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Numerical Solution for a Class of Time-Fractional Stochastic Delay Differential Equations with Fractional Brownian Motion

S. Banihashemi
University of Mazandaran

H. Jafari*
University of Mazandaran

A. Babaei
University of Mazandaran

Abstract. In this article, a numerical scheme is proposed to solve a class of time-fractional stochastic delay differential equations (TFSDDEs) with fractional Brownian motion (fBm). First, we convert the TFSDDE into a non-delay equation by using a step-by-step scheme. Then, by applying a collocation method based on Jacobi polynomials (JPs) in each step, the non-delay equation is reduced to a nonlinear system of algebraic equations. The convergence analysis of the presented scheme is evaluated. Finally, two numerical test examples are presented to highlight the applicability and efficiency of the investigated method.

AMS Subject Classification: 60G22; 26A33; 65C30.
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*Corresponding Author

1 Introduction

Stochastic differential equations (SDEs) are very important in modeling and analysis of many phenomena because of their significant uncertainty. During the recent decades, SDEs have been received the attention of many researchers in various fields such as engineering [12, 9], finance [23] and population dynamics [28, 24]. A stochastic model for pricing of financial instruments is considered in [1]. The authors in [5] described a stochastic model for the spread of Coronavirus. Also, delay SDEs are often used to model the phenomena where the future state of a system depends not only on the present condition but also on its past situations. For example, the authors in [14] proposed a nonlinear Ait-Sahalia model of the spot interest rate with delayed volatility function.

Fractional differential equations are the generalization of ordinary differential equations to arbitrary orders and are appropriate tools for the explanation of hereditary properties of various materials and processes [3, 17, 21, 4]. The implication of fractional order derivatives and integrals was applied to accurately investigate the dynamical behaviors of a linear triatomic molecule in [7]. Mohammadi et al. presented a Caputo fractional-order SIRD mathematical model for the transmission of COVID-19 between humans [18]. Also, Baleanu et al. described the existence of a unique solution of a fractional model for human liver involving Caputo-Fabrizio derivative in [6]. In [26], a nonlinear quantum boundary value problem was proposed in the sense of quantum Caputo derivative, with fractional q-integro difference conditions. Doan et al. considered a SDEs with fractional derivative in the Caputo sense in [13] and described the existence and uniqueness of solution. Aryani et al. described a system of fractional stochastic integro-differential equations in [2]. A nonlinear stochastic differential equation of fractional order involving a constant delay is considered in [8]. Moreover, Chaudhary et al. described the existence and uniqueness of the mild solution of stochastic fractional neutral integro-differential equation with nonlocal conditions in [11].

In this article, we consider a step-by-step numerical technique based on the Jacobi collocation method for the time-fractional stochastic delay differential equation (TFSDDE) with fBm in the form:

$$ {}_0\mathrm{D}_t^{\alpha} y(t) = \mathrm{F}(t, y(t), y(t-\tau_1)) + \mathrm{G}(t, y(t), y(t-\tau_2))\dot{\mathcal{B}}^{\mathrm{H}}(t), \tag{1} $$

on the domain $t \in \bar{\Omega} := (0, \mathtt{T}_{\max}]$ with the following initial condition

$$ y(t) = \varphi(t), \qquad \text{on} \quad [-\tau, 0], \quad \tau \in \mathbb{R}^+, \tag{2} $$

where τ_1 and τ_2 are non-negative constants, $\tau = \max\{\tau_1, \tau_2\}$, $\varphi(t)$ is a continuous stochastic process on $t \in [-\tau, 0]$ and $y(t)$ is an unknown stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover ${}_0\mathrm{D}_t^{\alpha}[\cdot]$ denotes the Caputo fractional derivative of order $\alpha \in (0, 1)$ and $\mathcal{B}^{\mathrm{H}}(t)$ is a standard fBm in which $\dot{\mathcal{B}}^{\mathrm{H}}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{B}^{\mathrm{H}}(t)\right)$. Throughout the article, we assume that the following conditions are presented:

Assumption 1. The fBm with Hurst parameter $\mathrm{H} \in (0, 1)$ is a continuous and centered Gaussian process $\mathcal{B}^{\mathrm{H}} = \{\mathcal{B}^{\mathrm{H}}(t) : t \in \bar{\Omega}\}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the covariance

$$ \mathbb{E}\left(\mathcal{B}^{\mathrm{H}}(s)\mathcal{B}^{\mathrm{H}}(t)\right) = \frac{1}{2}\left(s^{2\mathrm{H}} + t^{2\mathrm{H}} - |t-s|^{2\mathrm{H}}\right), \quad s, t \in \bar{\Omega}. $$

It is known that, when the Hurst index $\mathrm{H} = \frac{1}{2}$, the fBm is exactly the standard Brownian motion without memory, while for $\mathrm{H} \neq \frac{1}{2}$, it is a Gaussian process with memory (is not a semi-martingale) [16, 27]. The fBm is called a long memory process if $\mathrm{H} > \frac{1}{2}$, while it's called a short memory process if $\mathrm{H} < \frac{1}{2}$.

Assumption 2. The measurable functions $\mathrm{F}, \mathrm{H} : \bar{\Omega} \times \mathbb{R} \times \mathbf{C}([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ satisfy the following Lipschitz conditions:

$$ \begin{aligned} |\mathrm{F}(t, x, \tilde{x}) - \mathrm{F}(t, z, \tilde{z})| &\le \theta_{\mathrm{F}}|x - z| + \mu_{\mathrm{F}}|\tilde{x} - \tilde{z}|, \\ |\mathrm{G}(t, x, \tilde{x}) - \mathrm{G}(t, z, \tilde{z})| &\le \theta_{\mathrm{G}}|x - z| + \mu_{\mathrm{G}}|\tilde{x} - \tilde{z}|, \end{aligned} $$

where $\theta_{\mathrm{F}}, \theta_{\mathrm{G}}, \mu_{\mathrm{F}}, \mu_{\mathrm{G}} \in \mathbb{R}^+$, for $x, \tilde{x}, z, \tilde{z} \in \mathbb{R}$, $t \in \bar{\Omega}$ and $\mathbf{C}([-\tau, 0]; \mathbb{R})$ is the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}$.

The main objectives of the present paper are to construct a convergence spectral scheme for TFSDDE with fBm. In order to approximate the numerical solution of (1)-(2), first, we convert the TFSDDE into a non-delay equation using a step-by-step scheme. Then, by applying the collocation method based on JPs in each step, the non-delay equation is reduced to a nonlinear system of algebraic equations. Also, the convergence analysis of the presented scheme is evaluated.

This paper is structured as follows: In Section 2, the necessary definitions and fundamental properties of the JPs have been introduced. In Section 3, the step-by-step collocation method is proposed, and Section 4 contains a discussion of error analysis of this numerical method. The numerical algorithm is implemented for two test examples in Section 5. Finally, the conclusion is presented in Section 6.

2 Basic Definitions

In this section, we consider some fundamental definitions and necessary properties that are applied throughout the paper.

Definition 2.1. *The Caputo fractional derivative of order* $\alpha > 0$ *is defined as* [25]

$$
\begin{aligned}
{}_a\mathrm{D}_t^{\alpha} y(t) &= \frac{1}{\Gamma(m-\alpha)} \int_a^t y^{(m)}(\zeta)(t-\zeta)^{m-\alpha-1} \mathrm{d}\zeta, \quad && m-1<\alpha<m, \\
{}_a\mathrm{D}_t^{\alpha} y(t) &= \frac{d^m}{dt^m} y(t), && \alpha = m,
\end{aligned}
$$

where $t > a$, $m \in \mathbb{N}$ *and* $\Gamma(\cdot)$ *shows the Gamma function.*

Definition 2.2. *The Riemann-Liouville (RL) fractional integral of order* $\alpha \in (0,1)$ *is defined as* [25]

$$
{}_a\mathrm{I}_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{y(\zeta)}{(t-\zeta)^{1-\alpha}} \mathrm{d}\zeta, \qquad t > a.
$$

Definition 2.3. ([19]) *The family of JPs* $\{\mathbf{J}_i^{(\eta,\gamma)}(t)\}_{i=0}^{\infty}$, *with* $\eta, \gamma > -1$ *satisfy the following formula*

$$
\mathbf{J}_{i+1}^{(\eta,\gamma)}(t) = (\phi_i t - \mu_i)\mathbf{J}_i^{(\eta,\gamma)}(t) - \varrho_i \mathbf{J}_{i-1}^{(\eta,\gamma)}(t), \quad i = 1, 2, \ldots.
$$

on $t \in [-1, 1]$, *such that*

$$
\phi_i := \frac{(\varsigma_i+1)(\varsigma_i+2)}{2\upsilon_i(i+1)}, \quad \mu_i := \frac{(\varsigma_i+1)(\gamma^2-\eta^2)}{2(i+1)\upsilon_i\varsigma_i}, \quad \varrho_i := \frac{(\varsigma_i+2)(\gamma+i)(\eta+i)}{(i+1)\upsilon_i\varsigma_i},
$$

in which $\upsilon_i := i + \eta + \gamma + 1$, $\varsigma_i := 2i + \eta + \gamma$ *and*

$$\mathbf{J}_0^{(\eta,\gamma)}(t) = 1, \qquad \mathbf{J}_1^{(\eta,\gamma)}(t) = \frac{\eta + \gamma + 2}{2} t + \frac{\eta - \gamma}{2}.$$

Remark 2.4. ([20]) *The JPs* $\mathbf{J}_i^{(\eta,\gamma)}(t)$, $i = 0, 1, 2, \ldots$, *are orthogonal over* $t \in [-1, 1]$ *according to the weight function*

$$\omega(t; \eta, \gamma) = (1 - t)^{\eta}(1 + t)^{\gamma}.$$

The orthogonality condition of JPs is satisfied as follows

$$\int_{-1}^{1} \omega(t; \eta, \gamma) \mathbf{J}_i^{(\eta,\gamma)}(t) \mathbf{J}_j^{(\eta,\gamma)}(t) \mathrm{d}t = \lambda_i^{(\eta,\gamma)} \delta_{i,j},$$

in which

$$\lambda_i^{(\eta,\gamma)} := \frac{2^{\eta+\gamma+1}\Gamma(i + \eta + 1)\Gamma(i + \gamma + 1)}{(\varsigma_i + 1)\Gamma(i + 1)\Gamma(\upsilon_i)},$$

and $\delta_{i,j}$ *is the Kronecker function.*

Definition 2.5. *The shifted JPs* $\mathbf{J}_i^{(\eta,\gamma)}(t; a, b)$ *on* $[a, b]$ *are defined by*

$$\mathbf{J}_i^{(\eta,\gamma)}(t; a, b) = \mathbf{J}_i^{(\eta,\gamma)}\Big(\frac{2}{b - a}(t - b) + 1\Big), \quad i = 0, 1, 2, \ldots.$$

The explicit form of the shifted JPs satisfies the following formula [10]

$$\mathbf{J}_i^{(\eta,\gamma)}(t; a, b) = \sum_{k=0}^{i} \Lambda_{k,i}^{(\eta,\gamma)} \left(\frac{1}{b - a}(t - b) + 1\right)^k, \tag{3}$$

where

$$\Lambda_{k,i}^{(\eta,\gamma)} = \frac{(-1)^{i-k}\ \Gamma(i + \gamma + 1)\ \Gamma(i + k + \eta + \gamma + 1)}{(i - k)!\ k!\ \Gamma(k + \gamma + 1)\ \Gamma(i + \eta + \gamma + 1)}.$$

Also

$$\mathbf{J}_i^{(\eta,\gamma)}(a; a, b) = \frac{(-1)^i\ \Gamma(1 + i + \gamma)}{i!\ \Gamma(1 + \gamma)}, \qquad \mathbf{J}_i^{(\eta,\gamma)}(b; a, b) = \frac{\Gamma(i + \eta + 1)}{i!\ \Gamma(\eta + 1)}.$$

Remark 2.6. *The shifted JPs* $\mathbf{J}_i^{(\eta,\gamma)}(t;a,b)$, $i=0,1,2,...,$ *are orthogonal on* $[a,b]$ *according to the weight function*

$$ {}_a^b\omega(t;\eta,\gamma)=(b-t)^{\eta}(t-a)^{\gamma}, $$

such that

$$ \int_a^b {}_a^b\omega(t;\eta,\gamma)\mathbf{J}_i^{(\eta,\gamma)}(t;a,b)\mathbf{J}_j^{(\eta,\gamma)}(t;a,b)\mathrm{d}t=\frac{(b-a)^{\eta+\gamma+1}}{2^{\eta+\gamma+1}}\lambda_i^{(\eta,\gamma)}\delta_{i,j}. $$

Let $\Upsilon:=(a,b)\subseteq\mathbb{R}$, the Sobolev space $\mathbf{H}^{\mathrm{M}}_{\omega^{(\eta,\gamma)}}(\Upsilon)$, $\mathrm{M}\in\mathbb{Z}^+$ is defined by

$$ \mathbf{H}^{\mathrm{M}}_{\omega^{(\eta,\gamma)}}(\Upsilon)=\Big\{g:\ g \text{ is } \textbf{measurable} \text{ and } \|g\|_{\mathrm{M},\omega^{(\eta,\gamma)}}<\infty\Big\}, $$

in which

$$ \|g\|^2_{\mathrm{M},\omega^{(\eta,\gamma)}}=\sum_{k=0}^{\mathrm{M}}\|g^{(k)}\|^2_{\mathbf{L}^2_{\omega^{(\eta,\gamma)}}},\quad \|g\|^2_{\mathbf{L}^2_{\omega^{(\eta,\gamma)}}}=\int_{\Upsilon}{}_a^b\omega(s;\eta,\gamma)\ |g(s)|^2\ \mathrm{d}s. $$

A function $g\in\mathbf{H}^{\mathrm{M}}_{\omega^{(\eta,\gamma)}}(\Upsilon)$ can be written as expansion

$$ g(t)=\sum_{i=0}^{\infty}\mathrm{g}_i\ \mathbf{J}_i^{(\eta,\gamma)}(t;a,b), \tag{4} $$

where the coefficients g_i, $i=0,1,2,...,$ are given by

$$ \mathrm{g}_i=\frac{1}{{}_a^b\lambda_i^{(\eta,\gamma)}}\int_a^b g(s)\ {}_a^b\omega(s;\eta,\gamma)\ \mathbf{J}_i^{(\eta,\gamma)}(s;a,b)\ \mathrm{d}s, $$

such that ${}_a^b\lambda_i^{(\eta,\gamma)}:=\frac{(b-a)^{\eta+\gamma+1}}{2^{\eta+\gamma+1}}\lambda_i^{(\eta,\gamma)}$. Using first $n+1$ terms of (4), we can approximate $g(t)$ on the interval $[a,b]$ as follows

$$ g(t)\simeq g_n(t)=\sum_{i=0}^{n}\mathrm{g}_i\ \mathbf{J}_i^{(\eta,\gamma)}(t;a,b)\triangleq\mathbf{G}^{\mathrm{T}}\ {}_a^b\Psi^{(\eta,\gamma)}(t), \tag{5} $$

where

$$ \mathbf{G}:=\left[\mathrm{g}_0,\mathrm{g}_1,\cdots,\mathrm{g}_n\right]^{\mathrm{T}}, $$

and

$$ {}_a^b\Psi^{(\eta,\gamma)}(t):=\Big[\mathbf{J}_0^{(\eta,\gamma)}(t;a,b),\mathbf{J}_1^{(\eta,\gamma)}(t;a,b),\cdots,\mathbf{J}_n^{(\eta,\gamma)}(t;a,b)\Big]^{\mathrm{T}}. \tag{6} $$

Theorem 2.7. *Let* ${}_a^b\Psi^{(\eta,\gamma)}(t)$ *be the shifted JPs vector defined in (6), then*

$$ {}_0\mathrm{D}_t^\alpha \left({}_a^b\Psi^{(\eta,\gamma)}(t) \right) = {}_a^b\Delta(\eta,\gamma;\alpha;t), $$

where

$$ {}_a^b\Delta(\eta,\gamma;\alpha;t) := \left[{}_a^b\phi_0^{(\eta,\gamma)}(\alpha;t), {}_a^b\phi_1^{(\eta,\gamma)}(\alpha;t), ..., {}_a^b\phi_n^{(\eta,\gamma)}(\alpha;t) \right]^{\mathrm{T}}, $$

and

$$ {}_a^b\phi_i^{(\eta,\gamma)}(\alpha;t) = \sum_{k=1}^{i}\sum_{j=0}^{k-1}\sum_{r=0}^{j} \frac{\Lambda_{k,i}^{(\eta,\gamma)}(-b)^{j-r}k!(b-a)^{-(j+1)}}{\Gamma(r-\alpha+2)(k-j-1)!(j-r)!} t^{r-\alpha+1}. $$

Proof. From Definition 2.1 and the relation (3), for $i=0$, we get

$$ {}_a^b\phi_0^{(\eta,\gamma)}(\alpha;t) := {}_0\mathrm{D}_t^\alpha \left(\mathbf{J}_0^{(\eta,\gamma)}(t;a,b) \right) = \sum_{k=0}^{i} \Lambda_{k,i}^{(\eta,\gamma)} {}_0\mathrm{D}_t^\alpha(1) = 0. $$

For $i = 1, ..., n$,

$$ \begin{aligned} {}_a^b\phi_i^{(\eta,\gamma)}(\alpha;t) &:= {}_0\mathrm{D}_t^\alpha \left(\mathbf{J}_i^{(\eta,\gamma)}(t;a,b) \right) \\ &= \sum_{k=1}^{i} \Lambda_{k,i}^{(\eta,\gamma)} {}_0\mathrm{D}_t^\alpha \left(\frac{t-b}{b-a} + 1 \right)^k . \end{aligned} \tag{7} $$

Also

$$ \begin{aligned} {}_0\mathrm{D}_t^\alpha\left(\frac{t-b}{b-a}+1\right)^k &= \frac{k}{(b-a)\Gamma(1-\alpha)} \int_0^t (t-\zeta)^{-\alpha} \left(\frac{\zeta-b}{b-a} + 1 \right)^{k-1} \mathrm{d}\zeta \\ &= \sum_{j=0}^{k-1} \chi_{j,k}^{a,b}(\alpha) \int_0^t (t-\zeta)^{-\alpha}(\zeta-b)^j \mathrm{d}\zeta \\ &= \sum_{j=0}^{k-1} \chi_{j,k}^{a,b}(\alpha) \left[\sum_{r=0}^{j} \frac{(-b)^{j-r}\, j!}{r!(j-r)!} \int_0^t (t-\zeta)^{-\alpha}\zeta^r \mathrm{d}\zeta \right], \end{aligned} \tag{8} $$

where $\chi_{j,k}^{a,b}(\alpha)=\frac{k!}{\Gamma(1-\alpha)j!(k-j-1)!(b-a)^{j+1}}$. Let $\zeta=\lambda t$, $\lambda\in[0,1]$. Thus

$$\begin{aligned}\int_0^t (t-\zeta)^{-\alpha}\zeta^r \mathrm{d}\zeta &= \int_0^1 (t-\lambda t)^{-\alpha}(\lambda t)^r t \mathrm{d}\lambda\\ &= t^{r-\alpha+1}\mathfrak{B}(r+1,1-\alpha)\\ &= t^{r-\alpha+1}\frac{\Gamma(r+1)\Gamma(1-\alpha)}{\Gamma(r-\alpha+2)},\end{aligned}$$

where $\mathfrak{B}(x,y)$ is the Beta function. Hence, from (8)

$${}_0\mathrm{D}_t^{\alpha}(\frac{t-b}{b-a}+1)^k=\sum_{j=0}^{k-1}\chi_{j,k}^{a,b}(\alpha)\left[\sum_{r=0}^{j}\frac{(-b)^{j-r}\ j!}{(j-r)!}\frac{\Gamma(1-\alpha)}{\Gamma(r-\alpha+2)}t^{r-\alpha+1}\right]. \tag{9}$$

Therefore, from Eqs. (7) and (9), we obtain

$${}_a^b\phi_i^{(\eta,\gamma)}(\alpha;t)=\sum_{k=1}^{i}\sum_{j=0}^{k-1}\sum_{r=0}^{j}\frac{\Lambda_{k,i}^{(\eta,\gamma)}(-b)^{j-r}k!(b-a)^{-(j+1)}}{\Gamma(r-\alpha+2)(k-j-1)!(j-r)!}t^{r-\alpha+1}.$$

□

3 Numerical Scheme

In this section, we describe a step collocation scheme based on Jacobi polynomials to solve problem (1)-(2). For this reason, let $\mathtt{M}=[\frac{\mathrm{T_{max}}}{\tau}]$, then, we find the numerical solution of the considered problem by Jacobi collocation scheme in each subinterval $[(j-1)\tau, j\tau]$, $j=2,...,\mathtt{M}$. In first step, we want to solve the intended problem on the interval $[0,\tau]$. Thus, we have a non-delay problem as follows

$${}_0\mathrm{D}_t^{\alpha}y(t)=\mathrm{F}(t,y(t),\varphi(t-\tau_1))+\mathrm{G}(t,y(t),\varphi(t-\tau_2))\dot{\mathcal{B}}^{\mathrm{H}}(t), \tag{10}$$

$$y(0)=\varphi(0). \tag{11}$$

To obtain the numerical solution of (10)-(11) by applying Jacobi collocation scheme, we suppose

$$y(t)\simeq y_n^1(t)=\sum_{i=0}^{n}\mathrm{x}_i^1\ \mathbf{J}_i^{(\eta,\gamma)}(t;0,\tau)\triangleq \mathbf{X}_1^{\mathrm{T}}\ {}_0^{\tau}\Psi^{(\eta,\gamma)}(t), \tag{12}$$

where $\mathbf{X}_1 = \left[\mathtt{x}_0^1, \mathtt{x}_1^1, ..., \mathtt{x}_n^1\right]^{\mathrm{T}}$. According to Theorem 2.7, Eqs. (10) and (12)

$$\mathbf{R}_1^n(t) \triangleq \mathbf{X}_1^{\mathrm{T}} \, {}_0^{\tau}\Delta(\eta,\gamma;\alpha;t) - \mathrm{F}(t, \mathbf{X}_1^{\mathrm{T}} \, {}_0^{\tau}\Psi^{(\eta,\gamma)}(t), \varphi(t-\tau_1)) - \mathrm{G}(t, \mathbf{X}_1^{\mathrm{T}} \, {}_0^{\tau}\Psi^{(\eta,\gamma)}(t), \varphi(t-\tau_2))\dot{\mathcal{B}}^{\mathrm{H}}(t) \simeq 0. \tag{13}$$

Also, from the initial condition (11) and the relation (12)

$$\Phi_1^n \triangleq \mathbf{X}_1^{\mathrm{T}} \, {}_0^{\tau}\Psi^{(\eta,\gamma)}(0) - \varphi(0) \simeq 0. \tag{14}$$

Let ${}_1\mathbf{t}_0^{(\eta,\gamma)} := 0$, ${}_1\mathbf{t}_n^{(\eta,\gamma)} := \tau$ and $\{{}_1\mathbf{t}_i^{(\eta,\gamma)} : i = 1, ..., n-1\}$ be the roots of $\mathbf{J}_{n-1}^{(\eta,\gamma)}(t;0,\tau)$. Then, by evaluating (13) at n nodal points ${}_1\mathbf{t}_i^{(\eta,\gamma)}$, $i = 1, ..., n$, and the relation (14), we get

$$\begin{cases} \left({}_1\mathbf{t}_i^{(\eta,\gamma)} - {}_1\mathbf{t}_{i-1}^{(\eta,\gamma)}\right) \mathbf{R}_1^n(t)\left({}_1\mathbf{t}_i^{(\eta,\gamma)}\right) = 0, & i = 1, ..., n, \\ \Phi_1^n = 0, \end{cases} \tag{15}$$

in which

$$\dot{\mathcal{B}}^{\mathrm{H}}\left({}_1\mathbf{t}_i^{(\eta,\gamma)}\right) = \frac{\mathcal{B}^{\mathrm{H}}\left({}_1\mathbf{t}_i^{(\eta,\gamma)}\right) - \mathcal{B}^{\mathrm{H}}\left({}_1\mathbf{t}_{i-1}^{(\eta,\gamma)}\right)}{{}_1\mathbf{t}_i^{(\eta,\gamma)} - {}_1\mathbf{t}_{i-1}^{(\eta,\gamma)}}.$$

Hence, (15) gives a system of $n+1$ nonlinear algebraic equations. This system can be solved by applying a root-finding technique such as the Newton's iterative method, for the unknown coefficients $\mathtt{x}_i^1$, $i = 0, 1, ..., n$.

Generally, to obtain an approximate solution of the problem (1)-(2) on the interval $[(j-1)\tau, j\tau]$, $j = 2, ..., \mathtt{M}$, we need to solve the following equation

$${}_0\mathrm{D}_t^{\alpha} y^j(t) = \mathrm{F}(t, y^j(t), y^{j-1}(t-\tau_1)) + \mathrm{G}(t, y^j(t), y^{j-1}(t-\tau_2))\dot{\mathcal{B}}^{\mathrm{H}}(t), \tag{16}$$

$$y^j(0) = y^{j-1}(\tau), \tag{17}$$

where $y^j(t) := y((j-1)\tau + t)$. To obtain the numerical solution of (16)-(17) by applying Jacobi collocation scheme, we suppose

$$y^j(t) \simeq y_n^j(t) = \sum_{i=0}^{n} \mathtt{x}_i^j \, \mathbf{J}_i^{(\eta,\gamma)}(t;(j-1)\tau, j\tau) \triangleq \mathbf{X}_j^{\mathrm{T}} \, {}_{(j-1)\tau}^{j\tau}\Psi^{(\eta,\gamma)}(t), \tag{18}$$

where $\mathbf{X}_j = \left[\mathtt{x}_0^j, \mathtt{x}_1^j, ..., \mathtt{x}_n^j\right]^{\mathrm{T}}$. According to Theorem 2.7, Eqs. (16) and (18)

$$\begin{aligned}\mathbf{R}_j^n(t) &\triangleq \mathbf{X}_j^{\mathrm{T}}\, {}_{(j-1)\tau}^{\;\;j\tau}\Delta(\eta,\gamma;\alpha;t) - \mathrm{F}(t, \mathbf{X}_j^{\mathrm{T}}\, {}_{(j-1)\tau}^{\;\;j\tau}\Psi^{(\eta,\gamma)}(t), y_n^{j-1}(t-\tau_1)) \\ &\quad - \mathrm{G}(t, \mathbf{X}_j^{\mathrm{T}}\, {}_{(j-1)\tau}^{\;\;j\tau}\Psi^{(\eta,\gamma)}(t), y_n^{j-1}(t-\tau_2))\dot{\mathcal{B}}^{\mathrm{H}}(t) \simeq 0. \end{aligned} \tag{19}$$

Let ${}_j\mathbf{t}_0^{(\eta,\gamma)} := (j-1)\tau$, ${}_j\mathbf{t}_n^{(\eta,\gamma)} := j\tau$ and $\{{}_j\mathbf{t}_i^{(\eta,\gamma)} : i = 1, ..., n-1\}$ are the roots of $\mathbf{J}_{n-1}^{(\eta,\gamma)}(t; (j-1)\tau, j\tau)$. Then, from (19)

$$\left({}_j\mathbf{t}_i^{(\eta,\gamma)} - {}_j\mathbf{t}_{i-1}^{(\eta,\gamma)}\right)\mathbf{R}_j^n\left({}_j\mathbf{t}_i^{(\eta,\gamma)}\right) = 0, \qquad i = 1, ..., n, \tag{20}$$

with

$$\dot{\mathcal{B}}^{\mathrm{H}}\left({}_j\mathbf{t}_i^{(\eta,\gamma)}\right) = \frac{\mathcal{B}^{\mathrm{H}}\left({}_j\mathbf{t}_i^{(\eta,\gamma)}\right) - \mathcal{B}^{\mathrm{H}}\left({}_j\mathbf{t}_{i-1}^{(\eta,\gamma)}\right)}{{}_j\mathbf{t}_i^{(\eta,\gamma)} - {}_j\mathbf{t}_{i-1}^{(\eta,\gamma)}}.$$

Also, from (17) and (18)

$$\Phi_j^n \triangleq \mathbf{X}_j^{\mathrm{T}}\, {}_{(j-1)\tau}^{\;\;j\tau}\Psi^{(\eta,\gamma)}({}_j\mathbf{t}_0^{(\eta,\gamma)}) - y_n^{j-1}({}_j\mathbf{t}_0^{(\eta,\gamma)}) \simeq 0. \tag{21}$$

Hence, in each step $j = 2, ..., \mathtt{M}$, (20) and (21) give a system of $n+1$ nonlinear algebraic equations as follows:

$$\begin{cases}\left({}_j\mathbf{t}_i^{(\eta,\gamma)} - {}_j\mathbf{t}_{i-1}^{(\eta,\gamma)}\right)\mathbf{R}_j^n\left({}_j\mathbf{t}_i^{(\eta,\gamma)}\right) = 0, \qquad i = 1, ..., n, \\ \Phi_j^n = 0, \end{cases} \tag{22}$$

which can be solved by applying the Newton's iterative technique, for the unknown coefficients $\mathtt{x}_i^j$, $i = 0, 1, ..., n$.

After applying the described approach, we obtain

$$y(t) \simeq y_n(t) = \begin{cases} y_n^1(t), & t \in [0, \tau], \\ y_n^2(t), & t \in [\tau, 2\tau], \\ \quad\vdots & \qquad\vdots \\ y_n^{\mathtt{M}}(t), & t \in [(\mathtt{M}-1)\tau, \mathtt{T}_{\max}]. \end{cases} \tag{23}$$

4 Convergence Analysis

Lemma 4.1. *([15]) Let $g \in \mathbf{H}^{\mathrm{M}}_{\omega^{(\eta,\gamma)}}(\Upsilon)$, $\varsigma \in [0, \mathrm{M}]$ and g_n be the approximate of g obtained by (5). Then, there exists a positive constant ϑ such that*

$$\|g - g_n\|_{\mathrm{M},\omega^{(\eta,\gamma)}} \leq \vartheta \Big\{ (n + \eta + \gamma - 1)(n - 1) \Big\}^{\frac{\varsigma - \mathrm{M}}{2}} |g|_{\mathrm{M},\omega^{(\eta,\gamma)}},$$

where ϑ is independent of g, M, η, γ and the associated semi-norm is defined by

$$|g|_{\mathrm{M},\omega^{(\eta,\gamma)}} = \|g^{(\mathrm{M})}\|_{\mathbf{L}^2_{\omega^{(\eta+\mathrm{M},\gamma+\mathrm{M})}}}.$$

Remark 4.2. ([22]) *Let $g \in \mathbf{L}^2(\bar{\Omega})$, then*

$$\mathbb{E}\Big| \int_0^t g(\upsilon) \mathrm{d}\mathcal{B}^{\mathrm{H}}(\upsilon) \Big|^2 \leq t^{2\mathrm{H}-1} \int_0^t |g|^2 \mathrm{d}\upsilon.$$

Theorem 4.3. *Suppose that $y^j(t) \in \mathbf{H}^{\mathrm{M}}_{\omega^{(\eta,\gamma)}}(\Omega_j)$ is the exact solution of (1)-(2) in the subinterval $\Omega_j = [(j-1)\tau, j\tau] \subseteq \bar{\Omega}$, $j = 1, ..., M$, $y^j_n(t)$ is the numerical solution of (1)-(2) obtained by the proposed scheme in each step, and $\mathbf{R}^n_j(t)$ be the residual error function. Then*

$$\sup_{t \in \Omega_j} \mathbb{E}|\mathbf{R}^n_j(t)|^2 \leq (\hat{\mu}_j + \hat{\beta}_j t^{2\mathrm{H}-1}) \theta^{(\eta,\gamma)}_n \Big\{ |y^j|^2_{\mathrm{M},\omega^{(\eta,\gamma)}} + |y^{j-1}|^2_{\mathrm{M},\omega^{(\eta,\gamma)}} \Big\},$$

where

$$\theta^{(\eta,\gamma)}_n := \Big\{ (n + \eta + \gamma - 1)(n - 1) \Big\}^{\varsigma - \mathrm{M}},$$

and $\varsigma \in [0, \mathrm{M}]$. Also, $\hat{\mu}_j$ and $\hat{\beta}_j$, $j = 1, ..., M$, are positive constants.

Proof. According to the assumptions, by applying Definition 2.2, the numerical solution $y^j_n(t)$ is satisfied in the following equation

$$\begin{aligned} y^j_n(t) = \varphi(0) &+ {}_0\mathrm{I}^{\alpha}_t \left[\mathrm{F}(t, y^j_n(t), y^{j-1}_n(t - \tau_1)) \right] \\ &+ {}_0\mathrm{I}^{\alpha}_t \Big[\mathrm{G}(t, y^j_n(t), y^{j-1}_n(t - \tau_2)) \dot{\mathcal{B}}^{\mathrm{H}}(t) \Big] + \mathbf{R}^n_j(t), \end{aligned}$$

thus, from Eq. (16), we have

$$\mathbf{R}^n_j(t) = -\mathbf{e}^j_n(t) + {}^{\mathrm{F}}\mathbf{Y}^j_n(t) + {}^{\mathrm{G}}\mathbf{Y}^j_n(t), \tag{24}$$

in which $\mathbf{e}_n^j(t) = y^j(t) - y_n^j(t)$,

$$
\begin{aligned}
{}^{\mathrm{F}}\mathbf{Y}_n^j(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[& \mathrm{F}(s, y^j(s), y^{j-1}(s-\tau_1)) \\
& - \mathrm{F}(s, y_n^j(s), y_n^{j-1}(s-\tau_1)) \Big] \mathrm{d}s, \qquad (25)
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{\mathrm{G}}\mathbf{Y}_n^j(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[& \mathrm{G}(s, y^j(s), y^{j-1}(s-\tau_2)) \\
& - \mathrm{G}(s, y_n^j(s), y_n^{j-1}(s-\tau_2)) \Big] \mathrm{d}\mathcal{B}^{\mathrm{H}}(s). \qquad (26)
\end{aligned}
$$

By applying Lemma 4.1

$$
\sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^j(t)|^2 \le \vartheta_j \theta_n^{(\eta,\gamma)} |y^j|^2_{\mathrm{M},\omega^{(\eta,\gamma)}}, \qquad (27)
$$

where $\theta_n^{(\eta,\gamma)} := \Big\{ (n+\eta+\gamma-1)(n-1) \Big\}^{\varsigma-\mathrm{M}}$ and ϑ_j, $j = 1, ..., \mathtt{M}$, are some positive constants defined based on Lemma 4.2. Using the Cauchy-Schwartz inequality and Eq. (25), we get

$$
\begin{aligned}
\Big| {}^{\mathrm{F}}\mathbf{Y}_n^j(t) \Big|^2 \le & \frac{1}{\Gamma(\alpha)^2} \Big(\int_0^t |t-s|^{\alpha-1} \Big| \mathrm{F}(s, y^j(s), y^{j-1}(s-\tau_1)) \\
& - \mathrm{F}(s, y_n^j(s), y_n^{j-1}(s-\tau_1)) \Big| \mathrm{d}s \Big)^2 \\
\le & \frac{1}{\Gamma(\alpha)^2} \int_0^t |t-s|^{2(\alpha-1)} \mathrm{d}s \cdot \int_0^t \Big| \mathrm{F}(s, y^j(s), y^{j-1}(s-\tau_1)) \\
& - \mathrm{F}(s, y_n^j(s), y_n^{j-1}(s-\tau_1)) \Big|^2 \mathrm{d}s \\
\le & \, \mathtt{c}_1 \int_0^t \Big| \mathrm{F}(s, y^j(s), y^{j-1}(s-\tau_1)) - \mathrm{F}(s, y_n^j(s), y_n^{j-1}(s-\tau_1)) \Big|^2 \mathrm{d}s, \qquad (28)
\end{aligned}
$$

where $\mathtt{c}_1$ is a positive constant depended on α and $\mathtt{T}_{\max}$. Now, from the relation (28) and **Assumption 2**

$$
\Big| {}^{\mathrm{F}}\mathbf{Y}_n^j(t) \Big|^2 \le \mathtt{c}_1 \int_0^t \big(\theta_{\mathrm{F}} |\mathbf{e}_n^j(s)| + \mu_{\mathrm{F}} |\mathbf{e}_n^{j-1}(s-\tau_1)| \big)^2 \mathrm{d}s. \qquad (29)
$$

Taking mathematical expectation on both sides of Eq. (29), applying Minkowski's inequality and using Lemma 4.1, we obtain

$$\begin{aligned}\mathbb{E}\Big|{}^{\mathrm{F}}\mathbf{Y}_n^j(t)\Big|^2 &\le \mathtt{c}_2 \int_0^t \left(\mathbb{E}|\mathbf{e}_n^j(s)|^2 + \mathbb{E}|\mathbf{e}_n^{j-1}(s-\tau_1)|^2\right) \mathrm{d}s \\ &\le \mathtt{c}_2 \int_0^t \Big\{\sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^j(t)|^2 + \sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^{j-1}(t-\tau_1)|^2\Big\} \mathrm{d}s \\ &\le \mathtt{c}_2 \mathtt{T}_{\max} \{\sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^j(t)|^2 + \sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^{j-1}(t-\tau_1)|^2\} \\ &\le \mathtt{r}_j \theta_n^{(\eta,\gamma)} \Big\{|y^j|^2_{\mathrm{M},\omega^{(\eta,\gamma)}} + |y^{j-1}|^2_{\mathrm{M},\omega^{(\eta,\gamma)}}\Big\},\end{aligned} \tag{30}$$

where $\mathtt{r}_j = \mathtt{c}_2 \mathtt{T}_{\max} \xi_j$, $\mathtt{c}_2 = \mathtt{c}_1 \mathtt{c}_3^2$, $\mathtt{c}_3 = \max\{\theta_{\mathrm{F}}, \mu_{\mathrm{F}}\}$ and $\xi_j = \max_j\{\vartheta_j, \vartheta_{j-1}\}$. Moreover, from Eq. (26) and **Assumption 2**

$$\begin{aligned}\Big|{}^{\mathrm{G}}\mathbf{Y}_n^j(t)\Big|^2 \le& \frac{1}{\Gamma(\alpha)^2}\Big(\int_0^t |t-s|^{\alpha-1}\Big|\mathrm{G}(s, y^j(s), y^{j-1}(s-\tau_2)) \\ &- \mathrm{G}(s, y_n^j(s), y_n^{j-1}(s-\tau_2))\Big|\mathrm{d}\mathcal{B}^{\mathrm{H}}(s)\Big)^2 \\ \le& \frac{\mathtt{v}_1}{\Gamma(\alpha)^2}\Big(\int_0^t \left(\theta_{\mathrm{G}}|\mathbf{e}_n^j(s)| + \mu_{\mathrm{G}}|\mathbf{e}_n^{j-1}(s-\tau_2)|\right) \mathrm{d}\mathcal{B}^{\mathrm{H}}(s)\Big)^2,\end{aligned} \tag{31}$$

where $\mathtt{v}_1$ is a positive constant depended on α and $\mathtt{T}_{\max}$. Taking mathematical expectation on both sides of Eq. (31) and applying Remark 4.2, result

$$\begin{aligned}\mathbb{E}\Big|{}^{\mathrm{G}}\mathbf{Y}_n^j(t)\Big|^2 &\le \mathtt{v}_2 t^{2\mathrm{H}-1} \int_0^t \left(\mathbb{E}|\mathbf{e}_n^j(s)|^2 + \mathbb{E}|\mathbf{e}_n^{j-1}(s-\tau_2)|^2\right) \mathrm{d}s \\ &\le \mathtt{v}_2 t^{2\mathrm{H}-1} \int_0^t \Big(\sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^j(t)|^2 + \sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^{j-1}(t-\tau_2)|^2\Big) \mathrm{d}s \\ &\le \mathtt{v}_2 \mathtt{T}_{\max} t^{2\mathrm{H}-1} \Big(\sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^j(t)|^2 + \sup_{t\in\Omega_j} \mathbb{E}|\mathbf{e}_n^{j-1}(t-\tau_2)|^2\Big),\end{aligned} \tag{32}$$

where $\mathtt{v}_2 = \frac{\mathtt{v}_1 \mathtt{v}_3^2}{\Gamma(\alpha)^2}$ and $\mathtt{v}_3 = \max\{\theta_{\mathrm{G}}, \mu_{\mathrm{G}}\}$. So, from Lemma 4.1

$$\mathbb{E}\Big|{}^{\mathrm{G}}\mathbf{Y}_n^j(t)\Big|^2 \le \hat{\beta}_j t^{2\mathrm{H}-1} \theta_n^{(\eta,\gamma)} \Big\{|y^j|^2_{\mathrm{M},\omega^{(\eta,\gamma)}} + |y^{j-1}|^2_{\mathrm{M},\omega^{(\eta,\gamma)}}\Big\}, \tag{33}$$

in which $\hat{\beta}_j = \mathtt{v}_2 \mathtt{T}_{\max} \xi_j$. Thus, the relations (24), (27), (30) and (33) result

$$\mathbb{E}|\mathbf{R}_j^n(t)|^2 \leq (\hat{\mu}_j + \hat{\beta}_j t^{2\mathrm{H}-1}) \theta_n^{(\eta,\gamma)} \Big\{ |y^j|^2_{\mathrm{M},\omega^{(\eta,\gamma)}} + |y^{j-1}|^2_{\mathrm{M},\omega^{(\eta,\gamma)}} \Big\},$$

where $\hat{\mu}_j = \vartheta_j + \mathtt{r}_j$. Therefore, we obtain

$$\sup_{t\in\Omega_j} \mathbb{E}|\mathbf{R}_j^n(t)|^2 \leq (\hat{\mu}_j + \hat{\beta}_j t^{2\mathrm{H}-1}) \theta_n^{(\eta,\gamma)} \Big\{ |y^j|^2_{\mathrm{M},\omega^{(\eta,\gamma)}} + |y^{j-1}|^2_{\mathrm{M},\omega^{(\eta,\gamma)}} \Big\}.$$

□

5 Illustrative Test Examples

In this section, we implement the presented step collocation method to solve (1)-(2). Also, we calculate the numerical solution over $\mathcal{P}$ discretized fractional Brownian paths and the average of the obtained solutions on these paths will be considered as the final numerical solution.

In the following examples, we consider the $\mathbf{L}_\infty$-norm:

$$\|\mathbf{e}_n\|_\infty = \mathbb{E}\left[\max_{t\in[0,\mathtt{T}_{\max}]} |y(t) - y_n(t)| \right].$$

and the convergence order is defined as:

$$\mathbf{CO} = \log_{\frac{n_1}{n_2}} \frac{\|\mathbf{e}_{n_1}\|_\infty}{\|\mathbf{e}_{n_2}\|_\infty}.$$

The codes are written in Matlab software and the computations are performed on a machine using a 1.70 GHz processor.

Example 5.1. Consider the TFSDDE

$${}_0\mathrm{D}_t^\alpha y(t) = t y(t) y^2(t-\tau_1) + e^{y(t)} y(t-\tau_2) \dot{\mathcal{B}}^{\mathrm{H}}(t) + f(t),$$

with the initial condition $y(t) = 0$, $t \in [-\tau, 0]$. The exact solution is $y(t) = t^3 - t$.

Algorithm 1: The process of the proposed scheme for TFS-DDE with fBm (1) and the initial condition (2).

Input: $\mathtt{T_{max}}, \tau_1, \tau_2 \in \mathbb{R}^+$, $n \in \mathbb{Z}^+$, $\alpha \in (0,1)$, $\eta, \gamma > -1$, the functions F, G, φ and $\mathcal{B}^{\mathrm{H}}(t)$; Let $\tau = \max\{\tau_1, \tau_2\}$.

Step 1: Compute the shifted JPs $\mathbf{J}_i^{(\eta,\gamma)}(t; a, b)$ from Definition 2.5.

Step 2: Compute the vector of shifted JPs ${}_a^b\Psi^{(\eta,\gamma)}(t)$, from (6).

Step 3: Compute the vector ${}_a^b\Delta(\eta, \gamma; \alpha; t)$, from Theorem 2.7.

Step 4: Compute the collocation points ${}_1\mathbf{t}_i^{(\eta,\gamma)}$ for $i = 0, ..., n$.

Step 5: Solve the nonlinear system (15) and obtain the unknown vectors $\mathbf{X}_1$ by using **Step 2**, **Step 3** and **Step 4**.

Step 6: Let $y_n^1(t) = \mathbf{X}_1^{\mathrm{T}}\ {}_0^\tau\Psi^{(\eta,\gamma)}(t)$, on the interval $[0, \tau]$.

Step 7: Start temporal loop for $j = 2, ..., \mathtt{M}$, where $\mathtt{M} = \left[\frac{\mathtt{T_{max}}}{\tau}\right]$:

Step 7.1: Compute the collocation points ${}_j\mathbf{t}_i^{(\eta,\gamma)}$ for $i = 0, ..., n$.

Step 7.2: Solve the nonlinear system (22) and obtain the unknown vectors $\mathbf{X}_j$ by applying **Step 2**, **Step 3** and **Step 7.1**.

Step 7.4: Let $y_n^j(t) = \mathbf{X}_j^{\mathrm{T}}\ {}_{(j-1)\tau}^{j\tau}\Psi^{(\eta,\gamma)}(t)$, on the interval $[(j-1)\tau, j\tau]$.

Step 8: Post-processing the results.

Output: The approximate solution is: $y(t) \simeq y_n(t)$ from (23).

Figure 1 shows the exact and numerical solutions of $y(t)$ with $\mathcal{P} = 1$, when $\tau_1 = 0.5$, $\tau_2 = 1$, $n = 5$ and $\eta = \gamma = 0$. Figure 2 displays the absolute error of $y(t)$ with $\alpha = 0.45$, when $\mathrm{H} = 1/2$, $\tau_1 = 0.25$, $\tau_2 = 0.5$, $\mathcal{P} = 80$, $n = 9$ and $\eta = \gamma = 0.5$. Table 1 shows the values of errors, **CO** and CPU-time for $\eta = \gamma = -0.5$, $\tau_1 = 0.5$, $\tau_2 = 1$, $\mathcal{P} = 70$ and $n = \{6, 9, 12\}$. Finally, Figure 3 displays the logarithm of absolute error of $y(t)$ with $\alpha = 0.75$, when $\mathrm{H} = 3/4$, $\tau_1 = \tau_2 = 0.5$, $\mathcal{P} = 100$, $n = 8$ and $\eta = \gamma = 0$, $\eta = \gamma = 0.5$.

Example 5.2. Consider the TFSDDE

$$ {}_0\mathrm{D}_t^\alpha y(t) = -e^{y(t)} + \frac{y(t-\tau)}{1 + y^2(t-\tau)} + \varepsilon t y(t)\dot{\mathcal{B}}^{\mathrm{H}}(t) + f(t), $$

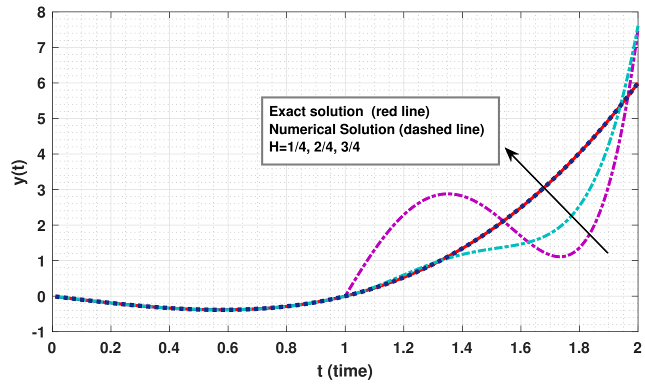

Figure 1: The exact and numerical solution of $y(t)$ in one trajectory of fBm for Example 5.1 with $\alpha = 0.55$.

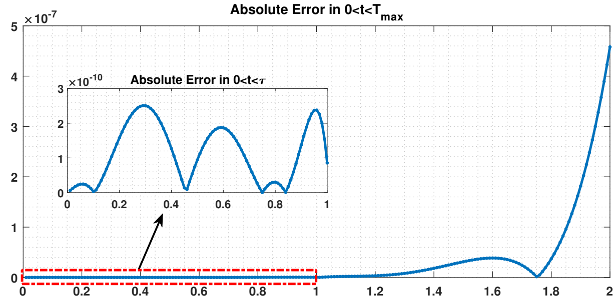

Figure 2: The absolute error of $y(t)$ for Example 5.1 with $\mathrm{H} = 1/2$.

where ε is a positive constant, with the initial condition $y(t) = 0$, $t \in [-\tau, 0]$. The exact solution is $y(t) = \sin(\pi t)$.

Figure 4 indicates the behaviour of the numerical solution of $y(t)$ with $\varepsilon = \{0.01, 0.1, 0.5, 0.8\}$ and Figure 5 shows the absolute error of $y(t)$ with $\varepsilon = 0.01$, when $\alpha = 0.5$, $\tau = 0.5$, $\mathcal{P} = 70$, $\eta = \gamma = -0.5$ and $n = 7$. Table 2 shows the values of errors, **CO** and CPU-time for $\eta = \gamma = 0$, $\tau = 1$, $\varepsilon = 0.02$, $\mathrm{H} = 0.7$, $\mathcal{P} = 100$ and $n = \{6, 9, 12\}$. Also, Figure 6 demonstrates the logarithm of absolute error for $y(t)$, when $\alpha = 0.75$, $\tau = 0.25$, $\varepsilon = 0.3$, $\eta = 0.5$, $\gamma = -0.5$, $\mathcal{P} = 30$ and $n = 4$.

Table 1: The $\mathbf{L}_\infty$-norm error, **CO** and CPU-time for Example 5.1.

	$\alpha = 0.25$		$\alpha = 0.75$		
n	$\|\mathbf{e}_n\|_\infty$	**CO**	$\|\mathbf{e}_n\|_\infty$	**CO**	CPU-time (s)
6	3.2490e − 03	−	2.2501e − 03	−	34.21
9	2.1328e − 07	23.7536	1.3628e − 07	23.9521	107.35
12	5.1405e − 11	28.9578	3.2280e − 11	29.0182	181.14

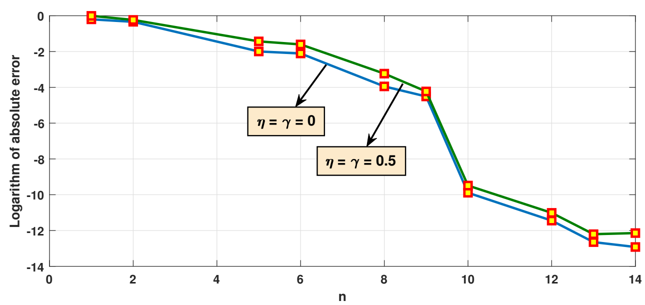

Figure 3: The logarithm of absolute error of $y(t)$ for different values of n for Example 5.1 with $\alpha = 0.75$ and $\mathrm{H} = 3/4$.

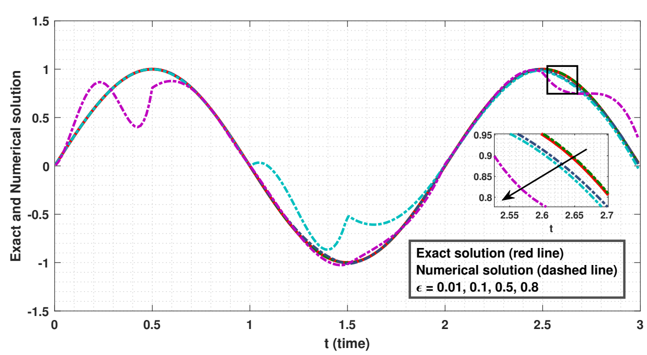

Figure 4: The exact and numerical solution of $y(t)$ with $\mathrm{H} = 0.45$ and $\alpha = 0.5$ in Example 5.2.

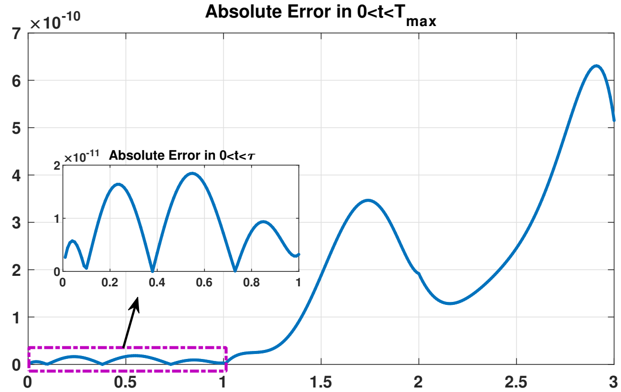

Figure 5: The absolute error of $y(t)$ in Example 5.2 with $\mathrm{H} = 0.5$ and $\varepsilon = 0.01$.

Table 2: The $\mathbf{L}_\infty$-norm error, **CO** and CPU-time for Example 5.2.

	$\alpha = 0.25$		$\alpha = 0.75$		
n	$\lVert\mathbf{e}_n\rVert_\infty$	**CO**	$\lVert\mathbf{e}_n\rVert_\infty$	**CO**	`CPU-time (s)`
6	3.6213e − 02	−	4.5601e − 03	−	46.27
9	4.3741e − 05	16.5708	9.4418e − 06	15.2416	128.12
12	6.1015e − 09	30.8587	8.2280e − 10	32.4940	195.64

6 Conclusion

A numerical scheme based on the JPs was investigated to solve a class of time-fractional stochastic delay differential equation with fractional Brownian motion. First, we converted the TFSDDE into a non-delay equation by a step-by-step scheme. Then, a collocation method based on the JPs was applied to find the numerical solution of the resulted problem in each step. Convergence analysis of the presented approach was evaluated. Finally, two test examples were implemented to highlight the efficiency and accuracy of the proposed algorithm. The obtained outputs confirm that this numerical scheme is a useful tool for solving such nonlinear stochastic delay equations. For future research works, it

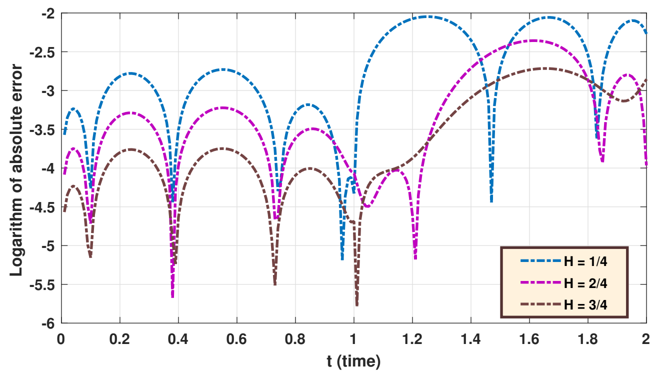

Figure 6: The logarithm of absolute error of $y(t)$ for Example 5.2 with $\alpha = 0.75$ and $\varepsilon = 0.3$.

may be possible to apply this algorithm for solving the TFSDDEs with variable-order fractional Brownian motion. Also, using this technique for similar models with distributed order fractional derivative could be the subject of some future researches.

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Seddigheh Banihashemi
Department of Applied Mathematics
Ph.D. Student of Applied Mathematics
University of Mazandaran
Babolsar, Iran
E-mail: s.banihashemi@stu.umz.ac.ir

Hossein Jafari
Department of Applied Mathematics
Professor of Applied Mathematics
University of Mazandaran
Babolsar, Iran
E-mail: jafari@umz.ac.ir

Afshin Babaei
Department of Applied Mathematics
Associate Professor of Applied Mathematics
University of Mazandaran
Babolsar, Iran
E-mail: babaei@umz.ac.ir