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On the Solution of a Nonconvex Fractional Quadratic Problem

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Abstract. In this paper, we give an algorithm for solving a class of nonconvex quadratic fractional problems that may arise during a correction of inconsistent set of linear inequalities. First, we show that for rank deficient matrices, an optimal solution for a nonconvex fractional minimization problem can be obtained via convex optimization approach. Then an iterative algorithm is designed to solve the problem in the full rank case. Finally, an illustrative numerical example is presented.

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1. Introduction

Inconsistent set of linear inequalities might frequently arise in real world problems [11] by various reasons such as error in data, wrong formulation, and many others. It might be the case that starting the model from

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the beginning might be expensive and time consuming, thus correcting an inconsistent set of linear inequalities does make sense and is studied in [2, 10]. Correcting such systems to consistent systems with minimal changes in problem data, requires to solve the following nonconvex fractional problem [10]:

$$\min_{x\in R^n} \frac{||(Ax-b)_+||^2}{1+||x||^2}. \tag{1}$$

However, it might be the case that the solution norm of (1) is too large and meaningless from practical point of view, thus it should be controlled. There are two popular approaches to do this. The first one is the so called Tikhonov regularization of (1) as follows:

$$\min_{x\in R^n} \frac{||(Ax-b)_+||^2}{1+||x||^2} + \rho||x||^2, \tag{2}$$

where ρ is a positive constant and called the regularization parameter. The second one requires prior information on solution norm, namely a bound on it. If this information is available, then we can consider the following problem instead of (1)

$$\begin{aligned} \min_{x\in R^n} \quad & \frac{||(Ax-b)_+||^2}{1+||x||^2} \\ & ||x||^2 \leqslant \gamma. \end{aligned} \tag{3}$$

In this paper, our focus is on the problem (3). We show that for rank deficient matrices A, instead of (3) it is sufficient to solve two convex optimization problems to get an optimal solution of it. Then, we discus the case where A is full rank.

2. Residual Vector

Let us first focus on the following problem:

$$\min_{x\in R^n} \|(Ax-b)_+\|^2. \tag{4}$$

Obviously, this is a convex optimization problem and let us denotes its solution set by X^*. The dual of (4) is given by

$$max - 2b^T u - ||u||^2$$
$$s.t. A^T u = 0, \tag{5}$$
$$u \geqslant 0.$$

Lemma 2.1. *Let x^* be an optimal solution of problem (4). Then $u^* = (Ax^* - b)_+$ is an optimal solution of (5).*

Proof. See [10]. □

The following theorem is crucial for the rest of the paper.

Theorem 2.2. *Let x_1^* and x_2^* be two different solutions of (4), then $(Ax_1^* - b)_+ = (Ax_2^* - b)_+$.*

Proof. Since the dual of (4) is a strictly convex minimization problem, then its solution is unique. However, as stated in the previous lemma, if x^* is a solution of (4), then $u^* = (Ax^* - b)_+$ is optimal for (5). Since the solution of (5) is unique, then we have proved the theorem. □

Lemma 2.3. *Let x^* be an optimal solution of (4). If there is a nonzero vector $d \in R^n$ for which*

$$\begin{aligned} A_1 d &\leqslant 0, \\ A_2 d &= 0, \end{aligned} \tag{6}$$

where A_1 and A_2 are submatrices of A with

$$(A_1 x^* - b_1)_+ = 0, \quad (A_2 x^* - b)_+ = A_2 x^* - b_2,$$

then X^* is unbounded.

Proof. Since x^* is an optimal solution of (4), then for any nonzero vector d that satisfies (6) and any $\alpha \geqslant 0$ we have

$$(A_1 x^* - b_1)_+ = (A_1(x^* + \alpha d) - b) = 0, \;\; (A_2 x^* - b)_+ = (A_2(x^* + \alpha d) - b)_+ = A_2 x^* - b.$$

Corollary 2.4. *For rank deficient matrices A, X^* is unbounded.*

Example 2.5. Consider the following inequalities

$$x_1 \geqslant 1,$$
$$x_1 \leqslant -1,$$
$$x_2 \geqslant 0.$$

Obviously, the coefficient matrix is full rank and any point on the nonnegative part of the x_2 axis is an optimal solution. This shows that even the coefficient matrix is full rank, but the solution set of (4) is unbounded.

Example 2.6. Consider the following inequalities

$$x_1 \geqslant 1,$$
$$x_1 \leqslant -1,$$
$$x_2 \geqslant 0,$$
$$x_2 \leqslant 1.$$

Here, again the coefficient matrix is full rank while X^* is bounded.
Now to solve (3), let us focus on the following minimization problem

$$\begin{array}{ll} \min & \|(Ax-b)_+\|^2 \\ s.t. & \|x\|^2 \leqslant \beta. \end{array} \tag{7}$$

Obviously, this is a convex optimization problem and can be solved using existing efficient algorithms [4]. If for the optimal solution of (7) $\|x^*\|^2 = \beta$, then it is optimal for (3) as well. In the following lemma we discuss the case where $\|x^*\|^2 < \beta$.

Lemma 2.7. *Suppose x^* is an optimal solution for (7) with $\|x^*\|^2 < \beta$ and $d \in R^n$ be a nonzero vector satisfies (6). Then there exists an $\alpha \in R$, for which $\|x^* + \alpha d\|^2 = \beta$, and $x^* + \alpha d$ is an optimal solution for (3).*

Proof. For a nonzero vector d satisfying (6), the nominator of (3) is the same for both x^* and $x^* + \alpha d$, where $\alpha \in R$. Moreover since $\|x^*\|^2 < \beta$, then there exist an $\alpha \in R$, for which $\|x^* + \alpha d\|^2 = \beta$, which completes the proof. It can be found the formula of calculationg α in [8]. $\square$

Remark 2.8. *For rank deficient matrix A, there always exists a nonzero vector $d \in R^n$, such that satisfies (6).*

Remark 2.9. *Let us denote by $f(x)$ the objective function of (3) and x^* be an optimal solution of (7) with $\|x^*\|^2 < \beta$. Then if there exists a nonzero vector $d \in R^n$ satisfying (6), we may get it by solving the following linear programming problem:*

$$\begin{array}{ll} \min & \nabla f(x^*)^T d \\ & A_1 d \leqslant 0 \\ & A_2 d = 0 \\ & \nabla f(x^*)^T d \geqslant -t \end{array} \tag{8}$$

where t is a positive constant.

Lemma 2.10. *Let x^* be an optimal solution of (7) with $\|x^*\|^2 < \beta$. Then the optimal objective value of (8) is negative.*

Proof. It follows from our previous discussion that we can decrease f in any nonzero direction d satisfying (6). □

3. Full Rank Case

In this section, we assume that the coefficient matrix A is full rank. In [7], we used a parametric approach to reduce the quadratic fractional problem into finding a zero of a univariate equation. Here, we use similar method to reduce our problem. However, the method leads us to a minimization of a single variable function over a closed interval. In other word, we rewrite the problem (3) as

$$\min_{1 \leqslant \alpha \leqslant 1+\gamma} \min_{\|x\|^2 = \alpha - 1} \frac{\|(Ax-b)_+\|^2}{\alpha}, \tag{9}$$

or

$$\min_{1 \leqslant \alpha \leqslant 1+\gamma} G(\alpha), \tag{10}$$

where

$$G(\alpha) = \min_{\|x\|^2 = \alpha - 1} \frac{\|(Ax-b)_+\|^2}{\alpha}. \tag{11}$$

The main difficulty in solving (9) is to solve the following subproblem efficiently

$$\begin{array}{ll} \min & \|(Ax-b)_+\|^2 \\ s.t. & \|x\|^2 = \beta. \end{array} \tag{12}$$

The KKT conditions for (12) are given by

$$\begin{aligned} A^T(Ax^* - b)_+ + \lambda^* x^* = 0, \\ \|x^*\|^2 - \beta = 0. \end{aligned}$$

In what follows, we give necessary and sufficient conditions of optimality for this problem.

Theorem 3.1. *The following are necessary and sufficient conditions for x^* to be an optimal solution for (12):*

$$\begin{aligned} A^T(Ax^* - b)_+ + \lambda^* x^* = 0, \\ \|x^*\|^2 = \beta, \\ H(x^*) + \lambda^* I \succeq 0. \end{aligned} \tag{13}$$

Proof. Suppose that x^* is a solution of (12). Obviously the first two conditions of (13) hold, thus it suffices to prove the last one. To do so, we know that

$$f(x) = f(x^*) + \nabla f(x^*)^T(x - x^*) + \frac{1}{2}(x - x^*)^T H(x^*)(x - x^*) + \|x - x^*\|^2 \alpha(x^*, x - x^*),$$

where $\lim_{x \to x^*} \alpha(x^*, x - x^*) = 0$ and $H(x^*) = A^T D(x^*)A$ is the generalized Hessian. Assume that $\|x\|^2 = \beta$. Now, since x^* is a solution of (12), then $f(x) \geqslant f(x^*)$ and since f is convex, then

$$f(x) - f(x^*) \geqslant \nabla f(x^*)^T(x - x^*) \geqslant 0.$$

This further implies that

$$\nabla f(x^*)^T(x - x^*) + \frac{1}{2}(x - x^*)^T H(x^*)(x - x^*) + \|x - x^*\|^2 \alpha(x^*, x - x^*) \geqslant 0.$$

Also, since $\frac{\lambda^*}{2}\|x - x^*\|^2 \geqslant 0$, we add $\frac{\lambda^*}{2}\|x - x^*\|^2$ to the previous inequality. Then, we have

$$\frac{1}{2}(x - x^*)^T(H(x^*) + \lambda^* I)(x - x^*) + \|x - x^*\|^2 \alpha(x^*, x - x^*) \geqslant 0,$$

which result to

$$(x - x^*)^T(H(x^*) + \lambda^* I)(x - x^*) \geqslant 0.$$

Now suppose (13) holds for x^*, then we show that it is optimal for (12). Let for x we have $\|x\|^2 = \beta$ and

$$\frac{1}{2}\|(Ax - b)_+\|^2 + \lambda^* \|x\|^2 < \frac{1}{2}\|(Ax^* - b)_+\|^2 + \lambda^* \|x^*\|^2.$$

Consider the following function

$$g(x) = \frac{1}{2} \left\|(Ax - b)_+\right\|^2 + \lambda^* \left\|x\right\|^2.$$

We have

$$\left\|(Ax - b)_+\right\|^2 < \left\|(Ax^* - b)_+\right\|^2$$

and let $z = \alpha x^* + (1 - \alpha)x$. Since g is convex, then for $\alpha \in (0, 1)$ we have

$$f(z) < f(x^*)$$

and

$$f(z) + \lambda^* \left\|z\right\|^2 < f(x^*) + \lambda^* \left\|x^*\right\|^2.$$

This contradicts with x^* being a local minimum. Thus it is the global solution for (12). □

In the following theorem, we discuss some properties of $G(\alpha)$.

Theorem 3.2. *Function $G(\alpha)$ has the following properties:*

- *$G(\alpha)$ is continuous in $[1, 1 + \gamma]$.*
- *for $\alpha > 1$, $G^{'}(\alpha) = \frac{-\lambda(\alpha)}{\alpha} - \frac{\|(Ax(\alpha)-b)_+\|^2}{\alpha^2}$.*
  where $x(\alpha)$ and $\lambda(\alpha)$ are a solution of (11) and its Lagrange multiplier respectively.

Proof.

- First, we show the continuity from left at 1. For $\alpha \geqslant 1$ we have

$$\begin{aligned}
|G(\alpha) - G(1)| &= \left| \frac{\left\|(Ax(\alpha) - b)_+\right\|^2}{\alpha} - \left\|(-b)_+\right\|^2 \right| \\
&= \left| \frac{\left\|(Ax(\alpha) - b)_+\right\|^2 - \left\|(-b)_+\right\|^2 + (1 - \alpha)\left\|(-b)_+\right\|^2}{\alpha} \right| \\
&\leqslant \left| \frac{\left\|(Ax(\alpha) - b)_+\right\|^2 - \left\|(-b)_+\right\|^2}{\alpha} \right| + \left| \frac{(1 - \alpha)\left\|(-b)_+\right\|^2}{\alpha} \right| \\
&\leqslant \frac{\left\|(Ax(\alpha) - b)_+ - (-b)_+\right\| \left(\left\|(Ax(\alpha) - b)_+\right\| + \left\|(-b)_+\right\|\right)}{\alpha} + \left| \frac{(1 - \alpha)\left\|(-b)_+\right\|^2}{\alpha} \right| \\
&\leqslant \frac{\left\|Ax(\alpha)\right\| \left(\left\|(Ax(\alpha) - b)_+\right\| + \left\|(-b)_+\right\|\right)}{\alpha} + \left| \frac{(1 - \alpha)\left\|(-b)_+\right\|^2}{\alpha} \right|
\end{aligned}$$

$$\leqslant \frac{\|A\|\sqrt{\alpha-1}\,(\|(Ax(\alpha)-b)_+\|+\|(-b)_+\|)}{\alpha}+\left|\frac{(1-\alpha)\,\|(-b)_+\|^2}{\alpha}\right|. \quad (14)$$

Obviously as $\alpha \to 1^+$, the right hand side of (14) approaches to zero, which implies the continuity of $G(.)$ at 1 from right. Analogously we have the continuity at the interval $[1, 1+\gamma]$.
For the second part of the theorem suppose

$$G(\alpha) = \frac{\|(Ax(\alpha)-b)_+\|^2}{\alpha}.$$

Then

$$G^{'}(\alpha) = \frac{2(x^{'}(\alpha))^T A^T (Ax(\alpha)-b)_+\alpha - \|(Ax(\alpha)-b)_+\|^2}{\alpha^2}.$$

Moreover since $\|x(\alpha)\|^2 = \alpha - 1$, then $2x(\alpha)^T x^{'}(\alpha) = 1$ and $A^T(Ax(\alpha)-b)_+ + \lambda(\alpha)x(\alpha) = 0$, then we have

$$G^{'}(\alpha) = -\frac{\lambda(\alpha)}{\alpha} - \frac{\|(Ax(\alpha)-b)_+\|^2}{\alpha^2}. \quad \square \quad (15)$$

Theorem 3.3. *The function $G(\alpha)$ is unimodal for every $\alpha > 1$.*

Proof. Now, in order to prove the unimodality of G, it is sufficient to prove if $G^{'}(\alpha) =$, then $G^{''}(\alpha) \geqslant 0$.
First, we rewrite (15) as

$$G^{'}(\alpha) = -\frac{\lambda(\alpha)}{\alpha} - \frac{G(\alpha)}{\alpha}. \quad (16)$$

By differentiating both sides of (16), we obtain

$$\begin{aligned} G^{''}(\alpha) &= -\frac{\lambda^{'}(\alpha)\alpha - \lambda(\alpha)}{\alpha^2} - \frac{G^{'}(\alpha)\alpha - G(\alpha)}{\alpha^2} \quad (17) \\ &= -\frac{1}{\alpha^2}\left(\lambda^{'}(\alpha)\alpha - 2\lambda(\alpha) - 2G(\alpha)\right), \end{aligned}$$

where the last equality is followed by (16). From the assomption $G^{'}(\alpha) = 0$, we have

$$\frac{\lambda(\alpha)}{\alpha} = -\frac{\|(Ax(\alpha)-b)_+\|^2}{\alpha^2}.$$

Therefore,

$$G^{''}(\alpha) = -\frac{\lambda^{'}(\alpha)}{\alpha}. \quad (18)$$

Notice that $A^T(Ax(\alpha)-b)_+ + \lambda(\alpha)x(\alpha) = 0$. Now, if we represent matrix A and vector b as the form $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, such that $A_1x(\alpha) > b_1$ and $A_2x(\alpha) \leqslant b_2$; we conclude that $A_1^T(A_1x(\alpha)-b_1) + \lambda(\alpha)x(\alpha) = 0$. By dierentiating both of sides, we have

$$(A_1^TA_1 + \lambda(\alpha)I)x^{'}(\alpha) + \lambda^{'}(\alpha)x(\alpha) = 0.$$

Multiplying $x^{'T}(\alpha)$ to both sides, it is obtained

$$x^{'T}(\alpha)(A_1^TA_1 + \lambda(\alpha)I)x^{'}(\alpha) + \lambda^{'}(\alpha)x^{'T}(\alpha)x(\alpha) = 0.$$

Finally, since $2x(\alpha)^Tx^{'}(\alpha) = 1$

$$\lambda^{'}(\alpha) = -\frac{1}{2}x^{'T}(\alpha)(A_1^TA_1 + \lambda(\alpha)I)x^{'}(\alpha) \leqslant 0.$$

The last inequality is a consequence of the nonnegative definiteness of $H(x(\alpha)) + \lambda(\alpha)^*I$. We know that $H(x(\alpha)) = A^TD(x(\alpha))A$ is a generalized Hesian matrix where $D(x(\alpha))$ is an $n \times n$ diagonal matrix whose ith diagonal entry is equal 1 if $(Ax-b)_i > 0$ and to 0 if $(Ax-b)_i \leqslant 0$. Therefore $H(x(\alpha)) + \lambda(\alpha)I = A_1^TA_1 + \lambda(\alpha)I$. We conclude that, when $G^{'}(\alpha) = 0$, then $G^{''}(\alpha) \geqslant 0$.
Now we outline our algorithm to solve (9). □

Algorithm 1

inputs

$$A \in R^{m\times n}, m \geqslant n, b \in R^m, \gamma > 0;$$

Set $\alpha_{\min} = 1$ and $\alpha_{\max} = 1+\gamma$.

$$|\alpha_{\max} - \alpha_{\min}| \geqslant \epsilon$$

Set $\alpha = \frac{\alpha_{\max}+\alpha_{\min}}{2}$;

Solve the following problem

$$\min_{\|x\|^2=\alpha-1} \frac{\|(Ax-b)_+\|^2}{\alpha}.$$

If $-\frac{\lambda(\alpha)}{\alpha} - \frac{\|(Ax(\alpha)-b)_+\|^2}{\alpha^2} > 0$, then $\alpha_{\max} = \alpha$, else $\alpha_{\min} = \alpha$.

4. Numerical Results

In this section we present a numerical example used in [1] to show the performance of our algorithm compared to the approach of Amaral et al., [1]. Let

$$A = \begin{bmatrix} -0.10433318 & -0.3349605 \\ -2.31759372 & -2.0354161 \\ -0.67781831 & 0.6546597 \\ 1.05241872 & -0.4327864 \\ 0.01449416 & -1.93122220 \\ 0.24375548 & 0.5536801 \end{bmatrix}, \quad b = \begin{bmatrix} -2.24440190 \\ 0.7579334 \\ 0.4302541 \\ 2.5746725 \\ -2.6003448 \\ 0.4284550 \end{bmatrix},$$

where A is full rank and $Ax \leqslant b$ is inconsistent. The solution obtained by our algorithm in 16 seconds is $(x_1, x_2) = (3.921407, 2.544738)$ and the corresponding objective function is 0.2152437 while the solution obtained in [2] after 37 seconds is $(x_1, x_2) = (3.923086, 2.548969)$ with the objective function equal to 0.2152440.

5. Conclusions

In this paper a new algorithm is introduced to solve a nonconvex fractional quadratic minimization problem. Then, we considered two possible cases. First, if the coefficient matrix A is rank deficient, the method corresponds to any nonzero direction d satisfying (6). Second, if the coefficient matrix A is full rank, we meet an iterative algorithm. In this case, we reduce the problem to minimization of a single variable function on a closed interval while the computation of value of this function consists of solving a minimization of quadratic function with a quadratic constraint. To solve the inner minimization problem, we use the method of Moré, D.C. Sorensen (see [6]). A numerical example is given to show that the proposed method is efficient in practice as well.

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