative ring $R$. A bilinear map on $M$ is a function $\mu : M \times M \longrightarrow M$ such that for all $m, n, m_i, n_i \in M$ and $r \in R$ :

(1) $\mu(m_1 + m_2, n) = \mu(m_1, n) + \mu(m_2, n)$;

(2) $\mu(m, n_1 + n_2) = \mu(m, n_1) + \mu(m, n_2)$;

(3) $\mu(rm, n) = \mu(m, rn) = r\mu(m, n)$.

Now for any two bilinear maps $\mu$ and $\nu$ on $M$, and $r \in R$ we define

$$(\mu + \nu)(m, n) = \mu(m, n) + \nu(m, n)$$

$$(r\mu)(m, n) = r \cdot \mu(m, n).$$

The set of all bilinear maps on $M$ forms an $R$-module which we denote by $Mult_R(M)$ and call any element of it a multiplication on the $R$-module $M$. Following [3], for a given submodule $N$ of an $R$-module $M$, the module $M$ is said to be nil modulo $N$ if $\mu(M \times M) \leq N$ for all $\mu \in Mult_R(M)$. Moreover, the square submodule of $M$ denoted by $\square_R M$ is defined as $\square_R M = \sum \mathrm{Im}(\varphi)$, where $\varphi$ runs in $Mult_R(M)$. We use the symbol $\square M$ if no ambiguity arises. An $R$-module $M$ is called nil if $\square M = 0$. Clearly $M$ is a nil $R$-module if and only if $M$ is nil modulo 0. Furthermore, $\square M$ is the intersection of all submodules $N$ of $M$ such that $M$ is nil modulo $N$, i.e., the smallest $R$-submodule $N$ of $M$ such that $M$ is nil modulo $N$.

**Theorem 2.1.** *Let $M$ be an $R$-module over a commutative ring $R$ with $S = End_R(M)$. Then $\square M = Tr_M(M \otimes_R M) = Tr_S(M)M$.*

**Proof.** See [3, Theorem 2.3]. $\square$

# 3 Main Results

In this section, we organize the main results of the paper in 3 subsections. The first one includes some general results about the square submodule and its dual notion. The remaining subsections discuss the situations in which either the square submodule or its dual notion is a specific submodule of the module under investigation.

## 3.1 Generalities on the square submodule and its dual notion

In this subsection, we express the square submodule and its dual notion in terms of some algebraic concepts. Moreover, the situations in which the module under discussion is either a nil or a co-nil module are investigated. We begin with a result which deals with a condition that the square submodule of a given module, is equal to the whole module.

**Proposition 3.1.** *Let $M$ be a regular module over a commutative ring $R$ with $S = End_R(M)$. Then $\square M = M$. Moreover, if $M$ is a regular $S$-module, then $\square M = M$.*

**Proof.** The hypothesis that $M$ is a regular $R$-module, implies that for any element $m$ of $M$, there exists an $R$-homomorphism $\alpha : M \to R$ such that $m = \alpha(m)m$. But $R$ is commutative, hence the multiplication by every element $r$ of $R$ induces an $R$-homomorphism of $M$. Now, a similar reasoning as [3, Proposition 3.6] yields the result. The last assertion is clear. □

The next result expresses the square submodule of a module in terms of some algebraic concepts.

**Theorem 3.2.** *Let $M$ be a module over a commutative ring $R$ with $S = End_R(M)$. If $S$ is a co-multiplication $R$-module, then*

$$\square M = (0 :_S Ann(Hom_R(M, S)))M.$$

**Proof.** The hypothesis that $N$ is a co-multiplication $R$-module implies that $Tr_N(M) = (0 :_N Ann(T))$, where $T = Hom_R(M, N)$. In fact, in view of the fact that $Ann(T)\alpha(M) = (Ann(T)\alpha)M = 0$ for all $\alpha \in T$, we reach to this conclusion that $Tr_N(M)$ is contained in $(0 :_N Ann(T))$. Conversely, there exists an ideal $I$ of $R$ such that $Tr_N(M) = (0 :_N I)$. Now, if $\beta \in T$ is an arbitrary element, then $\beta(M) = Im(\beta) \subseteq Tr_N(M) = (0 :_N I)$. We have, $I\beta(M) = 0$. Hence, $(\beta I)M = \beta(IM) = 0$. Consequently, $I \subseteq Ann_R(T)$ since $\beta \in T$ is arbitrary. We get, $(0 :_N Ann_R(T)) \subseteq (0 :_N I) = Tr_N(M)$. Consequently, $Tr_N(M) = (0 :_N Ann_R(T))$. Now, the assertion follows if we put $N = S$. □

**Definition 3.3.** Let $M$ be a module over a commutative ring $R$. Moreover, let $N$ be a submodule of $M \otimes_R M$. We say that $M$ is co-nil modulo $N$ exactly if $N \subseteq \ker(\varphi)$ for all $\varphi : M \otimes_R M \to M$.

Clearly, if $M$ is co-nil modulo both $N_1$ and $N_2$, then $M$ is co-nil modulo $N_1 + N_2$. Therefore, we may define a dual notion for the square submodule denoted by $\square^d M$ as follows.

**Definition 3.4.** Let $M$ be a module over a commutative ring $R$. Then, the dual notion of the square submodule of $M$ denoted by $\square^d M$ is defined as $\square^d M = \sum N$, where $N$ runs over submodules $N$ of $M \otimes_R M$ such that $M$ is co-nil modulo $N$.

It is obvious that $\square^d M$ is the largest submodule $X$ of $M \otimes_R M$ such that $M$ is co-nil modulo $X$.

**Definition 3.5.** Let $M$ be a module over a commutative ring $R$. Then, we say that $M$ is a co-nil module exactly if $\square^d M = 0$.

In the next two results, we express $\square^d M$ in terms of some algebraic concepts.

**Proposition 3.6.** *Let $M$ be an $R$-module over a commutative ring $R$. Then, $\square^d M = Rej_{M \otimes_R M}(M)$.*

**Proof.** We observe that if $M$ is co-nil modulo $N$, then $N \subseteq \ker(\varphi)$ for all $\varphi : M \otimes_R M \to M$. Hence, $N \subseteq \cap_{\varphi : M \otimes_R M \to M} \ker(\varphi)$. This means that $\square^d M \subseteq \cap_{\varphi : M \otimes_R M \to M} \ker(\varphi)$. Consequently, $\square^d M \subseteq Rej_{M \otimes_R M}(M)$. Conversely, we prove that $M$ is co-nil modulo $Rej_{M \otimes_R M}(M)$. This is clear, since for all $\varphi : M \otimes_R M \to M$, we have $Rej_{M \otimes_R M}(M) \subseteq \ker(\varphi)$. We conclude the result. $\square$

**Theorem 3.7.** *Let $M$ be an $R$-module over a commutative ring $R$ with $S = End_R(M)$. Then $\square^d M = M \otimes_R (0 :_M Tr_S(M)) = (0 :_M Tr_S(M)) \otimes_R M$.*

**Proof.** Let $\theta \in Hom_R(M, S)$ be arbitrary. For any $x, y \in M$, the map $f : M \times M \to M$ defined as $f(x, y) = \theta(x)(y)$, is a bilinear map which induces the $R$-homomorphism, $\varphi : M \otimes_R M \to M$ with $\varphi(x \otimes y) = \theta(x)(y)$. Now, let $m \otimes n$ be an arbitrary generator of $\square^d M$. Hence, $m \otimes n$ belongs to $\cap_{\varphi : M \otimes_R M \to M} \ker(\varphi)$. Now, we prove that $n \in 0 :_M Tr_S(M)$. In other words, $Tr_S(M).n = 0$. To do this, let $\theta_m : M \to S$ be an arbitrary element of $Tr_S(M)$. Then in view of the observation at the beginning of the proof, we have $\theta_m.n = \theta_m(n) = \varphi(m \otimes n) = 0$. Conversely, suppose that $m \otimes n$ be an arbitrary generator of $M \otimes_R (0 :_M Tr_S(M))$. Hence, $m \in M$ and $n \in Tr_S(M)$. We shall prove that $m \otimes n \in Rej_{M \otimes_R M}(M) = \cap_{\varphi : M \otimes_R M \to M} \ker(\varphi)$. To do this, let $\varphi : M \otimes_R M \to M$

be arbitrary. We have $\varphi(m \otimes n) = \theta_m(n) = \theta_m.n = 0$. We conclude the result. $\Box$

The next result shows that $\Box^d M$ inherits the divisibility of $M$.

**Proposition 3.8.** *Let $M$ be a module over a commutative ring $R$ with $S = End_R(M)$. If $M$ is a divisible $R$-module, then so is $\Box^d M$.*

**Proof.** Let $x \otimes y \in \Box^d M = (0 :_M Tr_S(M)) \otimes_R M$ for arbitrary $x \in (0 :_M Tr_S(M))$ and $y \in M$. Then, the divisibility of $M$ implies that $y = rz$ for some $z \in M$ and $r \in R$. Hence, we have $x \otimes y = x \otimes rz = r(x \otimes z) \in r\Box^d M$. $\Box$

At this point, we discuss the conditions that make an arbitrary module $M$ to be either a nil or a co-nil module.

**Proposition 3.9.** *Let $M$ be an $R$-module over a commutative ring $R$. Then, $M$ is a nil module exactly if $\Box^d M = M \otimes_R M$.*

**Proof.** Follows from the fact that $Tr_N(M) = 0$ exactly if $Rej_M(N) = M$ for any $R$-module $N$. $\Box$

**Definition 3.10.** Let $M$ be a module over a commutative ring $R$. Let $N$ be a submodule of $M \otimes_R M$. We define $[[M : N]]_l$ to be the $R$-submodule of $M$ generated by the set of all $m \in M$ such that $y \otimes m \in N$ and $\varphi(y \otimes m) = 0$ for all $y \in M$ and all $\varphi : M \otimes_R M \to M$. In a similar way, we define $[[M : N]]_r$.

**Proposition 3.11.** *Let $M$ be an $R$-module over a commutative ring $R$ and $N$ be a submodule of $M \otimes_R M$. Then $M$ is co-nil modulo $N$ exactly if either $N \subseteq [[M : N]]_l \otimes_R M$ or $N \subseteq M \otimes_R [[M : N]]_r$.*

**Proof.** Straightforward. $\Box$

We end this part with some examples which illustrate the square submodule and its dual submodule in commutative domains. Recall that a commutative semigroup $S$ is a Clifford semigroup if every element $s$ of $S$ is regular (in the sense of von Neumann), i.e., $s = s^2 r$ for some $r \in S$. Following [11], a domain $R$ is a Clifford regular domain if the class semigroup $S(R)$ of $R$ is a Clifford semigroup.

**Example 3.12.** Let $I$ be an ideal of a commutative domain $R$ with quotient field $Q$ and $S = End_R(I)$. Then in view of [3, Proposition 3.4],

$\Box I = (S : I)I^2 = (I : I^2)I^2$. In particular, $\Box I = 0$ if and only if $I = 0$. Moreover, $\Box I = I$ for every ideal $I$ of $R$ exactly if $R$ is a Clifford regular domain.

Moreover, in view of Theorem 3.7 and Proposition 3.9 we get:

**Example 3.13.** Let $I$ be an ideal of a commutative domain $R$ with quotient field $Q$ and $S = End_R(I)$. Then we have:

$$\Box^d I = (0 :_I Tr_S(I)) \otimes I = (0 :_I (S : I)I) \otimes_R I. = (0 :_I (I : I^2)I) \otimes_R I.$$

Moreover, $\Box^d I = I \otimes_R I$ for every ideal $I$ of $R$ exactly if $I$ is nil.

## 3.2 Prime square submodule and its dual notion

In this subsection, we investigate the situations in which either the square submodule of a module $M$ or its dual notion is a prime submodule of $M$ or $M \otimes_R M$, respectively. First, we give several results about the prime submodules which may be of independent interest.

**Proposition 3.14.** *Let $M$ and $N$ be co-multiplication modules over a commutative ring $R$. If $Ann_R(M)$ is prime ideal of $R$, then any non-zero element $\alpha$ of $\mathrm{Hom}_R(M, N)$ satisfies $Im(\alpha) = (0 :_N Ann_R(M))$.*

**Proof.** The hypothesis that $M$ and $N$ are co-multiplication modules, implies that there exists two sided ideals $I, J$ such that $M.\alpha = (0 :_N I)$ and $\ker \alpha = (0 :_M J)$. We have, $0 = I(0 :_N I) = I(M\alpha)$. Hence, $IM \subseteq \ker \alpha = (0 :_M J)$. This implies that $JIM = 0$. We have $JI \subseteq Ann_R(M)$. The hypothesis that $Ann_R(M)$ is prime ideal of $R$, implies that $I \subseteq Ann_R(M)$ or $J \subseteq Ann_R(M)$. By the second inclusion and $\ker \alpha = (0 :_M J)$ we get $\ker \alpha = M$. This means that $\alpha = 0$, a contradiction. Therefore, $I \subseteq Ann_R(M)$. Consequently, by the relation $M.\alpha = (0 :_N I)$ we get $(0 :_N Ann_R(M)) \subseteq (0 :_N I) = M.\alpha$. On the other hand, $Ann_R(M)(M\alpha) = 0$, hence $M.\alpha \subseteq (0 :_N Ann_R(M))$ which implies that $(0 :_N Ann_R(M)) = M.\alpha$. $\Box$

**Corollary 3.15.** *Let $M$ be a co-multiplication module over a commutative ring $R$ with $S = End_R(M)$. Moreover, suppose that $S$ is a co-multiplication $R$-module. If $Ann_R(M)$ is prime ideal of $R$, then $\Box_R M = (0 :_S Ann_R(M))$.*

**Proof.** Clearly, if $M$ and $N$ are co-multiplication modules over a commutative ring $R$ such that $Ann_R(M)$ is a prime ideal of $R$, then $Tr_N(M) = (0 :_N Ann_R(M))$. Now we have $\Box_R M = Tr_S(M) = (0 :_S Ann_R(M))$. $\Box$

**Corollary 3.16.** *Let $M$ be co-multiplication module over a commutative ring $R$. Let $N$ be a non-trivial submodule of $M$. If $Ann_R(M)$ is prime ideal of $R$, then $Tr_N(M) = N$.*

**Proof.** Let $M$ be a co-multiplication module over a commutative ring $R$ with $S = End_R(M)$. Moreover, suppose that $N$ is a non-trivial submodule of $M$. If $Ann_R(M)$ is a prime ideal of $R$, then by a similar reasoning as Proposition 3.14 any non-zero element of $\mathrm{Hom}_R(M, N)$ is surjective. Now our assertion is concluded. $\Box$

**Proposition 3.17.** *Let $M$ be a module over a commutative ring $R$ and $K$ be a submodule of $M$. Let $\alpha \in Hom_R(M, R)$ be a such that $m - \alpha(m)m \in K$ for all $m \in M$. If $K$ is a prime submodule of $M$, then any proper submodule $N$ of $M$ containing $K$ is a prime submodule of $M$.*

**Proof.** Let $r \in R$ and $m \in M$ be such that $rm \in N$. Moreover, suppose that $m$ does not belong to $N$. We shall prove that $rM \subseteq N$. The hypothesis implies that $rm - \alpha(rm)rm \in K$. But, $K$ is a prime submodule of $M$, hence either $m - \alpha(rm)m \in K$ or $rM \subseteq K$. If $m - \alpha(rm)m \in K$, then $m - \alpha(rm)m \in N$. This yields $m \in N$, a contradiction. Therefore, $rM \subseteq K \subseteq N$. We conclude the result. $\Box$

**Proposition 3.18.** *Let $M$ be a regular module over a commutative ring $R$ such that $\{0\}$ is a prime submodule of $M$. Then every proper submodule of $M$ is prime.*

**Proof.** We observe that in view of Proposition 3.17, if $M$ is a regular module over a commutative ring $R$ and $K$ is a prime submodule of $M$, then every proper submodule of $M$ containing $K$ is prime. Now our assertion is clear. $\Box$

**Corollary 3.19.** *Let $M$ be a regular module over an integral domain $R$. Then every proper submodule of $M$ is prime.*

**Proof.** Clearly, $\{0\}$ is a prime submodule of $M$. Therefore, in view of Proposition 3.18, we are done. $\Box$

**Lemma 3.20.** *Let $M$ be a module over a commutative ring $R$ with $S = End_R(M)$. If $N$ is a prime $S$-submodule of $M$, then it is a prime $R$-submodule of $M$.*

**Proof.** Let $r \in R$ and $m \in M$ such that $rm \in N$. Suppose that $m \notin N$. Since $R$ is commutative, the map $\theta(x) = rx$ for all $x \in M$, is an element of $S$. We have $rm = \theta(m) = m.\theta \in N$. Since, $N$ is a prime $S$-submodule of $M$ and $m \notin N$, so $M.\theta \subseteq N$. This implies that $rM = M.\theta \subseteq N$. We conclude the result. $\square$

The next two results investigate the situations in which the square submodule of a given module $M$ is a prime submodule of $M$.

**Theorem 3.21.** *Let $M$ be a module over a commutative ring $R$ with $S = End_R(M)$. If $S$ is a regular integral domain, then $MJ$ is a prime submodule of $M$ for any ideal $J$ of $S$ with $MJ \neq M$.*

**Proof.** The hypothesis that $S$ is regular implies that for any arbitrary ideal $J$ of $S$, the epimorphism $\mu_J : M \otimes J \to MJ$ defined as $m \otimes s \mapsto ms$ for all $m \in M$ and $s \in J$ is a monomorphism. This implies that $M$ is a flat $S$-module. Now, by Corollary 3.19, $J$ is a prime ideal of $S$. Hence, in view of [9, Corollary 2.6], $MJ$ is a prime $S$-submodule of $M$. Consequently, $MJ$ is a prime $R$-submodule of $M$ by Lemma 3.20. $\square$

**Corollary 3.22.** *Let $M$ be a module over a commutative ring $R$ such that $S = End_R(M)$ is a regular integral domain. If $\square M \neq M$, then $\square M$ is a prime submodule of $M$.*

**Proof.** It follows from Theorem 3.21 and the fact that $\square M = Tr_S(M).M$, where $Tr_S(M)$ is the two sided ideal of $S$. $\square$

The remaining results of this subsection investigate the situations in which $\square^d M$ is a prime submodule of $M \otimes_R M$.

**Proposition 3.23.** *Let $M$ be a prime module over a commutative ring $R$. Then $0 :_M I$ is a prime submodule of $M$ for any ideal of $R$ provided that $IM \neq 0$.*

**Proof.** The hypothesis that $IM \neq 0$ implies that $0 :_M I \neq M$. Suppose that $rm \in 0 :_M I$ and $m \notin 0 :_M I$. We prove $rIM = 0$. Let $t \in I$ be arbitrary. Since $rm \in 0 :_M I$, we get $rtm = 0$. Now the hypothesis that $\{0\}$ is a prime submodule of $M$ implies that $m = 0$ or $rtM = 0$. If $m = 0$, then $mI = 0$, a contradiction. Therefore, $rtM = 0$. We conclude the result. $\square$

**Lemma 3.24.** *Let $M$ be a non-nil prime module over a commutative ring $R$ with $S = End_R(M)$. Then $0 :_M Tr_S(M)$ is a prime $R$-submodule of $M$.*

**Proof.** The hypothesis that $M$ is non-nil implies that $Tr_S(M).M \neq 0$. Therefore, $0 :_M Tr_S(M) \neq M$. Now, the assertion follows from a similar argument as Proposition 3.23. $\square$

**Corollary 3.25.** *Let $M$ be a flat and prime module over a commutative ring $R$ with $S = End_R(M)$. If $M$ is non-nil, then $\square^d M$ is a prime submodule of $M \otimes_R M$ provided that $(0 :_M Tr_S(M)) \otimes_R M$ is a proper submodule of $M \otimes_R M$.*

**Proof.** It follows from Theorems 3.7, Lemma 3.24 and [9, Theorem 2.4.]. $\square$

### 3.3 Small and essential square submodule and its dual notion

In this subsection, we investigate the situation in which the square submodule of an arbitrary module $M$ and its dual notion are either an a-small or an $S$-essential submodule of $M$ or $M \otimes_R M$, respectively. Moreover, we have some results related to the notions of small and essential submodules of a module $M$ which may be of independent interest. We recall some definitions from [4] and [5]. Let $M$ be a module over a commutative ring $R$ with $S = End_R(M)$. A submodule $N$ of $M$ is called small or superfluous, denoted by $N \ll M$, if for any submodule $X$ of $M$, $X + N = M$ implies that $X = M$. Following Amouzegar-Kalati and Keskin-Tütüncü [4], $N$ is called annihilator small (a-small), denoted by $N \ll_a M$, if for any submodule $X$ of $M$, $X + N = M$ implies that $(0 :_S M) = 0$. Moreover, $N$ is called essential, denoted by $N \subseteq_e M$, provided that for each submodule $L$ of $M$, $N \cap L = 0$ implies that $L = 0$. Following Amouzegar [5], $N$ is called $S$-essential, denoted by $N \subseteq_{s-e} M$, provided that for each submodule $X$ of $M$, $N \cap X = 0$

implies that $(X :_S M) = 0$. An $R$-module $M$ is called retractable if for any non-zero submodule $N$ of $M$ there exists a non-zero $\alpha : M \to N$. Moreover, $M$ is called co-retractable if for any proper submodule $N$ of $M$, there exists a non-zero $\alpha : M \to M$ such that $\alpha(N) = 0$. Let $U$ and $M$ be $R$-modules. Then, $U$ is said to be $M$-projective in case for each $R$-epimorphism $g : M \to N$ and each $R$-homomorphism $f : U \to N$ there exists an $R$-homomorphism $h : U \to M$ such that $f = gh$. If $U$ is $U$-projective, then $U$ is said to be quasi-projective. If $U$ is $M$-projective for every $R$-module $M$, then $U$ is said to be projective. A projective cover of a module $M$ is an epimorphism $p : P \to M$ such that $P$ is a projective module and $\ker(p) \ll P$.

At first, we prove some results related to generalized small and essential submodules of an arbitrary module $M$. In particular, we have some generalization to [6, Corollary 5.15 and Proposition 5.14].

**Theorem 3.26.** *Let $M$ and $P$ be modules over a commutative ring $R$ and $P \xrightarrow{\theta} M$. Let $N$ be a submodule of $M$ such that $\theta(P)+N = M$. Then, $\theta^{-1}(N) \ll P$ exactly if any submodule $P' \subseteq P$ satisfying $\theta(P')+N = M$ implies that $P' = P$.*

**Proof.** $(\Longrightarrow)$ : Let $\theta^{-1}(N)$ be a small submodule of $P$. Moreover, let $P' \subseteq P$ be such that $\theta(P') + N = M$. We prove that $P' = P$. We claim that $P' + \theta^{-1}(N) = P$. To do this, suppose that $y \in P$ be arbitrary. Therefore, we have $\theta(y) \in M = \theta(P') + N$. Hence, there exist elements $z \in P'$ and $n \in N$ such that $\theta(y) = \theta(z) + n$. This implies that $y - z \in \theta^{-1}(N)$. Therefore, $y = z + (y - z) \in P' + \theta^{-1}(N)$. Consequently, $P \subseteq P' + \theta^{-1}(N)$. We conclude that $P' + \theta^{-1}(N) = P$. Now the hypothesis that $\theta^{-1}(N)$ is a small submodule of $P$, yields $P' = P$.

$(\Longleftarrow)$ : We prove that $\theta^{-1}(N)$ is a small submodule of $P$. Let $L$ be a submodule of $P$ such that $\theta^{-1}(N)+L = P$. This implies that $\theta(\theta^{-1}(N)+L) = \theta(P)$. Hence $\theta(\theta^{-1}(N)) + \theta(L) = \theta(P)$. We get $\theta(\theta^{-1}(N)) + \theta(L) + N = \theta(P) + N = M$. The fact that $\theta(\theta^{-1}(N)) \subseteq N$, implies that $\theta(L) + N = M$. Consequently, the hypothesis implies that $L = P$. We conclude the result. $\square$

**Corollary 3.27.** *Let $P \xrightarrow{\theta} M \to 0$ be an exact sequence of $R$-modules. Then, $\ker(\theta)$ is a small submodule of $P$ exactly if any submodule $P' \subseteq P$ satisfying $\theta(P') = M$ implies that $P' = P$. In particular, $P \xrightarrow{\theta} M \to 0$*

*is a projective cover for $M$ exactly if any submodule $P' \subseteq P$ satisfying $\theta(P') = M$ implies that $P' = P$.*

**Proof.** Put $N = 0$ in Theorem 3.26. □

**Theorem 3.28.** *Let $M$ and $P$ be modules over a commutative ring $R$ and $P \xrightarrow{\theta} M$. Let $N$ be a submodule of $M$ such that $\theta(P) + N = M$. Then, $\theta^{-1}(N) \ll P$ exactly if any module $A$ and any $R$-homomorphism $A \xrightarrow{h} P$ satisfying $\theta h(A) + N = M$ imply that $h(A) = P$.*

**Proof.** $(\Longrightarrow)$ : Let $\theta^{-1}(N)$ be a small submodule of $P$. In view of Theorem 3.26, we put $P' = h(A)$. Then, we have $h(A) = P$.

$(\Longleftarrow)$ : We prove that $\theta^{-1}(N)$ is a small submodule of $P$. In view of Theorem 3.26, let $\theta(P') + N = M$ for some submodule $P'$ of $P$. This implies that $\theta i(P') + N = M$, where $P' \xrightarrow{i} P \xrightarrow{\theta} M$. The hypothesis implies that $i(P') = P$. Consequently, $P' = P$. □

**Corollary 3.29.** *[6, Corollary 5.15] An epimorphism $g : M \to N$ is superfluous if and only if for all homomorphisms (equivalently, monomorphisms) $h$, if $gh$ is epic, then $h$ is epic.*

**Proof.** Put $N = 0$ in Theorem 3.28. □

**Proposition 3.30.** *[6, Proposition 5.14] For a submodule $K$ of $M$ the following statements are equivalent:*

1. *$K \ll M$.*
2. *The natural map $p_K : M \to M/K$ is a superfluous epimorphism.*
3. *For every module $N$ and for every $h \in Hom(N, M)$ the relation $Imh + K = M$ implies $Imh = M$.*

**Proof.** Put $N = 0$ in Theorem 3.28 and suppose that $\theta = p_K$. Then we have $p_K^{-1}(0) \ll M$ exactly if any module $A$ and any $R$-homomorphism $A \xrightarrow{h} M$ satisfying $p_k h(A) + 0 = M/K$ implies that $h(A) = M$. But we have $M/K = p_k h(A) = (h(A) + K)/K$. This implies that $K \ll M$ exactly if any module $A$ and any $R$-homomorphism $A \xrightarrow{h} M$ satisfying $h(A) + K = M$ we have $h(A) = M$. We conclude the result. □

The next two results investigate the situations in which the square submodule of a given module $M$ is either an essential or an $S-$essential submodule of $M$.

**Lemma 3.31.** *Let $M$ be a module over a commutative ring $R$ with $S = End_R(M)$. Let $T$ be a submodule of $M$ such that $(T :_S M)$ is an essential ideal of $S$. Then, $(T :_S M)M$ is an S-essential submodule of $M$. In particular, $T$ is an S-essential submodule of $M$.*

**Proof.** Let $(T :_S M)M \cap X = 0$ for some submodule $X$ of $M$. Clearly, we have $(T :_S M) \cap (X :_S M) = 0$. Now, the hypothesis that $(T :_S M)$ is an essential ideal of $S$, implies that $(X :_S M) = 0$. We conclude that $(T :_S M)M$ is an S-essential submodule of $M$.

Moreover, we observe that if $N$ is a submodule of $T$ such that $N$ is an S-essential submodule of $M$, then $T$ is an S-essential submodule of $M$. Now, the last assertion is clear. $\square$

**Proposition 3.32.** *Let $M$ be a module over a commutative ring $R$ with $S = End_R(M)$. If $Tr_S(M)$ is an essential (left) ideal of $S$, then we have the following.*

1. *$(\square M :_S M)M$ is an $S-$essential submodule of $M$. In particular $\square M$ is an $S-$essential submodule of $M$.*
2. *If $M$ is a retractable module, then $\square M$ is an essential submodule of $M$.*

**Proof.** First, we observe that $Tr_S(M) \subseteq (\square M :_S M) \subseteq S$. Infact, given any arbitrary generator $\varphi(m) \in Tr_S(M)$ for $m \in M$ and $\varphi : M \to S$, we have $\varphi(m)(n) \in \square M$ for all $n \in M$. This implies that $\varphi(m)(M) \subseteq \square M$, hence we get the desired inclusion. Now the hypothesis that $Tr_S(M)$ is an essential ideal of $S$, implies that $(\square M :_S M)$ is an essential ideal of $S$. Now our assertion is clear by Lemma 3.31.

For the second assertion, we suppose that $\square M \cap X = 0$. We shall prove that $X = 0$. By the way of contradiction, suppose that $X \neq 0$. Then, the hypothesis that $M$ is retractable, implies that there exists a non-zero homomorphism $\beta : M \to X$. This means $\beta \in (X :_S M)$. Consequently, $(X :_S M) \neq 0$, a contradiction. $\square$

The final two results investigate the situations in which $\square^d M$ is a small submodule of $M \otimes_R M$.

**Proposition 3.33.** *Let $M$ be a module over a commutative ring $R$ with $S = End_R(M)$. If $Tr_S(M)$ is an essential (left) ideal of $S$, then we have the following.*

1. $(0 :_M Tr_S(M))$ *is an $a-$small submodule of $M$.*
2. *If $M$ is a co-retractable module, then $(0 :_M Tr_S(M))$ is a small submodule of $M$.*

**Proof.** Let $(0 :_M Tr_S(M)) + X = M$ for some submodule $X$ of $M$. Then, we have $0 = (0 :_S M) = (0 :_S [(0 :_M Tr_S(M)) + X]) = [0 :_S (0 :_M (Tr_S(M)))] \cap (0 :_S X)$. Now in view of the fact that $Tr_S(M) \subseteq [0 :_S (0 :_M (Tr_S(M)))]$, we have $Tr_S(M) \cap (0 :_S X) \subseteq [0 :_S (0 :_M (Tr_S(M)))] \cap (0 :_S X) = 0$. Therefore, $Tr_S(M) \cap (0 :_S X) = 0$. But, $Tr_S(M)$ is an essential ideal of $S$, hence $(0 :_S X) = 0$ as desired.

For the second assertion, suppose that $(0 :_M Tr_S(M)) + X = M$. By the way of contradiction, suppose that $X \neq M$. Since $M$ is co-retractable, there exists a non-zero $\beta : M \to M$ such that $\beta(M) = 0$. Therefore, $\beta \in (0 :_S X)$. This means $(0 :_S X) \neq 0$, a contradiction. $\square$

**Corollary 3.34.** *Let $M$ be a co-retractable faithful finitely generated multiplication flat module over a commutative ring $R$ with $S = End_R(M)$. If $Tr_S(M)$ is an essential (left) ideal of $S$, then $\square^d M$ is a small submodule of $M \otimes_R M$.*

**Proof.** In view of Proposition 3.33, we get that $(0 :_M Tr_S(M))$ is an small submodule of $M$. Now that assertion follows from Theorem 3.7 and [10, Corollary 13]. $\square$

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