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Original Research Paper

Hybrid Fractional Diffusion Problem with Dirichlet Boundary Conditions

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Abstract. The aim of this research is to establish the analytic solution of partial differential equations with homogenous initial boundary conditions. Having hybrid fractional order derivative allows us to have classical boundary and initial conditions. The solution of the problem is obtained in terms of bivariate Mittag-Leffler function as a Fourier series by utilizing separation of variables method (SVM) and the inner product defined on $L^2[0,l]$. The presented examples illustrate the accuracy and effectiveness of the SVM for the fractional diffusion problems. The accuracy of the obtained solution can also be seen from the observation that as the fractional order α tends to 1, the solution of the fractional diffusion problem tends to the solution of the diffusion problem.

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1 Introduction

Since mathematical models including fractional derivatives play a vital role, fractional derivatives have drawn increasing attention from many researchers in various branches of science. Therefore, there are many different fractional derivatives such as Caputo, Riemann-Liouville, Atangana-Baleanu defined as follows [17]:

The Caputo fractional derivative of order q is defined as

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s) ds, t \in [t_0, t_0 + T]$$

where $u^{(n)}(t) = \frac{d^n u}{dt^n}$, $n-1 < q < n$.

The Riemann-Liouville fractional derivative of order q is defined as

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-q-1} u(s) ds, t \in [t_0, t_0 + T]$$

where $n-1 < q < n$.

However these fractional derivatives do not satisfy the most important properties of the ordinary derivative, which leads to many difficulties in analyzing or obtaining the solution of fractional mathematical models.
As a result, many scientists focus on defining new fractional derivatives to cover the setbacks of the defined ones. Moreover, the success of mathematical modelling of systems or processes depends on the fractional derivative it involves, since the correct choice of the fractional derivative allows us to model the real data of systems or processes accurately. In order to the define new fractional derivatives, various methods exist, and these are classified based on their features and formation such as non-local fractional derivatives and local fractional derivatives. The constant proportional Caputo hybrid operator is a newly defined fractional derivative that is a combination of the Caputo derivative and the proportional derivative and is defined as:

$$\begin{aligned} {}_0^{CPC}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(K_1(\alpha) f(\tau) + K_0(\alpha) f'(\tau)\right)(t-\tau)^{-\alpha} d\tau \\ &= K_1(\alpha)\, {}_0^{RL}I_t^{1-\alpha} f(t) + K_0(\alpha)\, {}_0^{C}D_t^{\alpha} f(t) \end{aligned}$$

where the functions K_0 and K_1 satisfy certain properties in terms of limit [4]. The domain of this operator contains functions f on positive reals such that f and its derivative f' are locally L^1 functions. Moreover, ${}_0^{RL}I_t^{\alpha}$ and ${}_0^{C}D_t^{\alpha}$ represent the Riemann–Liouville integral and Caputo derivative, respectively.Note that this hybrid fractional operator can be enounced as a linear combination of the Caputo fractional derivative and the Riemann–Liouville fractional integral. Notice that the constant proportional Caputo hybrid operator is obtained by adding a non-locality property to a proportional derivative operator, which allows us to model processes with non-local behaviour more efficiently which is the most important advantage of it. The non-locality property of the constant proportional Caputo hybrid operator is a result of the Riemann-Liouville integral which is defined as:
The Riemann-Liouville time-fractional integral of a real valued function $u(x,t)$ is defined as

$$I_t^{\alpha} u\left(x,t\right) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(t-s\right)^{\alpha-1} u(x,s) ds$$

where $\alpha > 0$ denotes the order of the integral.
This new fractional derivative have been drawing the attention of many researchers in various branches of science. As a result, there is a substantial amount of study in the literature such as on the hybrid fractional derivative [14, 23, 24, 15, 3], heat and mass transportation [2], dynamics of processes [1], fractional Schrodinger and Bogoyavlenskii equations [20], Modified Zakharov-Kuznetsov equation [16, 22] and Konopelchenko-Dubrovsky and Kadomtsev-Petviashvili equations [21].
The choice of functions K_0 and K_1 included in the definition of the constant proportional Caputo hybrid operator motivates us to analyze the solution of fractional diffusion equations with initial and boundary conditions for various functions K_0 and K_1 and compare them. In this study, we focus on obtaining the solution of the following fractional diffusion equation with various the constant proportional Caputo hybrid operator by making use of the SVM:

$$\begin{aligned} {}_0^{CPC}D_{\alpha}^{\;1}\, f\left(t\right) &= \left(1-\alpha\right)\, {}_0^{RL}I_t^{1-\alpha} f\left(t\right) + \alpha\, {}_0^{C}D_t^{\alpha} f\left(t\right), \\ {}_0^{CPC}D_{\alpha}^{\;2}\, f\left(t\right) &= \left(1-\alpha^2\right)\, {}_0^{RL}I_t^{1-\alpha} f\left(t\right) + \alpha^2\, {}_0^{C}D_t^{\alpha} f\left(t\right). \end{aligned}$$

where $0 < \alpha < 1, 0 \leq x \leq l,\ 0 \leq t \leq T$. Here we use the following forms of the proportional derivatives: We especially consider the following ones:

$$
\begin{aligned}
{}_0^{CPC}D_{\alpha}^{\ 1}\ f(t) &= (1-\alpha)\ {}_0^{RL}I_t^{1-\alpha} f(t) + \alpha {}_0^{C}D_t^{\alpha} f(t)\,,\\
{}_0^{CPC}D_{\alpha}^{\ 2}\ f(t) &= \left(1-\alpha^2\right)\ {}_0^{RL}I_t^{1-\alpha} f(t) + \alpha^2 {}_0^{C}D_t^{\alpha} f(t)\,.
\end{aligned}
$$

The novelty of this study is the application of the SVM to a time fractional diffusion equation including the constant proportional Caputo hybrid derivative operator. As a result, the implementation of this method and its effectiveness and accuracy are presented explicitly.

From a physical aspect, the intrinsic nature of the physical system can be reflected to the mathematical model of the system by using fractional derivatives. Therefore, the solution of the fractional mathematical model is in excellent agreement with the predictions and experimental measurement of it. The systems whose behaviour is non-local can be modelled better by fractional mathematical models, and the degree of its nonlocality can be arranged by the order of fractional derivative. In order to analyze the diffusion in a non-homogenous medium that has memory effects, it is better to analyze the solution of the fractional mathematical model for this diffusion. As a result, in order to model a process, the correct choice of fractional derivative and its order must be determined. In the mathematical modelling of a diffusion problem for different matters such as liquid, gas and temperature, the suitable fractional order α is chosen, since the diffusion coefficient depends on the order α of fractional derivative [5]. This mathematical modelling describes the behaviour of matter in a phase. There is a vast amount of published work on the diffusion of various matters in science, especially in fluid mechanics and gas dynamics [6, 7, 8, 9, 10, 11, 12, 18, 13]. From this aspect, the analysis of this problem plays an important role in its application. Moreover, sub-diffusion cases for which $0 < \alpha < 1$ are under consideration. The solution of the fractional mathematical model of sub-diffusion cases behaves much slower than the solution of the integer-order mathematical model unlike the fractional mathematical model for super-diffusion [19].

The focus of the current work is to establish the solutions of the following problem:

$$\begin{aligned} {}_{0}^{CPC}D_t^{\alpha} u(x,t) &= \gamma^2 u_{xx}(x,t), && (1) \\ u(0,t) &= u(l,t) = 0, && (2) \\ u(x,0) &= f(x) && (3) \end{aligned}$$

where $0 < \alpha < 1, 0 \leq x \leq l, 0 \leq t \leq T, \gamma \in \mathbb{R}$.

2 Main Results

The analytic form of the solution for the problem (1)-(3) is established by employing the well known method SVM.

$$u(x,t;\alpha) = X(x)\ T(t;\alpha) \tag{4}$$

where $0 \leq x \leq l, 0 \leq t \leq T$.
Utilizing (4) in (1) and arranging leads to the following:

$$\frac{{}_{0}^{CPC}D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = \gamma^2 \frac{X''(x)}{X(x)} = -\lambda^2. \tag{5}$$

Taking the right hand side of equation (5) and related boundary conditions (2) into account the following problem is obtained:

$$\begin{aligned} X''(x) + \lambda^2 X(x) = 0, && (6) \\ X(0) = X(l) = 0 && (7) \end{aligned}$$

which has the solution $X(x) = e^{rx}$. As a result, the following characteristic equation is reached $r^2 + \lambda^2 = 0$.
Case 1. We have coincident roots $r_1 = r_2$ which happens when $\lambda = 0$. Therefore, the solution of the problem (6)-(7) becomes $X(x) = k_1 x + k_2$. The first boundary condition yields $X(0) = k_2 = 0$ which indicates that $X(x) = k_1 x$. In a similar manner, utilizing last condition yields $X(l) = k_1 l = 0 \Rightarrow k_1 = 0$, leading to $X(x) = 0$. Therefore, there is no solution for $\lambda = 0$.
Case 2. We have two distinct real roots r_1, r_2 which happens when $\lambda > 0$. As a result, the solution of the problem (6)-(7) becomes $X(x) = c_1 e^{r_1 x} +$

$c_2 e^{r_2 x}$. By utilizing the first condition, we have $X(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$ which leads to the solution $X(x) = c_1 \left(e^{r_1 x} - e^{r_2 x}\right)$. In a similar manner, second boundary condition yields $X(l) = c_1 \left(e^{r_1 l} - e^{r_2 l}\right) = 0$. Since $e^{r_1 l} \neq e^{r_2 l}$, the equation $X(l) = c_1 \left(e^{r_1 l} - e^{r_2 l}\right) = 0$ is satisfied if and only if $c_1 = 0$ which indicates $X(x) = 0$. Therefore, there is no solution for $\lambda > 0$.

Case 3. We have two complex roots which happens when $\lambda < 0$. Consequently, the solution of the problem (6)-(7) becomes $X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$. Utilization of the first condition allows us to obtain $X(0) = c_1 = 0$ which leads to the solution $X(x) = c_2 \sin(\lambda x)$. In a similar manner, utilization of the last condition yields $X(l) = c_2 \sin(\lambda l) = 0$, indicating that $\sin(\lambda l) = 0$. Therefore, the following eigenvalues are obtained:

$$\lambda_n = \frac{w_n}{l}, \lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

where $w_n = n\pi$ satisfy the equation $sin(w_n) = 0$.
Therefore, the following solution is established:

$$X_n(x) = c_2 \sin\left(w_n \left(\frac{x}{l}\right)\right), n = 1, 2, 3, \ldots$$

The other equation in (5) with λ_n leads to the following fractional differential equation:

$$\frac{{}_0^{CPC}D_t^{\alpha}\left(T(t;\alpha)\right)}{T(t;\alpha)} = -\gamma^2 \lambda_n^2$$

which yields the following solution [4]

$$T_n(t;\alpha) = E^1_{\alpha,1,\ 1}\left(\frac{-\gamma^2\lambda_n^2}{K_0(\alpha)} t^{\alpha}, \frac{-K_1(\alpha)}{K_0(\alpha)} t\right), n = 0, 1, 2, 3, \ldots$$

where a bivariate Mittag-Leffler function $E^{(\gamma)}_{\alpha,\ \beta,\ \kappa}(x, y)$ proposed by Özarslan and Kürt [14], is represented in double power series as follows:

$$E^{(\gamma)}_{\alpha,\beta,\ \kappa}(x, y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}}{\Gamma(\alpha r + \beta s + \kappa)} \frac{x^r}{r!} \frac{y^s}{s!}$$

where $\alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha), Re(\beta), Re(\kappa) > 0$. The solution for every eigenvalue λ_n is constructed as

$$\begin{aligned} u_n(x,t;\alpha) &= X_n(x)\,T_n(t;\alpha) \\ &= E^1_{\alpha,1,1}\left(\tfrac{-\gamma^2\lambda_n^2}{K_0(\alpha)}t^\alpha, \tfrac{-K_1(\alpha)}{K_0(\alpha)}t\right)\sin\left(w_n\left(\tfrac{x}{l}\right)\right), n = 0,1,2,3,\ldots \end{aligned}$$

which yields general solution as

$$u(x,t;\alpha) = \sum_{n=0}^{\infty} A_n \sin\left(w_n\left(\frac{x}{l}\right)\right) E^1_{\alpha,1,\,1}\left(\frac{-\gamma^2\lambda_n^2}{K_0(\alpha)}t^\alpha, \frac{-K_1(\alpha)}{K_0(\alpha)}t\right). \quad (8)$$

The convergence of the series in (8) is given in [4].
Notice that this solution fulfills fractional differential equation and boundary conditions.
Making use of initial condition yields the following:

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \sin\left(w_n\left(\frac{x}{l}\right)\right).$$

By taking the inner product in $L^2[0,l]$ into account, A_n for $n = 0,1,2,3,\ldots$ are acquired:

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(w_n\left(\frac{x}{l}\right)\right).$$

The advantage of this method comparing with the homotopy method or other numerical methods is that exact solutions of the fractional differential equations are established by the SVM, while their approximate solutions are acquired by homotopy and other numerical methods. Although the SVM is a very common method to construct the solution of partial differential equations, applying it to fractional differential equations is not included in a vast number of studies in literature.

3 Illustrative Example

Let the following mathematical problem be considered:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t), \\ u(0,t) &= u(1,t), \\ u(x,0) &= \sin(\pi x) \end{aligned} \quad (9)$$

whose solution is

$$u\left(x,t\right)=\sin\left(\pi x\right)e^{-\pi^{2}t}$$

where $0\leq x\leq 1, 0\leq t\leq T$.
Let us consider the following fractional diffusion problem:

$$\begin{aligned} {}_{0}^{CPC}D_{t}^{\alpha}u\left(x,t\right) &= u_{xx}\left(x,t\right), &(10)\\ u\left(0,t\right) &= u\left(1,t\right), &(11)\\ u\left(x,0\right) &= \sin\left(\pi x\right) &(12)\end{aligned}$$

where $0<\alpha<1, 0\leq x\leq 1,\ 0\leq t\leq T$.
The method SVM yields the following equations:

$$\frac{{}_{0}^{CPC}D_{t}^{\alpha}\left(T\left(t;\alpha\right)\right)}{T\left(t;\alpha\right)}=\frac{X''\left(x\right)}{X\left(x\right)}=-\lambda^{2}. \tag{13}$$

Taking the right hand side of equation (13) and related boundary conditions (11) into account yields:

$$\begin{aligned} X''\left(x\right)+\lambda^{2}X\left(x\right)=0, &\quad(14)\\ X\left(0\right)=X\left(1\right)=0, &\quad(15)\end{aligned}$$

The solution of the problem (14)-(15) is obtained as

$$X_{n}\left(x\right)=\sin\left(n\pi x\right), n=1,2,3,\ldots$$

The other equation (13) for each eigenvalue λ_n leads to the following:

$$\frac{{}_{0}^{CPC}D_{t}^{\alpha}\left(T\left(t;\alpha\right)\right)}{T\left(t;\alpha\right)}=-\lambda^{2}$$

which yields the following solution

$$T_{n}\left(t;\alpha\right)=E_{\alpha,1,\ 1}^{1}\left(\frac{-n^{2}\pi^{2}}{K_{0}\left(\alpha\right)}t^{\alpha},\frac{-K_{1}\left(\alpha\right)}{K_{0}\left(\alpha\right)}t\right), n=0,1,2,3,\ldots$$

Corresponding to λ_n, the following solution is obtained

$$u_{n}\left(x,t;\alpha\right)=E_{\alpha,1,\ 1}^{1}\left(\frac{-n^{2}\pi^{2}}{K_{0}\left(\alpha\right)}t^{\alpha},\frac{-K_{1}\left(\alpha\right)}{K_{0}\left(\alpha\right)}t\right)\sin\left(n\pi x\right), n=0,1,2,3,\ldots$$

Hence Superposition Principle leads to the following sum:

$$u\left(x,t;\alpha\right)=\sum_{n=0}^{\infty}A_n\sin\left(n\pi x\right)\ E^{1}_{\alpha,1,\ 1}\left(\frac{-n^2\pi^2}{K_0\left(\alpha\right)}t^{\alpha},\frac{-K_1\left(\alpha\right)}{K_0\left(\alpha\right)}t\right).$$

Utilizing initial condition (12) yields the following:

$$u\left(x,0\right)=\sum_{n=0}^{\infty}A_n\sin\left(n\pi x\right).$$

Taking the inner product into account allows us to determine A_n for $n=0,1,2,3,\dots$ in the following form:

$$A_n=2\int_0^1\sin\left(\pi x\right)\sin\left(n\pi x\right)dx.$$

For $n\neq 1$, $A_n=0$. For $n=1$, we get

$$A_1=2\int_0^1\sin^2\left(\pi x\right)dx=1.$$

Thus

$$u\left(x,t;\alpha\right)=\sin\left(\pi x\right)E^{1}_{\alpha,1,\ 1}\left(\frac{-\pi^2}{K_0\left(\alpha\right)}t^{\alpha},\frac{-K_1\left(\alpha\right)}{K_0\left(\alpha\right)}t\right). \tag{16}$$

The accuracy of the obtained solution is checked by substituting $\alpha=1$ into (16) which yields the solution of (9).

Particularly, the problem (10)-(12) have the following solution for the specific functions K_0 and K_1:

Case 1: For $K_0\left(\alpha\right)=\alpha, K_1\left(\alpha\right)=1-\alpha$, the solution becomes

$$u\left(x,t;\alpha\right)=\sin\left(\pi x\right)E^{1}_{\alpha,1,\ 1}\left(\frac{-\pi^2}{\alpha}t^{\alpha},\frac{\alpha-1}{\alpha}t\right).$$

Case 2: For $K_0\left(\alpha\right)=\alpha^2, K_1\left(\alpha\right)=1-\alpha^2$, the solution becomes

$$u\left(x,t;\alpha\right)=\sin\left(\pi x\right)E^{1}_{\alpha,1,\ 1}\left(\frac{-\pi^2}{\alpha^2}t^{\alpha},\frac{\alpha^2-1}{\alpha^2}t\right).$$

The graphics of solutions for Case 1, Case 2 and Problem (9) in 2D are given in Fig.1-4 for various values of α.

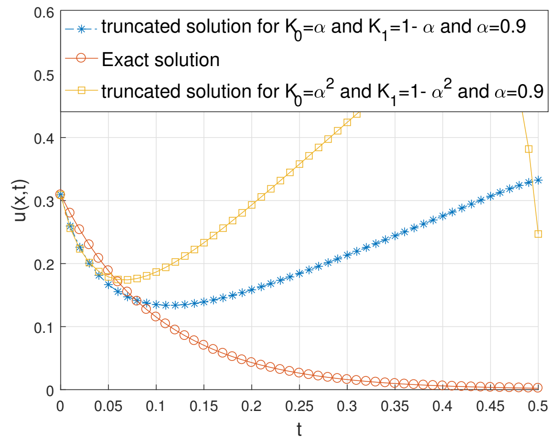

Figure 1: The graphics of solutions for Example for different functions $K_0(\alpha)$ and $K_1(\alpha)$ in 2D at $x = 0.1$ and for $\alpha = 0.9$.

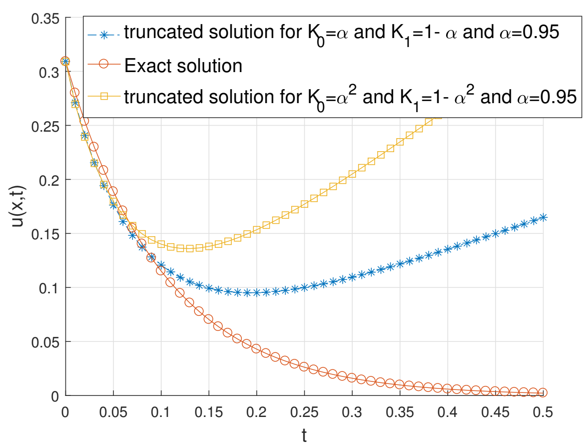

Figure 2: The graphics of solutions for Example for different functions $K_0(\alpha)$ and $K_1(\alpha)$ in 2D at $x = 0.1$ and for $\alpha = 0.95$.

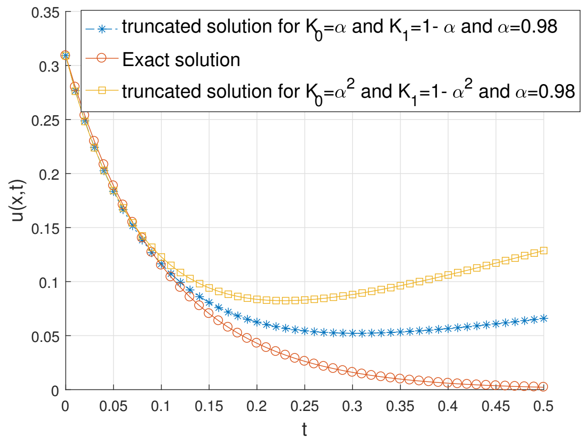

Figure 3: The graphics of solutions for Example for different functions $K_0\left(\alpha\right)$ and $K_1\left(\alpha\right)$ in 2D at $x = 0.1$ and for $\alpha = 0.98$.

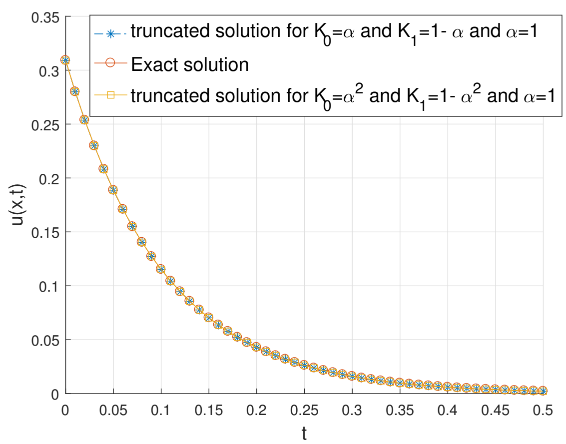

Figure 4: The graphics of solutions for Example for different functions $K_0\left(\alpha\right)$ and $K_1\left(\alpha\right)$ in 2D at $x = 0.1$ and for $\alpha = 1$.

Note that by truncated solution, we mean the approximate solution. It is clear from figures 1-4 that the solution of time fractional diffusion equation including the constant proportional Caputo hybrid derivative operator ${}_{0}^{CPC}D_{\alpha}^{\ 1}\ f\left(t\right)$ converges the solution of the integer order diffusion equation as α tends to 1 faster than its solution including the constant proportional Caputo hybrid derivative operator ${}_{0}^{CPC}D_{\alpha}^{\ 2}\ f\left(t\right)$. As a result, we conclude that the choice of the functions $K_0\left(\alpha\right)$ and $K_1\left(\alpha\right)$ in ${}_{0}^{CPC}D_{\alpha}^{\ 1}\ f\left(t\right)$ are better than those in ${}_{0}^{CPC}D_{\alpha}^{\ 2}\ f\left(t\right)$. Moreover, the graphs of the solutions move away from the solution of the corresponding integer order differential equation, as the fractional order α decreases away from 1.

4 Conclusion

The solution of the mathematical problem with the hybrid time fractional derivative is constructed by the SVM in terms of the bivariate Mittag-Leffler function. Besides, the accuracy of the solution is tested by taking $\alpha = 1$ in the solution which yields the solution of the mathematical problem with ordinary derivative. As a result, the illustrative example indicates that the SVM plays an influential role in the construction of mathematical problems including fractional derivatives.
Based on the analytic solution, we reach the conclusion that diffusion processes decay over time until an initial condition is reached when α is less than a certain value of α for Case 1 but diffusion processes decay with time for all values of α between 0 and 1 for Case 2. As α tends to 0, the rate of decaying increases. This implies that in the mathematical model for diffusion of the matter which has a small diffusion rate, the value of α must be close to 0. This model can account for various diffusion processes of diverse methods.

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