Journal of Mathematical Extension
Vol. 15, SI-NTFCA, (2021) (14)1-17
URL: https://doi.org/10.30495/JME.SI.2021.2081
ISSN: 1735-8299
Original Research Paper

Approximate Solution for High Order Fractional Integro-Differential Equations Via an Optimum Parameter Method

B. Agheli*
Qaemshahr Branch, Islamic Azad University

R. Darzi
Neka Branch, Islamic Azad University

A. Dabbaghian
Neka Branch, Islamic Azad University

Abstract. The most significant objective of this article is the adoption of a method with a free parameter known as "The Optimum Asymptotic Homotopy Method" which has been utilized in order to obtain solutions for integral differential equations of high-order non integer derivative. The process in this method is more favorable than "Homotopy Perturbation Method" as it has a more rapid convergence compared to the mentioned method or even the similar methods. Another advantage of this method is that the convergence rate is recognized as control area. It is worth mentioning that Caputo derivative is adopted in this article. A number of instances are provided to better understand the method and its level of efficiency compared to other same methods.

AMS Subject Classification: 47G20; 65F10; 26A33.
Keywords and Phrases: Integro-differential equations; Optimum asymptotic homotopic method (OAHM); Caputo derivative.

Received: March 2021; Published: August 2021
*Corresponding Author

1 Introduction

The issue of fractional arithmetic and differential equations of non integer order in particular have been efficient and taken into consideration in various sciences such as mathematical physics [13], chemistry [31, 32], economy [35], traffic model [30], medicine [20], dynamic issues [8, 9, 29], fluid flows (waterfall) [23], optimal control [6] and etc [7, 10, 11, 24–26, 33]. Researchers who are interested in such spheres can investigate and study books and articles written in terms of such fields, sources that comprehensively discuss how to generalize a regular arithmetic.

In this work, we apply an optimum asymptotic homotopy method (OAHM) to gain the approximate solution of SFIBVPs

$$D^{\vartheta}\mathfrak{y}(t) - \eta\int_0^t \mathfrak{k}(t,s)G[\mathfrak{y}(s)]ds = \mathfrak{g}(t), \quad p < \vartheta \leq p+1, \ p \in \mathbb{Z}^+, \tag{1}$$

subject to

$$\mathfrak{y}(0) = \gamma_0, \ \mathfrak{y}^{(i)}(0) = \gamma_i, \ \mathfrak{y}(b) = \theta_0, \ \mathfrak{y}^{(i)}(b) = \theta_i, \tag{2}$$

where $0 < t < b$ and D^{ϑ} denotes the fractional differential operator of order ϑ and given by

$$D^{\vartheta}\mathfrak{y}(t) = \frac{1}{\Gamma(p+1-\vartheta)}\int_0^t (t-s)^{\vartheta-1}\mathfrak{y}^{(p)}(s)ds, \tag{3}$$

in which $p < \vartheta \leq p+1$ and $p \in Z^+$.

The OAHM was presented and developed by Marinca et al. [21] and it can be shown that HPM is a special case of OAHM. Several authors have proved the effectiveness, generalization and reliability of this method. The advantage of OAHM is built in convergence criteria, which is controllable. In OAHM, the control and adjustment of the convergence region are provided in a convenient way. In next section we present description of OAHM.

Many mathematical models in the domain of engineering [18] and other scientific fields of study [36, 38] can be expressed by fractional-integral equations. From among them we can point out some issues in the modeling of turbulent aerodynamic phenomena, issues of dynamism in a particular population and the issue of heat transfer in composite material with special properties.

In spite of all investigations and theoretical numerical researches in the field of boundary value issues related to fractional differential equations and differential-integral equations [28], it is indicated that investigations in this sphere are still in their initial stages.

It should be noted that, a large number of FIDEs and IDEs find the exact solution is difficult or not possible. This makes use of the approximate or numerical solution methods to support obtaining a solution to this problem. Using approximate or numerical solution methods in order to solve the FDEs, FIDEs and SFIBVPs has been proposed by the scholars who have recorded that including the following: homotopy analysis method and q-homotopy analysis method [1, 4], variational iteration method [3, 27], Laplace transform method [16], Adomian's decomposition method [3], homotopy perturbation method and optimal homotopy perturbation method [27] and collocation method [12, 40] and so on [19, 22, 39].

This paper is organized as follows: in Section 2, description if OAHM is given. In Section 3, we have expressed the convergence of OHAM. In Section 4, the application of OAHM to the Eqs. 1 and 2 are illustrated, and some numerical examples are presented. And conclusions are drawn in Section 5.

2 Dissection Of *OAHM*

The general dimension of the proposed approach [2] in this part is given below and represented in the following differential equation

$$(\mathfrak{L}+\mathfrak{N})(\zeta(\tau))+\mathfrak{g}(\tau)=0,\ \tau\in\Omega, \tag{4}$$

containing

$$B(\zeta,\frac{\partial\zeta}{\partial\tau})=0,\ \tau\in\Gamma,$$

as boundary conditions.

In which $\mathfrak{L}$ and $\mathfrak{N}$ are linear and nonlinear operators respectively, $\zeta(\tau)$ is an undefined function, ζ is an independent variable representation and ultimately $\mathfrak{g}(\tau)$ is a defined function.

Now the approach is described:
First, based on the above-mentioned approach, the following homotopic structure is considered

$$\hbar\left(v(\tau;\varrho),\varrho\right):\Omega\times[0,1]\to\mathbb{R},$$

Now the following zero-order equation will be considered:

$$(1-\varrho)\Big(\mathbf{L}(v(x;\varrho))+\mathfrak{g}(x)\Big)=\mathfrak{H}(p)\Big(A(v(x;\varrho))+\mathfrak{g}(x)\Big),\tag{5}$$

Here $\varrho\in[0,1]$ is an embedded parameter and $\tau\in\mathbb{R}$, for $\varrho\neq 0$, $\mathfrak{H}(\varrho)$ is auxiliary function and $\mathfrak{H}(0)=0$. Assuming the conditions $\varrho=0$ and $\varrho=1$, respectively, the following relations are established
$v(\tau;0)=\zeta_0(\tau),\quad v(\tau;1)=\zeta(\tau).$

Therefore, when ϱ grow from zero to one, $v(\tau;\varrho)$ changes from the initial conjecture $\zeta_0(\tau)$ to the solution $\zeta(\tau)$. It should be noted that the initial conjecture $\zeta_0(\tau)$ is accurate in the initial condition and

$$\mathfrak{L}\big(\zeta_0(\tau)\big)+\mathfrak{g}(\tau)=0.\tag{6}$$

It is assumed that the ratio to ϱ has the following Taylor series expansion:

$$\mathfrak{H}(\varrho)=\varrho c_1+\varrho^2 c_2+\varrho^3 c_3+\dots,\tag{7}$$

where $c_1,c_2,c_3,\dots$ are defined the convergence control parameters that are unknown and their calculation approaches will be clarified in the examples that are provided in the continuation of the work.

In order to calculate the approximate solution of the problem, the expansion of the Taylor series around the point p for $v(\tau;\varrho,c_k)$, will be written:

$$v(\tau;\varrho,c_k)=\zeta_1(x)+\sum_{r=1}^{\infty}\zeta_k(\tau;c_r)\,\varrho^r,\quad r=1,2,\dots.\tag{8}$$

Defining the vectors

$$\vec{c_l}=\{c_1,c_2,\dots,c_l\},\tag{9}$$

and

$$\vec{\zeta_s}=\big\{\zeta_0(\tau),\zeta_1(\tau;\vec{c_1}),\dots,\zeta_s(\tau;\vec{c_s})\big\}.$$

the first and second order equations will be considered as (Equation (6), is characterized as the zero order.):

$$\mathfrak{L}\big(\zeta_1(\tau)\big) = c_1\,\mathfrak{N}_0(\vec{\zeta_0}) + \mathfrak{g}(\tau) \tag{10}$$

and second-order equation by

$$\mathfrak{L}\big(\zeta_2(\tau)\big) - \mathfrak{L}\big(\zeta_1(\tau)\big) = c_2\,\mathfrak{N}_0(\vec{\zeta_0}) + c_1\Big(\mathfrak{L}\left(\zeta_1(\tau)\right) + \mathfrak{N}_1\left(\vec{\zeta_1}\right)\Big). \tag{11}$$

In general, the equations for the script k, ie u_k, are as follows:

$$\mathfrak{L}\big(\zeta_\iota(\tau)\big) - \mathfrak{L}\big(\zeta_{\iota-1}(\tau)\big) = c_\iota\,\mathfrak{N}_0\big(\zeta_0(\tau)\big) + \sum_{m=1}^{\iota-1} c_m\Big(\mathfrak{L}\left(\zeta_{\iota-m}(\tau)\right) + \mathfrak{N}_{\iota-m}\left(\vec{\zeta}_{\iota-1}\right)\Big), \tag{12}$$

in which $\iota = 2, 3, \ldots$ and $\mathfrak{N}_m\big(\zeta_0(\tau), \zeta_1(\tau), \ldots, \zeta_m(\tau)\big)$ is the coefficient of "ϱ^m", in the extension of $\mathfrak{N}\big(v(\zeta;\varrho)\big)$, about the index parameter "ϱ" and we have

$$\mathfrak{N}\big(v(\zeta;\varrho,c_\iota)\big) = \mathfrak{N}_0\big(\zeta_0(\tau)\big) + \sum_{m=1}^{\infty} \mathfrak{N}_m\left(\vec{\zeta}_m\right)\varrho^m. \tag{13}$$

It is easy to realize that the convergence of the series (8), pertain the coefficients $c_1, c_2, \ldots$.

$$\tilde{v}(\tau;\varrho;c_i) = \zeta_0(\tau) + \sum_{k=1}^{m} \zeta_\iota(\tau;c_i)\varrho^\iota, \quad i = 1, 2, \ldots. \tag{14}$$

The below residual is a result of embedded (14) in (4):

$$R(\tau;c_i) = \mathfrak{L}\big(\tilde{v}(\tau;\varrho,c_i)\big) + \mathfrak{g}(\tau) + \mathfrak{N}\big(\tilde{v}(\tau;\varrho,c_i)\big), \quad i = 1, 2, \ldots. \tag{15}$$

If $R = 0$, then $\tilde{v}$ will be the accurate solution 4.

By the use of the minimum squares methodology and recognition of the accurate solution to the problem, the the L^2-norm of the error

$$E\,v_m(c_1,\ c_2,\ c_3, \ldots,\ c_m).$$

can be minimized,

The L^2-norm of the eror is signified as

$$\|\mathfrak{E}\,\widetilde{v}_m(c_1,\ldots,\ c_m)\|_2 = \left(\int_\Omega\int_\Gamma \widetilde{v}_m^2(\tau,t)\,dt\,d\tau\right)^{\frac{1}{2}},$$

in which $\mathfrak{E}\,\widetilde{v}_m(\tau,t) = |\widetilde{v}_{accurate}(\tau,t) - \widetilde{v}_m(\tau,t;\,c_1,\ldots,\ c_m)|$.

3 Convergence Of *OAHM*

Topics in this part are prepared for analysis and expression of convergence for the *OAHM*.

Theorem 3.1. *[14] Let the solution components $\zeta_0,\zeta_1,\zeta_2,\ldots,$ be defined as given in Eqs.(11)-(12). The series solution $\sum_{\iota=0}^{m-1}\zeta_\iota(\tau,t)$ defined in 14 converges, if $\exists\ 0<\kappa<1$ such that $\|\zeta_{\iota+1}\|\leq\kappa\|\zeta_\iota\|\ \forall\iota\geq\iota_0$ for some $\iota_0\in\mathbb{N}$.*

Proof. Under consideration

$\mathfrak{T}_0 = \zeta_0$

$\mathfrak{T}_1 = \zeta_0+\zeta_1$

$\mathfrak{T}_2 = \zeta_0+\zeta_1+\zeta_2$

$\ldots$

$\mathfrak{T}_n = \zeta_0+\zeta_1+\zeta_2+\ldots+\zeta_n,$

as the sequence $\{\mathfrak{T}_n\}_{n=0}^{\infty}$. Evidence is sufficient to show that the sequence $\{\mathfrak{T}_n\}_{n=0}^{\infty}$ in the Hilbert space $\mathbb{R}$ is a Cauchy sequence. To achieve this, consider

$$\begin{aligned}\|\mathfrak{T}_{n+1}-\mathfrak{T}_n\| &= \|\zeta_{n+1}\|\\ &\leq \kappa\|\zeta_n\|\\ &\leq \kappa^2\|\zeta_{n-1}\|\\ &\vdots\\ &\leq \kappa^{n-\iota_0+1}\|\zeta_{\iota_0}\|.\end{aligned}$$

Assuming that $n \geq m > \iota_0$ and for every $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\|\mathfrak{T}_n - \mathfrak{T}_m\| &= \|(\mathfrak{T}_n - \mathfrak{T}_{n-1}) + (\mathfrak{T}_{n-1} - \mathfrak{T}_{n-2}) + \ldots + (\mathfrak{T}_m - \mathfrak{T}_{m-1})\| \\
&\leq \|(\mathfrak{T}_n - \mathfrak{T}_{n-1})\| + \|(\mathfrak{T}_{n-1} - \mathfrak{T}_{n-2})\| + \ldots + \|(\mathfrak{T}_m - \mathfrak{T}_{m-1})\| \\
&\leq \kappa^{n-\iota_0}\|\zeta_{\iota_0}\| + \kappa^{n-\iota_0-1}\|\zeta_{\iota_0}\| + \ldots + \kappa^{m-\iota_0+1}\|\zeta_{\iota_0}\| \\
&= (\frac{1-\kappa^{n-m}}{1-\kappa})\kappa^{m-\iota_0+1}\|\zeta_{\iota_0}\|.
\end{aligned}$$

According to the $0 < \kappa < 1$, it results that $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} \|\mathfrak{T}_n - \mathfrak{T}_m\| = 0$. Thereupon, in the Hilbert space $\mathbb{R}$, sequence $\{\mathfrak{T}_n\}_{n=0}^{\infty}$ is a Cauchy sequence and this implies that series solution converges to series $\sum\limits_{\iota=0}^{\infty} \zeta_\iota(\tau, t)$.

4 Test Example

In all these examples in this section, mathematical software *Mathematica* is used for calculations and graphs. Also, approximate solutions are obtained compared with methods *VHPIM* [30] and *Oq.HAM*.

Example 4.1. We offer the Volterra integro-differential equation [28]:

$$D^{\vartheta}\zeta(\varsigma) + \int_0^{\varsigma} \zeta(\varsigma)d\varsigma = 1,\ 0 \leq \varsigma \leq 1,\ 0 \leq \vartheta \leq 1, \tag{16}$$

with the precise solution $\zeta(\varsigma) = sin(\varsigma)$ for $\vartheta = 1$ and the primary condition:

$$\zeta(0) = 0. \tag{17}$$

Following the OAHM, according to what was formulated and presented in section 2 for Eqs.(16-17), we get:

$$\begin{aligned}
\zeta_0(\varsigma) &= \frac{\varsigma^{\vartheta}}{\Gamma(\vartheta+1)}, \\
\zeta_1(\varsigma) &= \frac{c_1\varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)}, \\
\zeta_2(\varsigma) &= \frac{c_1^2\varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)} + \frac{c_1\varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)} + \frac{c_2\varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)} + \frac{c_1^2\varsigma^{3\vartheta+2}}{\Gamma(3\vartheta+3)} \\
&\ldots .
\end{aligned}$$

In the below, estimates of solution for Eq.(16) with the first three sentences is given:

$$\zeta(\varsigma) \approx \frac{c_1^2 \varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)} + \frac{2c_1 \varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)} + \frac{c_2 \varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)} + \frac{c_1^2 \varsigma^{3\vartheta+2}}{\Gamma(3\vartheta+3)} + \frac{\varsigma^{\vartheta}}{\Gamma(\vartheta+1)}. \tag{18}$$

According to least square method(LSM) for the calculations of the constants c_1 and c_2, we can gain

$$c_1 = -0.978948, \quad c_2 = 0.000501513.$$

Table 1: The estimated solutions to $\vartheta = 1$ and different values of ζ 4.1.

x	u_{VHPIM}	$u_{Oq.HAM}$	u_{OAHM}	$Exact$
0.0	0.0	0.0	0.0	0.0
0.2	0.198669	0.198669	0.19867	0.198669
0.4	0.389418	0.389419	0.389425	0.389418
0.6	0.564642	0.564648	0.564655	0.564642
0.8	0.717356	0.717397	0.717364	0.717356
1.0	0.841470	0.841667	0.841477	0.841471

Example 4.2. We propound the Volterra integro-differential equation [28]:

$$D^{\vartheta}\zeta(\varsigma) - \int_0^{\varsigma} (\varsigma - \tau)\zeta(\tau)d\tau = 1, \; 0 \le \varsigma \le 1, \; 1 \le \vartheta \le 2, \tag{19}$$

given that the primary condition

$$\zeta(0) = 1, \qquad \zeta'(0) = 0. \tag{20}$$

From the OAHM, like to what was introduced in section 2 for Eqs.

(19-20), we get:

$$\begin{aligned}
\zeta_0(x) =& 1 + \frac{\varsigma^{\vartheta}}{\Gamma(\vartheta+1)},\\
\zeta_1(x) =& \frac{c_1\varsigma^{\vartheta+2}}{\Gamma(\vartheta+3)} - \frac{c_1\varsigma^{2\vartheta+2}}{\Gamma(2\vartheta+3)},\\
\zeta_2(x) =& \frac{c_1^2\varsigma^{\vartheta+2}}{\Gamma(\vartheta+3)} - \frac{c_1\varsigma^{\vartheta+2}}{\Gamma(\vartheta+3)} - \frac{c_2\varsigma^{\vartheta+2}}{\Gamma(\vartheta+3)} + \frac{c_1^2\varsigma^{3\vartheta+4}}{\Gamma(3\vartheta+5)} +\\
& \frac{c_1^2\varsigma^{2\vartheta+2} - c_2\varsigma^{2\vartheta+2} + \frac{c_1^2\varsigma^{2\vartheta+4}}{2(\vartheta+2)(2\vartheta+3)} - c_1\varsigma^{2\vartheta+2}}{\Gamma(2\vartheta+3)},\\
& \dots .
\end{aligned}$$

Then, the first three terms as assessment of solution for Eq.(19) is as

$$\begin{aligned}
\zeta(\varsigma) \approx\ & 1 + \frac{c_1^2\varsigma^{3\vartheta+4}}{\Gamma(3\vartheta+5)} + \frac{\varsigma^{\vartheta}\left((\vartheta+1)(\vartheta+2) + \left((c_1-2)\,c_1 - c_2\right)\varsigma^2\right)}{\Gamma(\vartheta+3)} +\\
& \frac{\varsigma^{2\vartheta+2}\left(-4(\vartheta+2)(2\vartheta+3)c_1 - 2(\vartheta+2)(2\vartheta+3)c_2\right)}{\Gamma(2\vartheta+5)} +\\
& \frac{\varsigma^{2\vartheta+2}\left(c_1^2\left(2(\vartheta+2)(2\vartheta+3) + \varsigma^2\right)\right)}{\Gamma(2\vartheta+5)}.
\end{aligned} \tag{21}$$

We gain the constants c_1, c_2 using the LSM, as folows

$$c_1 = -0.999895, \quad c_2 = 3.99958.$$

In Table 2, we can view the accurate and assessment solutions featuring $\vartheta = 2$ through applying OAHM.

With $\vartheta = 2$, the assessment solution gained by the mentioned method corresponds to the accurate solution $\zeta(\varsigma) = \cosh(\varsigma)$.

Example 4.3. For the fourth instance, consider the Volterra integro-differential equation [28]:

$$D^{\vartheta}\zeta(\varsigma) + \int_0^{\varsigma}(\varsigma-\tau)\zeta(\tau)d\tau = -1,\ 0 \le \varsigma \le 1,\ 3 \le \vartheta \le 4, \tag{22}$$

although the primary condition

$$\zeta(0) = -1, \quad \zeta'(0) = 1, \quad \zeta''(0) = -1, \quad \zeta'''(0) = 1. \tag{23}$$

Table 2: assessment result of test example 4.2.

x	u_{VHPIM}	$u_{Oq.HAM}$	u_{OAHM}	$Exact$
0.0	1.0	1.0	1.0	1.0
0.2	1.020066941	1.02007	1.01993	1.02007
0.4	1.081085602	1.08107	1.07893	1.08107
0.6	1.185642306	1.18547	1.17454	1.18547
0.8	1.338637450	1.33743	1.30257	1.33743
1.0	1.548685515	1.54308	1.45697	1.54308

From the OAHM, like to what was introduced in section 2 for Eqs.(22-23), we get:

$$
\begin{aligned}
\zeta_0(\varsigma) &= -1+\varsigma-\frac{\varsigma^2}{2}\frac{\varsigma^3}{6}-\frac{\varsigma^{\vartheta}}{\Gamma(\vartheta+1)},\\
\zeta_1(\varsigma) &= -\frac{c_1\varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)}+\\
&\quad \frac{c_1\varsigma^{\vartheta+1}}{\Gamma(\vartheta+5)}\left(-\vartheta^3-9\vartheta^2-26\vartheta+\varsigma^3-\vartheta\varsigma^2-4\varsigma^2+\vartheta^2\varsigma+7\vartheta\varsigma+12\varsigma-24\right)\\
&\quad \ldots .
\end{aligned}
$$

Then, the first three terms as assessment of solution for Eq.(22) is as:

$$
\begin{aligned}
\zeta(\varsigma) \approx& -1+\varsigma-\frac{\varsigma^2}{2}\frac{\varsigma^3}{6}-\frac{\varsigma^{\vartheta}}{\Gamma(\vartheta+1)}-\frac{c_1\varsigma^{2\vartheta+1}}{\Gamma(2\vartheta+2)}+\\
&\frac{c_1\varsigma^{\vartheta+1}}{\Gamma(\vartheta+5)}\left(-\vartheta^3-9\vartheta^2-26\vartheta+\varsigma^3-\vartheta\varsigma^2-4\varsigma^2+\vartheta^2\varsigma+7\vartheta\varsigma+12\varsigma-24\right).
\end{aligned}
\tag{24}
$$

We gain the constants c_1 and c_2 using the LSM, as folows

$$c_1 = 0, \quad c_2 = 0.999393.$$

With $\vartheta = 4$, the assessment solution gained by the mentioned method corresponds to the accurate solution $\zeta(\varsigma) = \sinh(\varsigma) - \cosh(\varsigma)$

Table 3: Approximate result of test example 4.3.

ς	ζ_{VHPIM}	ζ_{OAHM}	*accurate*	*Absolute error*
0.0	-1.0	-1.0	-1.0	0.0
0.2	-0.817731	-0.818731	-0.818731	1.56698×10^{-9}
0.4	-0.67022	-0.67032	-0.67032	1.56698×10^{-8}
0.6	-0.548612	-0.548812	-0.548812	3.55738×10^{-7}
0.8	-0.44922	-0.44933	-0.449329	1.43263×10^{-6}
1.0	-0.367669	-0.367884	-0.367879	4.07205×10^{-6}

5 Conclusion

We have successfully applied OAHM to obtain approximate solution of the fractional We have successfully applied OAHM to obtain approximate solution of the fractional integro-differential equations. The result indicate that a few iteration of OAHM will result in some useful solutions.

Finally, it should be added that the suggested technique has the potentials to be practical in solving other similar nonlinear and linear problems in partial differential equations featuring fractional derivative.

Appendix A: Illustration Of OAHM With Details

Consider test example (4.1):

$$D^{\vartheta}\zeta(\varsigma) + \mu \int_0^{\varsigma} K(\varsigma)G(\zeta(\tau))\, d\tau = \mathfrak{g}(\varsigma),\ 0 \leq \varsigma \leq 1,\ L \leq \vartheta \leq L+1, \quad (25)$$

With considering

$$\phi(\varsigma; \varrho, c_1, c_2, \cdots) = \zeta_0 + \sum_{i=1}^{\infty} \zeta_i \varrho^i, \quad (26)$$

and

$$\mathfrak{H}(\varrho) = \varrho\, c_1 + \varrho^2 c_2 + \cdots, \quad (27)$$

and using of OAHM featuring equations (25), (26) and (27)

$$\begin{aligned}\Big(D^{\vartheta}\zeta_0+ \varrho D^{\vartheta}\zeta_1+ \varrho^2 D^{\vartheta}\zeta_2 + \cdots - \mathfrak{g}(\varsigma)\Big) - \\ \Big(\varrho D^{\vartheta}\zeta_0 + \varrho^2 D^{\vartheta}\zeta_1 + \varrho^3 D^{\vartheta}\zeta_2 + \cdots - \varrho\, \mathfrak{g}(\varsigma)\Big) - \\ c_1\varrho\left(D^{\vartheta}\zeta_0+ \varrho D^{\vartheta}\zeta_1 + \varrho^2 D^{\vartheta}\zeta_2 + \cdots\right) - \\ c_1\varrho\,\mu\left(\int_0^{\varsigma} K(\varsigma)G(\zeta_0 + \varrho\,\zeta_1 + \varrho^2\zeta_2 + \cdots)\,d\tau\right) - \\ - c_1\varrho\, g(\varsigma)c_2\varrho^2\left(D^{\vartheta}u_0+ \varrho D^{\vartheta}\zeta_1 + \varrho^2 D^{\vartheta}\zeta_2 + \cdots\right) - \\ c_2\varrho^2\,\mu\left(\int_0^{\varsigma} K(\varsigma)G(\zeta_0 + \varrho\,\zeta_1 + \varrho^2\zeta_2 + \cdots)\,d\tau\right) - \\ c_2\varrho^2\,\mathfrak{g}(\varsigma) + \cdots = 0.\end{aligned}$$

Zero th order problem can be obtained with equating the coefficients of different power in "ϱ".

References

[1] S. Abbasbandy, M. S. Hashemi, & I. Hashim, On convergence of homotopy analysis method and its application to fractional integro-differential equations, *Quaestiones Mathematicae*, 36 (1) (2013), 93-105.

[2] L. Ali, S. Islam, T. Gul, I. Khan, & L. C. C. Dennis, New version of Optimal Homotopy Asymptotic Method for the solution of nonlinear boundary value problems in finite and infinite intervals, *Alexandria Engineering Journal*, 55 (3) (2016), 2811-2819.

[3] S. M. Amer, M. H. Saleh, M. A. Mohamed,& N. S. Abdelrhman, Variational Iteration Method and Adomian Decomposition Method for Fourth-Order Fractional Integro-Differential Equations, *International Journal of Computer Applications*, 80 (6) (2013).

[4] D. Baleanu, R. Darzi, & B. Agheli, New study of weakly singular kernel fractional fourth-order partial integro-differential equations

based on the optimum q-homotopic analysis method, *Journal of Computational and Applied Mathematics*, 320 (2017), 193-201.

[5] D. Baleanu, & A. C. Luo, *Discontinuity and Complexity in Nonlinear Physical Systems*, J. T. Machado (Ed.), Springer, (2014).

[6] D. Baleanu, S. S. Sajjadi, J. H. Asad, A. Jajarmi, & E. Estiri, Hyperchaotic behaviors, optimal control, and synchronization of a nonautonomous cardiac conduction system, *Advances in Difference Equations*, 2021 (1) (2021), 157.

[7] D. Baleanu, A. Jajarmi, H. Mohammadi, & S. Rezapour, A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative, *Chaos, Solitons & Fractals*, 134 (2020), 109705.

[8] D. Baleanu, S. S. Sajjadi, A. Jajarmi, O. Defterli, & J. H. Asad, The fractional dynamics of a linear triatomic molecule, *Rom. Rep. Phys*, 73 (1) (2021), 105.

[9] D. Baleanu, S. S. Sajjadi, A. Jajarmi, & O. Defterli, On a nonlinear dynamical system with both chaotic and nonchaotic behaviors: a new fractional analysis and control, *Advances in Difference Equations*, 1 (2021), 234.

[10] Z. Bouazza, S. Etemad, M. S. Souid, S. Rezapour, F. Martínez, & M. K. Kaabar, A Study on the Solutions of a Multiterm FBVP of Variable Order, *Journal of Function Spaces*, 2021 (2021).

[11] A. Boutiara, M. M. Matar, M. K. Kaabar, F. Martínez, S. Etemad, S. Rezapour, (2021). Some Qualitative Analyses of Neutral Functional Delay Differential Equation with Generalized Caputo Operator, *Journal of Function Spaces*, (2021).

[12] M. R. Eslahchi, M. Dehghan,& M. Parvizi, Application of the collocation method for solving nonlinear fractional integro-differential equations, *Journal of Computational and Applied Mathematics*, 257 (2014), 105-128.

[13] O. Guner, & A. Bekir, The Exp-function method for solving nonlinear space-time fractional differential equations in mathematical physics, *Journal of the Association of Arab Universities for Basic and Applied Sciences*, (2017).

[14] A. K. Gupta, & S. S. Ray, Comparison between homotopy perturbation method and optimal homotopy asymptotic method for the soliton solutions of Boussinesq-Burger equations, *Computers & Fluids*, 103 (2014), 34-41.

[15] N. Herisanu,& V. Marinca, Explicit analytical approximation to large-amplitude non-linear oscillations of a uniform cantilever beam carrying an intermediate lumped mass and rotary inertia, *Meccanica*, 45 (6) (2010), 847-855.

[16] M. KA. Kaabar, F. Martínez, J. F. Gómez-Aguilar, B. Ghanbari, M. Kaplan, & H. Gunerhan, New approximate analytical solutions for the nonlinear fractional Schrodinger equation with second-order spatio-temporal dispersion via double Laplace transform method, *Mathematical Methods in the Applied Sciences*, (2021).

[17] A. A. Kilbas, H. M. Srivastava, d J.J. Trujillo, Theory and application of fractional differential equations, Elsevier B.V, *Netherlands*, (2006).

[18] E. G. Ladopoulos, *Singular integral equations: linear and non-linear theory and its applications in science and engineering*, Springer Science & Business Media, (2013).

[19] J. Korvenpaa, T. Kuusi, & G. Palatucci, Fractional superharmonic functions and the Perron method for nonlinear integro-differential equations, *Mathematische Annalen*, (2016), 1-47.

[20] R. L. Magin, O. Abdullah, D. Baleanu, & X. J. Zhou, Anomalous diffusion expressed through fractional order differential operators in the Bloch-Torrey equation, *Journal of Magnetic Resonance*, 190 (2) (2008), 255-270.

[21] V. Marinca, N. Herişanu, C. Bota, & B. Marinca, An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate, *Applied Mathematics Letters*, 22 (2) (2009), 245-251.

[22] F. Martínez, I. Martínez, M. K. Kaabar, & S. Paredes, Solving systems of conformable linear differential equations via the conformable exponential matrix, *Ain Shams Engineering Journal*, (2021).

[23] C. Ming, F. Liu, L. Zheng, I. Turner, & V. Anh, Analytical solutions of multi-term time fractional differential equations and application to unsteady flows of generalized viscoelastic fluid, *Computers & Mathematics with Applications*, 72 (9) (2016), 2084-2097.

[24] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, & S. Rezapour, Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives, *Advances in Difference Equations*, 2021 (1) (2021), 1-18.

[25] H. Mohammadi, M. K. Kaabar, J. Alzabut, A. Selvam, & S. Rezapour, A Complete Model of Crimean-Congo Hemorrhagic Fever (CCHF) Transmission Cycle with Nonlocal Fractional Derivative, *Journal of Function Spaces*, 2021 (2021).

[26] H. Mohammadi, S. Rezapour, & A. Jajarmi, On the fractional SIRD mathematical model and control for the transmission of COVID-19: the first and the second waves of the disease in Iran and Japan, *ISA transactions*, (2021), doi: 10.1016/j.isatra.2021.04.012.

[27] Y. Nawaz, Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations, *Computers & Mathematics with Applications*, 61 (8) (2011), 2330-2341.

[28] A. Neamaty, B. Agheli, & R. Darzi, Numerical solution of high-order fractional Volterra integro-differential equations by variational homotopy perturbation iteration method, *Journal of Computational and Nonlinear Dynamics*, 10 (6) (2015), 061023.

[29] A. Neamaty, M. Nategh, & B. Agheli, Local non-integer order dynamic problems on time scales revisited, *International Journal of Dynamics and Control*, (2018), 1-13.

[30] A. Neamaty, M. Nategh, & B. Agheli, Time-Space Fractional Burger's Equation on Time Scales, *Journal of Computational and Nonlinear Dynamics*, 12 (3) (2017), 031022.

[31] K. B. Oldham, Fractional differential equations in electrochemistry, *Advances in Engineering Software*, 41 (1) (2010), 9-12.

[32] M. A. Z. Raja, R. Samar, E. S. Alaidarous, & E. Shivanian, Bio-inspired computing platform for reliable solution of Bratu-type equations arising in the modeling of electrically conducting solids, *Applied Mathematical Modelling*, 40 (11) (2016), 5964-5977.

[33] S. Rezapour, H. Mohammadi, & A. Jajarmi,A new mathematical model for Zika virus transmission, *Advances in Difference Equations*, 2020 (1) (2020), 589.

[34] K. Sayevand, M. Fardi, E. Moradi, & F. H. Boroujeni, Convergence analysis of homotopy perturbation method for Volterra integro-differential equations of fractional order, *Alexandria Engineering Journal*, 52 (4) (2013), 807-812.

[35] E. Scalas, The application of continuous-time random walks in finance and economics, *Physica A: Statistical Mechanics and its Applications*, 362 (2) (2006), 225-239.

[36] V. E. Tarasov, Fractional integro-differential equations for electromagnetic waves in dielectric media, *Theoretical and Mathematical Physics*, 158 (3) (2009), 355-359.

[37] F. Wang, W. Li, & H. Zhang, A new extended homotopy perturbation method for nonlinear differential equations, *Mathematical and Computer Modelling*, 55 (3) (2012), 1471-1477.

[38] W. ,Wittayakiattilerd, & A. Chonwerayuth, Fractional integro-differential equations of mixed type with solution operator and optimal controls, *Journal of Mathematics Research*, 3 (3) (2011), 140.

[39] M. Yi, L. Wang, & J. Huang, Legendre wavelets method for the numerical solution of fractional integro-differential equations with weakly singular kernel, *Applied Mathematical Modelling*, 40 (4) (2016), 3422-3437.

[40] S. Yuzbasi, A collocation method based on Bernstein polynomials to solve nonlinear Fredholm-Volterra integro-differential equations, *Applied Mathematics and Computation*, 273 (2016), 142-154.

Bahram Agheli
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Qaemshahr Branch, Islamic Azad University,
Qaemshahr, Iran
E-mail: b.agheli@qaemiau.ac.ir

Rahmat Darzi
Department of Mathematics
Associate Professor of Mathematics
Department of Mathematics, Neka Branch, Islamic Azad University,
Neka, Iran
E-mail: r.darzi@iauneka.ac.ir

Abdolhadi Dabbaghian
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Neka Branch, Islamic Azad University,
Neka, Iran
E-mail: a.dabbaghian@iauneka.ac.ir