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$N-$Legendre and $N-$Slant Curves in the Unit Tangent Bundles of Minkowski Surfaces with Natural Diagonal Structures

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Abstract. This paper is devoted to discuss $N-$Legendre and $N-$slant curves in the unit tangent bundles of Minkowski surfaces. Unit tangent bundles are considered with a natural diagonal structure which generalizes the standard contact metric structure.

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1 Introduction

Let (M, g, ϕ, ξ, η) be a 3-dimensional contact metric manifold. The notion of slant curves in M was introduced by Cho *et al.* in [5]. A curve $\gamma(s)$ in M is said to be slant if its tangent vector field makes constant *contact angle* with the Reeb vector field ξ. In this paper, for a contact Riemann manifold, it was proved that a slant curve in a Sasakian 3-manifold is that its ratio of κ and $\tau - 1$ is constant. Slant curves with contact angle $\frac{\pi}{2}$ are called Legendre curves. In [2], Baikoussis and

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Blair showed that on a 3-dimensional Sasakian manifold, the torsion of a Legendre curve is $+1$.

Let (M, g) be a smooth Riemannian manifold and TM its tangent bundle. The best known Riemannian metric on the tangent bundle, which is called as Sasaki metric and denoted by ${}^{S}g$, was introduced by Sasaki in [11]. In addition to this, Tachibana and Okumura defined an almost complex structure J in TM which is compatible with ${}^{S}g$ in [12] (for a history of tangent bundles, see [1]).

The unit tangent bundle T_1M is a hypersurface in TM. In [13], Tashiro constructed an almost contact metric structure $(\bar{g}, \bar{\phi}, \bar{\xi}, \bar{\eta})$ in T_1M which is induced from the almost complex structure $(TM, {}^{S}g, J)$ in TM (see also [4]).

Some other contact metric structures on unit tangent bundles may be constructed by using a few lifts to the tangent bundle. One of them, which is known as the natural diagonal structure, was introduced and studied by Druta–Romaniuc and Oproiu in [6], [7] and [8].

Similar to slant and Legendre curves, $N-$Legendre and $N-$slant curves can be defined as follows: A curve $\gamma(s)$ in a contact metric manifold M is said to be $N-$slant if its normal vector field makes constant contact angle with the Reeb vector field ξ. $N-$slant curves with contact angle $\frac{\pi}{2}$ called $N-$Legendre curves [9]. $\tilde{N}-$Legendre and $\tilde{N}-$slant curves were discussed in the unit tangent bundles of Riemannian and Minkowski surfaces when the unit tangent bundles are endowed with the Sasaki metric in [9] and [3].

In the present paper, these curves are studied in the unit tangent bundles of Minkowski surfaces which are endowed with natural diagonal metric structures. Some geometric results are obtained when the surfaces are supposed to be de Sitter and not de Sitter spaces.

2 Preliminaries

2.1 Minkowski space

The Minkowski 3-space M_1^2 is a real vector space provided with the standard metric given by

$$g = dx_1^2 + dx_2^2 - dx_3^2,$$

in terms of the natural coordinate system (x_1, x_2, x_3). Recall that an arbitrary vector $x = (x_1, x_2, x_3)$ in M_1^2 can have one of three Lorentzian causal characters: It is said to be spacelike, timelike and lightlike (null), if $g(x, x) > 0$, $g(x, x) < 0$ and $g(x, x) = 0$. Similarly, an arc-parametrized curve $\alpha(s)$ in M_1^2 is spacelike, timelike and lightlike (null) if $g(\alpha'(s), \alpha'(s)) = 1$, $g(\alpha'(s), \alpha'(s)) = -1$ and $g(\alpha'(s), \alpha'(s)) = 0$, respectively.

For any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, the Minkowski pseudo vector product is defined by

$$x \times_1 y = \begin{vmatrix} i & j & -k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\{i, j, k\}$ canonical basis of M_1^2. The norm of a vector x is defined by $\|x\| = \sqrt{|g(x, x)|}$ and two vectors x and y in M_1^2 is said to be orthogonal if $g(x, y) = 0$. The sets

$$\begin{array}{rcll} S_1^2(\mathbf{r}) & = & \{x = (x_1, x_2, x_3) \in M_1^2 : g(x, x) = \mathbf{r}^2\} & \text{Lorentzian sphere} \\ H_1^2(\mathbf{r}) & = & \{x = (x_1, x_2, x_3) \in M_1^2 : g(x, x) = -\mathbf{r}^2\} & \text{Hyperbolic sphere} \end{array}$$

are called de Sitter and anti de Sitter spaces, respectively.

Let $\{T, N, B\}$ be the moving Frenet frame along an arc-parametrized curve $\alpha(s)$ in M_1^2. The Frenet formulae are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\epsilon_1\epsilon_2\kappa & 0 & \tau \\ 0 & -\epsilon_2\epsilon_3\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{1}$$

where $\tau(s)$ is torsion of the curve α at s and $g(T, T) = \epsilon_1 = \pm 1$, $g(N, N) = \epsilon_2 = \pm 1$, $g(B, B) = -\epsilon_1\epsilon_2$. We have the following relations for this moving frame

$$T \times_1 N = \epsilon_1\epsilon_2 B, \; N \times_1 B = -\epsilon_1 T \text{ and } B \times_1 T = -\epsilon_2 N.$$

When M_1^1 is a Minkowski surface endowed with the standard metric given by

$$g = dx_1^2 - dx_2^2,$$

then the moving Frenet formulae of an arc-parametrized curve $\alpha(s)$ in M_1^1 turn to be

$$\left(\begin{array}{c} T' \\ N' \end{array}\right) = \left(\begin{array}{cc} 0 & \epsilon_2\kappa \\ -\epsilon_1\kappa & 0 \end{array}\right)\left(\begin{array}{c} T \\ N \end{array}\right), \tag{2}$$

and $g(T,T) = \epsilon_1 = \pm 1$, $g(N,N) = \epsilon_2 = \pm 1$.

2.2 Unit tangent bundle

Let (M,g) be an $n-$dimensional pseudo-Riemannian manifold and let TM its tangent bundle with projection π. For a local coordinate neighborhood in (U,x^i) in M, there is a local coordinate neighborhood $(\pi^{-1}(U)$, $x^i,u^i)$ in TM. For a vector field X on M, its horizontal lift X^h is defined by $X^h = X^i\frac{\partial}{\partial x^i} - X^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}$, where Γ_{ij}^k denotes the Christoffel symbols of the Levi-Civita connection ∇ of g. The vertical lift X^v of X is given by $X^v = X^i\frac{\partial}{\partial u^i}$.

The canonical energy density function of the tangent vector u with respect to g is denoted by $t = \frac{1}{2}g(u,u)$.

An almost complex structure J on TM is defined by

$$\begin{aligned} JX^h &= a_1X^v + b_1 g(X,u)u^v, \\ JX^v &= -a_2X^h - b_2 g(X,u)u^h, \end{aligned} \tag{3}$$

where a_1, b_1, a_2, b_2 are smooth functions of t such that $a_2 = \frac{1}{a_1}, b_2 = -\frac{b_1}{a_1(a_1+2tb_1)}$ [6].

The natural diagonal lift metric g^d is defined by

$$\left\{\begin{array}{l} g^d(X^h,Y^h) = c_1 g(X,Y) + d_1 g(X,u)g(Y,u), \\ g^d(X^v,Y^h) = g^d(X^h,Y^v) = 0, \\ g^d(X^v,Y^v) = c_2 g(X,Y) + d_2 g(X,u)g(Y,u), \end{array}\right. \tag{4}$$

for every vector fields X,Y on M and every tangent vector u, where $t = g(u,u)/2$ and c_1, c_2, d_1, d_2 are smooth functions of t which satisfies $c_1 > 0, c_2 > 0, c_1 + 2td_1 > 0$ and $c_2 + 2td_2 > 0$.

The unit tangent bundle T_1M of M is a submanifold of TM defined by $T_1M = \{u \in TM : g(u,u) = \pm 1\}$, where u^v is a normal to T_1M. For a vector field X on M, its tangential lift is defined by $X^t = X^v - g(X,u)u^v$.

Hence, for a vector field $\tilde{X}$ on T_1M, we can write it uniquely as $\tilde{X} = X^h + X^t$.

The induced pseudo-Riemannian metric g_1^d on T_1M is uniquely determined by

$$\begin{cases} g_1^d(X^h, Y^h) = c_1 g(X,Y) + d_1 g(X,u) g(Y,u), \\ g_1^d(X^v, Y^h) = g_1^d(X^h, Y^v) = 0, \\ g_1^d(X^v, Y^v) = c_2[g(X,Y) - g(X,u) g(Y,u)], \end{cases} \tag{5}$$

for every vector fields X, Y on M and every tangent vector u, where c_1, d_1, c_2 are constants such that $c_1 > 0$, $c_2 > 0$, $c_1 + d_1 > 0$ [7].

A contact metric structure $(\varphi_1, \xi_1, \eta_1, g_1)$ on T_1M is given by the following relations:

$$\begin{aligned} \varphi_1(X^h) &= a_1 X^t,\ \varphi_1(X^t) = -a_2 X^h + a_2 g(X,u) u^h, \qquad (6) \\ \xi_1 &= \frac{1}{2\lambda\alpha} u^h,\ \eta_1(X^t) = 0,\ \eta_1(X^h) = 2\alpha\lambda g(X,u),\ g_1 = \alpha g_1^d, \end{aligned}$$

for every vector fields X, Y on M and every tangent vector u, where $\lambda > 0$ is a scalar, $\alpha = \frac{c_1+d_1}{4\lambda^2}$ and a_1, a_2 are functions defined in (3). This contact metric structure is said to be natural diagonal structure [6].

The Levi-Civita connection ∇_1 of (T_1M, g_1) is given by

$$\begin{aligned} \nabla_{1\ X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2}(R(X,Y)u)^t - \frac{d_1}{2c_2}[g(X,u)Y^t - g(Y,u)X^t], \\ \nabla_{1\ X^h} Y^t &= (\nabla_X Y)^t - \frac{c_2}{2c_1}(R(Y,u)X)^h + \frac{d_1}{2c_1} g(X,u) Y^h \\ &\quad + \frac{d_1}{2(c_1+d_1)} g(X,Y) u^h - \frac{d_1(2c_1+d_1)}{2c_1(c_1+d_1)} g(X,u) g(Y,u) u^h \\ &\quad - \frac{c_2 d_1}{2c_1(c_1+d_1)} g(Y, R(X,u)u) u^h, \\ \nabla_{1\ X^t} Y^h &= -\frac{c_2}{2c_1}(R(X,u)Y)^h + \frac{d_1}{2c_1} g(Y,u) X^h \\ &\quad + \frac{d_1}{2(c_1+d_1)} g(X,Y) u^h - \frac{d_1(2c_1+d_1)}{2c_1(c_1+d_1)} g(X,u) g(Y,u) u^h \\ &\quad - \frac{c_2 d_1}{2c_1(c_1+d_1)} g(X, R(Y,u)u) u^h, \\ \nabla_{1\ X^t} Y^t &= -g(Y,u) X^t, \qquad (7) \end{aligned}$$

for every vector fields X, Y on M and every tangent vector u, where R is the curvature tensor on M [6].

3 N-Legendre and N-slant curves

Let (M, g) be a surface and let $\gamma : I \subset \mathbb{R} \to M$ be a curve on M. Suppose that $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a curve on $(T_1M, g_1, \varphi_1, \xi_1, \eta_1)$, where the contact metric structure is given by (6). We define the Legendre and slant curves as follows:

Definition 3.1. [10] Let γ be a curve in an almost contact metric manifold $(M, g, \varphi, \xi, \eta)$. The curve γ is said to be Legendre (resp. slant) if the angle between its tangent vector field T of γ and the Reeb vector field ξ is $\frac{\pi}{2}$ (resp. $[0, \pi] - \{\pi/2\}$), i.e. $g(T, \xi) = 0$ (resp. $g(T, \xi) = c$), where c is a non-zero constant.

Definition 3.2. [9] Let γ be a curve in an almost contact metric manifold $(M, g, \varphi, \xi, \eta)$. The curve γ is said to be N-Legendre (resp. N-slant) if the angle between its normal vector field N of γ and the Reeb vector field ξ is $\pi/2$ (resp. $[0, \pi] - \{\pi/2\}$), i.e. $g(N, \xi) = 0$ (resp. $g(N, \xi) = c$), where c is a non-zero constant.

Let $\tilde{\gamma}$ be an arc-parametrized spacelike or timelike (non-null) curve in the unit tangent bundle $(T_1M_1^2, g_1, \varphi_1, \xi_1, \eta_1)$ and let $(\hat{T}, \tilde{N}, \tilde{B}, \tilde{\kappa}, \tilde{\tau})$ denotes the Frenet apparatus of $\tilde{\gamma}$. In this case,

$$
\begin{aligned}
\tilde{T}(s) &= \frac{d\gamma^i}{ds}\frac{\partial}{\partial x^i} + \frac{dX^i}{ds}\frac{\partial}{\partial u^i} \\
&= \frac{d\gamma^i}{ds}(\frac{\partial}{\partial x^i})^h(\tilde{\gamma}(s)) + (\frac{dX^i}{ds} + \frac{d\gamma^j}{ds}X^k\Gamma^i_{jk})\frac{\partial}{\partial u^i}(\tilde{\gamma}(s)) \\
&= (E^h + (\nabla_E X)^t)(\tilde{\gamma}(s)),
\end{aligned} \tag{8}
$$

where $E = \gamma'(s)$.

From equations (5) and (6), the Lorentzian angle between $\tilde{T}$ and $\xi_1 = \frac{1}{2\lambda\alpha}u^h$ is obtained by

$$
\frac{g_1(\tilde{T}, \xi_1)}{\left|\tilde{T}\right| |\xi_1|} = \sqrt{c_1 + d_1}g(E, X) = L(\theta), \tag{9}
$$

where $L(\theta)$ is

(i) $\cos\theta,$ if $\tilde{T}$ and ξ_1 are spacelike vectors that span a spacelike subspace,
(ii) $\cosh\theta,$ if $\tilde{T}$ and ξ_1 are spacelike vectors that span a timelike subspace,
(iii) $\sinh\theta,$ if $\tilde{T}$ and ξ_1 have different causal characters. (10)

Differentiating both side of equation (9) with respect to s and using equations (5), (6) and (8), we get

$$\begin{aligned}
\frac{d}{ds}g_1(\tilde{T},\xi_1) &= g_1(\nabla_{1\tilde{T}}\tilde{T},\xi_1)+g_1(\tilde{T},\nabla_{1\tilde{T}}\xi_1)\\
&= \tilde{\kappa}g_1(\tilde{N},\xi_1)+\frac{1}{2\lambda\alpha}g_1(\tilde{T},\nabla_{1E^h}X^h+\nabla_{1(\nabla_E X)^t}X^h)\\
&= \tilde{\kappa}g_1(\tilde{N},\xi_1)+\frac{1}{2\lambda\alpha^2}[(c_1+d_1)g(E,\nabla_E X)\\
&\quad -c_2R(E,X,X,\nabla_E X)]\\
&= |\xi_1|\,\theta' L'(\theta).
\end{aligned}$$

Therefore,

$$\begin{aligned}
g_1(\tilde{N},\xi_1) &= \frac{1}{2\lambda\alpha^2\tilde{\kappa}}\left(c_2R(E,X,X,\nabla_E X)-(c_1+d_1)g(E,\nabla_E X)\right)\\
&\quad +|\xi_1|\,\theta' L'(\theta),
\end{aligned} \tag{11}$$

where $\theta'=\frac{d\theta}{ds}$ and R is the curvature tensor of M_1^2.

If (T,N) is a Frenet frame on γ given in (2), then from equation (9), we have

$$X=\frac{1}{r\sqrt{c_1+d_1}}L(\theta)T+\beta N, \tag{12}$$

for a smooth function β, where $r=\|E\|$. Because of X is a unit vector, we write

$$\frac{\lambda^2\epsilon_1}{(c_1+d_1)r^2}L^2(\theta)+\epsilon_2\beta^2=\epsilon_X,$$

from which

$$\beta=\pm\frac{1}{r}\sqrt{\epsilon_X\epsilon_2 r^2-(\frac{\lambda}{\sqrt{c_1+d_1}})^2\epsilon_1\epsilon_2L^2(\theta)}, \tag{13}$$

where $\epsilon_X = g(X,X) = \pm 1$.

Differentiating equation (12) with respect to s, we occur

$$\begin{aligned}\nabla_E X &= \frac{1}{\sqrt{c_1+d_1}}(\frac{L(\theta)}{r})'T + \frac{\kappa L(\theta)}{\sqrt{c_1+d_1}}N + \beta' N + \epsilon_2 r\beta\kappa T \quad (14)\\ &= ((\frac{L(\theta)}{r\sqrt{c_1+d_1}})' + \epsilon_2 r\beta\kappa)T + (\frac{\kappa L(\theta)}{\sqrt{c_1+d_1}} + \beta')N.\end{aligned}$$

Since $g(X, \nabla_E X) = 0$, equations (9) and (14) lead to

$$E = \frac{L(\theta)}{\sqrt{c_1+d_1}}X + \frac{r}{g(\nabla_E X, \nabla_E X)}\left((\frac{L(\theta)}{r\sqrt{c_1+d_1}})' + \epsilon_2 r\beta\kappa\right)\nabla_E X. \quad (15)$$

So,

$$\begin{aligned}R(E,X,X,\nabla_E X) &= r\left((\frac{L(\theta)}{r\sqrt{c_1+d_1}})' + \epsilon_2 r\beta\kappa\right)\frac{R(\nabla_E X, X, X, \nabla_E X)}{g(\nabla_E X, \nabla_E X)}\\ &= r\left((\frac{L(\theta)}{r\sqrt{c_1+d_1}})' + \epsilon_2 r\beta\kappa\right)\sigma(s), \quad (16)\end{aligned}$$

where $\sigma(s)$ is the sectional curvature of M_1^2. Setting equations (13)-(16) in (11), we express the main equation

$$\begin{aligned}g_1(\tilde{N}, \xi_1) &= \frac{r(c_2\sigma(s) - (c_1+d_1))}{2\lambda\alpha^2\tilde{\kappa}}((\frac{L(\theta)}{r\sqrt{c_1+d_1}})'\\ &\quad \pm r\epsilon_2\kappa\sqrt{\epsilon_X\epsilon_2 r^2 - (\frac{\lambda}{\sqrt{c_1+d_1}})^2\epsilon_1\epsilon_2 L^2(\theta)})\\ &\quad + |\xi_1|\frac{\theta' L'(\theta)}{\tilde{\kappa}}. \quad (17)\end{aligned}$$

Now we can state the following propositions.

Proposition 3.3. *Let $T_1S_1^2$ be the unit tangent bundle of the de Sitter space S_1^2 with the natural diagonal metric structure given by (6) such that $c_2 = c_1 + d_1$. Then all Legendre and slant non-geodesic curves are $\tilde{N}-$Legendre curves.*

Proof. Let $\tilde{\gamma}(s) = (\gamma(s), X(s))$ be a Legendre or a slant curve with arc-parameter on the contact metric manifold $T_1S_1^2$ such that $c_2 = c_1 + d_1$. We know that the sectional curvature $\sigma(s)$ of the de Sitter space equals 1. Thus, from Definition 3.1 and equation (17), we see that

$$g_1(\tilde{N}, \xi_1) = 0.$$

This completes the proof. $\square$

Proposition 3.4. *Let $T_1S_1^2$ be the unit tangent bundle of the de Sitter space S_1^2 with the natural diagonal metric structure given by (6) such that $c_2 = c_1 + d_1$ and let $\tilde{\gamma}$ be a non-slant curve on $T_1S_1^2$. Then $\tilde{\gamma}$ is an $\tilde{N}-$slant curve if the angle θ satisfies*

(i) $\theta = \arccos c \int \tilde{\kappa}$, if $\tilde{T}$ and ξ_1 are spacelike vectors in the same space-conic,

(ii) $\theta = \arg\cosh c \int \tilde{\kappa}$, if $\tilde{T}$ and ξ_1 are spacelike or timelike vectors in the same time-conic,

(iii) $\theta = \arg\sinh c \int \tilde{\kappa}$, if $\tilde{T}$ is spacelike and ξ_1 is timelike vectors, where c is a non-zero constant.

Proof. Let $\tilde{\gamma}(s) = (\gamma(s), X(s))$ be a non-slant curve with arc-parameter on the contact metric manifold $T_1S_1^2$ such that $c_2 = c_1 + d_1$. Since the sectional curvature $\sigma(s)$ of the de Sitter space is 1, Definition 3.1 and equation (17) demonstrate that

$$g_1(\tilde{N}, \xi_1) = \frac{L(\theta)'}{\tilde{\kappa}} = c \;\; \textit{constant}.$$

If $\tilde{T}$ and ξ_1 are spacelike vectors in the same space-conic, then $L(\theta) = \cos\theta$, hence

$$g_1(\tilde{N}, \xi_1) = (\cos\theta)' = c\tilde{\kappa}.$$

This differential equation yields

$$\theta = \arccos c \int \tilde{\kappa}.$$

Thus, we prove the statement (i). The proof of the statements (ii) and (iii) is similar. $\square$

Proposition 3.5. *Let M_1^2 be not de Sitter space and $T_1M_1^2$ be the unit tangent bundle of M_1^2 with the natural diagonal metric structure given by (6) such that $c_2 = c_1 + d_1$. Assume that $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a slant curve on $T_1M_1^2$ and γ is a curve with constant velocity r_0. Then the curve $\tilde{\gamma}$ is $\tilde{N}-$slant if and only if*

$$\frac{(\sigma - 1)\kappa}{\widetilde{\kappa}}$$

is a non-zero constant.

Proof. Let M_1^2 be a not de Sitter space (i.e. $\sigma(s) \neq 1$). Suppose that the curve $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a slant curve in $(T_1M_1^2, g_1, \varphi_1, \xi_1, \eta_1)$ such that $c_2 = c_1 + d_1$, where γ has constant velocity of r_0. Then from (17), we get

$$\begin{aligned} g_1(\tilde{N}, \xi_1) &= \frac{r_0 c_2(\sigma(s) - 1)}{2\lambda\alpha^2\tilde{\kappa}} \left(\pm\epsilon_2\kappa\sqrt{\epsilon_X\epsilon_2^2 r_0^2 - (\frac{\lambda}{\sqrt{c_1 + d_1}})^2\epsilon_1\epsilon_2 L^2(\theta)} \right) \\ &= \pm\frac{r_0 c_2}{2\lambda\alpha^2} \left(\epsilon_2\sqrt{\epsilon_X\epsilon_2 \left(\frac{r_0}{2}\right)^2 - (\frac{\lambda}{c_1 + d_1})^2\epsilon_1\epsilon_2 L^2(\theta)} \right) \\ &\quad \times \frac{(\sigma(s) - 1)\kappa}{\widetilde{\kappa}} \\ &= \bar{c}\frac{(\sigma(s) - 1)\kappa}{\tilde{\kappa}}, \end{aligned}$$

where $\bar{c}$ is a non-zero constant. Thus, the proof is complete from Definition 3.2. $\square$

Example 3.6. Let $\tilde{\gamma}$ be an arbitrary slant curve on $H_1^2(\mathbf{r})$ or $S_1^2(\mathbf{r})$ ($\mathbf{r} \neq 1$). Then $\tilde{\gamma}$ is an $\tilde{N}-$slant curve if and only if $\frac{\kappa}{\tilde{\kappa}}$ is a non-zero constant.

Proposition 3.7. *Let M_1^2 be a timelike (resp. spacelike) surface and $T_1M_1^2$ be the unit tangent bundle of M_1^2 with the natural diagonal metric structure given by (6) such that $c_2 = c_1 + d_1$. Suppose that $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a curve on $T_1M_1^2$ and γ is a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$. If X has a different causal character with γ (resp. is a spacelike vector) and $\tilde{T}$, ξ_1 fulfill the relation (10 (iii)) (resp. 10 (i)), then*

(1) $\tilde{\gamma}(s)$ is an $\tilde{N}-$Legendre curve if and only if

$$\theta = \int \kappa\bar{\sigma}(s)ds,$$

(2) $\tilde{\gamma}(s)$ is a $\tilde{N}-$slant curve if and only if

$$L(\theta)' \pm \kappa\overline{\sigma}(s)L'(\theta) = c\widetilde{\kappa},$$

where $\bar{\sigma}(s) = \frac{\kappa\lambda^2 c_2(\sigma(s)-1)}{c_2(\sigma(s)-1)+2\alpha^2\lambda|\xi_1|\sqrt{c_1+d_1}}$ *and* $L(\theta)$ *equals* $\sinh\theta$ *(resp.* $\cos\theta$*).*

Proof. (1) Suppose that M_1^2 is a timelike (resp. spacelike) surface. Let $\tilde{\gamma}(s) = (\gamma(s), X(s))$ be a curve on $T_1M_1^2$, γ be a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$ and X has a different causal character with γ (resp. spacelike vector). We have

$$\epsilon_X = -\epsilon_1 = \epsilon_2 \text{ (resp. } \epsilon_X = \epsilon_1 = \epsilon_2).$$

From equation (17), we express

$$g_1(\tilde{N}, \xi_1) = \frac{c_2(\sigma(s)-1)}{2\alpha^2\tilde{\kappa}\sqrt{c_1+d_1}}\left(\frac{\theta' L'(\theta)}{\lambda} \pm \frac{\kappa\lambda}{c_1+d_1}\sqrt{1+L^2(\theta)}\right) + |\xi_1|\,\frac{\theta' L'(\theta)}{\tilde{\kappa}},$$

$$\begin{aligned}\text{(resp.) } g_1(\tilde{N}, \xi_1) &= \frac{c_2(\sigma(s)-1)}{2\alpha^2\tilde{\kappa}\sqrt{c_1+d_1}}\left(\frac{\theta' L'(\theta)}{\lambda} \pm \frac{\kappa\lambda}{c_1+d_1}\sqrt{1-L^2(\theta)}\right) \\ &\quad + |\xi_1|\,\frac{\theta' L'(\theta)}{\tilde{\kappa}}.\end{aligned}$$

When the vectors $\tilde{T}$ and ξ_1 fulfill the relation (10 (iii)) (resp. 10 (i)), the function $L(\theta)$ equals $\sinh\theta$ (resp. $\cos\theta$). Therefore, $\tilde{\gamma}(s)$ is an $\tilde{N}-$Legendre curve if and only if

$$g_1(\tilde{N}, \xi_1) = \frac{c_2(\sigma(s)-1)}{2\alpha^2\tilde{\kappa}\sqrt{c_1+d_1}}\left(\frac{\theta'}{\lambda} \pm \frac{\kappa\lambda}{c_1+d_1}\right) + |\xi_1|\,\frac{\theta'}{\tilde{\kappa}} = 0$$

and

$$\theta = \int \frac{\kappa\lambda^2 c_2(\sigma(s)-1)}{c_2(\sigma(s)-1) + 2\alpha^2\lambda\,|\xi_1|\,\sqrt{c_1+d_1}}ds.$$

Similarly, $\tilde{\gamma}(s)$ is an $\tilde{N}-$slant curve (i.e. $g(N,\xi_1)=c$) if and only if

$$L(\theta)' \pm \kappa \frac{\kappa\lambda^2 c_2(\sigma(s)-1)}{c_2(\sigma(s)-1)+2\alpha^2\lambda\left|\xi_1\right|\sqrt{c_1+d_1}} L'(\theta) = c\tilde{\kappa}.$$

Thus the proof ends. $\square$

Now, we will prove some propositions under the assumption the angle θ is linear (i.e. $\theta = es+f$). Furthermore, we suppose that the vector fields ξ_1 and X have same causal characters.

Proposition 3.8. *Let M_1^2 be a spacelike surface ($\sigma(s)\neq 1$) and $T_1M_1^2$ be the unit tangent bundle of M_1^2 with the natural diagonal metric structure given by (6) such that $c_2 = c_1+d_1$. Suppose that $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a non-slant curve on $T_1M_1^2$ and γ is a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$. Let X be a spacelike vector and the vectors $\tilde{T}$ and ξ_1 spacelike vectors which span a spacelike vector subspace. In this case, the curve $\tilde{\gamma}(s)$ is $\tilde{N}-$Legendre if and only if*

$$(\sigma(s)-1)\left(-\frac{e}{\lambda} \pm \frac{\lambda}{c_1+d_1}\kappa\right) = \frac{2e\left|\xi_1\right|\alpha^2\sqrt{c_1+d_1}}{c_2}.$$

Proof. Let M_1^2 be a spacelike surface, $\tilde{\gamma}(s) = (\gamma(s), X(s))$ be a non-slant curve on $T_1M_1^2$, γ be a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$ and X be a spacelike vector ($\epsilon_X = 1$). Then, from equation (17), we have

$$\begin{aligned}&\frac{c_2(\sigma(s)-1)}{2\alpha^2\tilde{\kappa}\sqrt{c_1+d_1}}(\frac{eL'(\theta)}{\lambda}\\ &\pm\epsilon_2\kappa\frac{\lambda}{c_1+d_1}\sqrt{\epsilon_X\epsilon_2-\epsilon_1\epsilon_2L^2(\theta)})\\ &\quad+\frac{e\left|\xi_1\right|L'(\theta)}{\tilde{\kappa}}=0\end{aligned}$$

and

$$\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(\frac{eL'(\theta)}{\lambda} \pm \kappa\frac{\lambda}{c_1+d_1}\sqrt{1-L^2(\theta)}\right) + e\left|\xi_1\right|L'(\theta) = 0.$$

Since the vectors $\tilde{T}$ and ξ_1 span a spacelike subspace, we have

$$L(\theta) = \cos\theta,\ L'(\theta) = -\sin\theta.$$

So,

$$\begin{aligned}
\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(\frac{eL'(\theta)}{\lambda}\pm\frac{\lambda\kappa}{c_1+d_1}\sqrt{1-L^2(\theta)}\right)+e\left|\xi_1\right|L'(\theta) &= 0\\
\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(-\frac{e\sin\theta}{\lambda}\pm\frac{\lambda}{c_1+d_1}\kappa\sin\theta\right)-e\left|\xi_1\right|\sin\theta &= 0\\
(\sigma(s)-1)\left(-\frac{e}{\lambda}\pm\frac{\lambda}{c_1+d_1}\kappa\right) = \frac{2e\left|\xi_1\right|\alpha^2\sqrt{c_1+d_1}}{c_2}.
\end{aligned}$$

Therefore, we prove the proposition. $\square$

Proposition 3.9. *Let M_1^2 be a timelike surface $(\sigma(s)\neq 1)$ and $T_1M_1^2$ be the unit tangent bundle of M_1^2 with the natural diagonal metric structure given by (6) such that $c_2=c_1+d_1$. Suppose that $\tilde{\gamma}(s)=(\gamma(s),X(s))$ is a non-slant curve on $T_1M_1^2$ and γ is a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$. Let X has a different causal character from γ, and $\tilde{T}$, ξ_1 fulfill (10 (iii)). In this case, the curve $\tilde{\gamma}(s)$ is $\tilde{N}-$Legendre if and only if*

$$(\sigma(s)-1)\left(\frac{e}{\lambda}\pm\frac{2\lambda}{c_1+d_1}\kappa\right)=-\frac{2e\alpha^2\sqrt{c_1+d_1}}{c_2}.$$

Proof. Let M_1^2 be a timelike surface, $\tilde{\gamma}(s)=(\gamma(s),X(s))$ be a non-slant curve on $T_1M_1^2$, γ be a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$ and X has a different causal character from γ. We have $\epsilon_X=-\epsilon_1=\epsilon_2$. Then, equation (17) gives

$$\frac{c_2(\sigma(s)-1)}{2\alpha^2\tilde{\kappa}\sqrt{c_1+d_1}}\left(\frac{eL'(\theta)}{\lambda}\pm\epsilon_2\kappa\frac{\lambda}{c_1+d_1}\sqrt{\epsilon_X\epsilon_2-\epsilon_1\epsilon_2L^2(\theta)}\right)+\frac{e\left|\xi_1\right|L'(\theta)}{\tilde{\kappa}}=0.$$

If the vectors $\tilde{T}$ and ξ_1 have different causal characters, we write

$$L(\theta)=\sinh\theta,\ L'(\theta)=\cosh\theta.$$

Therefore, we obtain

$$\begin{aligned}
\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(\frac{eL'(\theta)}{\lambda}\pm\frac{\lambda\kappa}{c_1+d_1}\sqrt{1+L^2(\theta)}\right)+e\left|\xi_1\right|L'(\theta) &= 0\\
\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(\frac{e\cosh\theta}{\lambda}\pm\frac{\lambda}{c_1+d_1}\kappa\cosh\theta\right)+e\left|\xi_1\right|\cosh\theta &= 0
\end{aligned}$$

$$(\sigma(s)-1)\left(\frac{e}{\lambda}\pm\frac{2\lambda}{c_1+d_1}\kappa\right)=-\frac{2e\left|\xi_1\right|\alpha^2\sqrt{c_1+d_1}}{c_2}.$$

So, the proposition is proved. $\square$

Example 3.10. Consider M_1^2 as the anti de Sitter space H_1^2. Since the sectional curvature of H_1^2 is -1, under the assumptions of Proposition 3.9, $\tilde{N}-$Legendre condition of the curve $\tilde{\gamma}$ is that its projection curve γ has a constant curvature $\kappa=\pm\frac{e(c_1+d_1)}{2\lambda}(\frac{|\xi_1|\alpha^2\sqrt{c_1+d_1}}{c_2}-\frac{1}{\lambda})$.

Proposition 3.11. *Let M_1^2 be a spacelike surface $(\sigma(s)\neq 1)$ and $T_1M_1^2$ be the unit tangent bundle of M_1^2 with the natural diagonal metric structure given by (6) such that $c_2=c_1+d_1$. Suppose that $\tilde{\gamma}(s)=(\gamma(s),X(s))$ is a non-slant curve on $T_1M_1^2$ and γ is a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$. If X is a spacelike vector, then the curve $\tilde{\gamma}(s)$ is $\tilde{N}-$slant if and only if*

$$\theta=\arcsin\frac{c\tilde{\kappa}}{-e(\frac{c_2(\sigma(s)-1)}{2\lambda\alpha^2\sqrt{c_1+d_1}}+|\xi_1|)\pm\frac{\lambda\kappa c_2(\sigma(s)-1)}{2\alpha^2(c_1+d_1)^{3/2}}},\ c \text{ non-zero constant,}$$

when $\tilde{T}$ and ξ_1 stay in the same spacelike subspace.

Proof. Under the assumptions M_1^2 is a spacelike surface, $\tilde{\gamma}(s)=(\gamma(s),X(s))$ is a non-slant curve on $T_1M_1^2$, γ is a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$ and X is a spacelike vector, using the relations $g(N,\xi_1)=c$; $\epsilon_X=\epsilon_1=\epsilon_2=1$, we get the following equation from (17)

$$\frac{c_2(\sigma(s)-1)}{2\alpha^2\tilde{\kappa}\sqrt{c_1+d_1}}\left(\frac{eL'(\theta)}{\lambda}\pm\epsilon_2\kappa\frac{\lambda}{c_1+d_1}\sqrt{\epsilon_X\epsilon_2-\epsilon_1\epsilon_2L^2(\theta)}\right)+\frac{e\left|\xi_1\right|L'(\theta)}{\tilde{\kappa}}=c.$$

and

$$\frac{c_2(\sigma(s)-1)}{2\alpha^2\tilde{\kappa}\sqrt{c_1+d_1}}\left(\frac{eL'(\theta)}{\lambda}\pm\kappa\frac{\lambda}{c_1+d_1}\sqrt{1-L^2(\theta)}\right)+\frac{e\left|\xi_1\right|L'(\theta)}{\tilde{\kappa}}=c.$$

If $\tilde{T}$ and ξ_1 stay in the same spacelike subspace, then

$$L(\theta)=\cos\theta,\ L'(\theta)=-\sin\theta,$$

and

$$\begin{aligned}
\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(-\frac{e\sin\theta}{\lambda}\pm\frac{\lambda}{c_1+d_1}\kappa\sin\theta\right)-e\,|\xi_1|\sin\theta &= c\tilde{\kappa}\\
\sin\theta\left(-e(\frac{c_2(\sigma(s)-1)}{2\lambda\alpha^2\sqrt{c_1+d_1}}+|\xi_1|)\pm\frac{\lambda\kappa}{c_1+d_1}\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\right) &= c\tilde{\kappa}\\
\arcsin\frac{c\tilde{\kappa}}{-e(\frac{c_2(\sigma(s)-1)}{2\lambda\alpha^2\sqrt{c_1+d_1}}+|\xi_1|)\pm\frac{\lambda\kappa}{c_1+d_1}\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}} &= \theta.
\end{aligned}$$

Thus, we prove the proposition. □

Proposition 3.12. *Let M_1^2 be a timelike surface ($\sigma(s)\neq 1$) and $T_1M_1^2$ be the unit tangent bundle of M_1^2 with the natural diagonal metric structure given by (6) such that $c_2=c_1+d_1$. Suppose that $\tilde{\gamma}(s)=(\gamma(s),X(s))$ is a non-slant curve on $T_1M_1^2$ and γ is a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$. Let X has a different causal character from γ. In this case, the curve $\tilde{\gamma}(s)$ is $\tilde{N}-$slant if and only if*

$$\theta=\arg\cosh\frac{c\tilde{\kappa}}{e(\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}+|\xi_1|)\pm\frac{\lambda\kappa c_2(\sigma(s)-1)}{2\alpha^2(c_1+d_1)^{3/2}}},\ c\text{ non-zero constant,}$$

when $\tilde{T}$ and ξ_1 fulfill the relation (10 (iii)).

Proof. Under the assumptions M_1^2 is a timelike surface, $\tilde{\gamma}(s)=(\gamma(s),X(s))$ is a non-slant curve on $T_1M_1^2$, γ is a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$ and X has a different causal character from γ and $\tilde{T}$, using the relations $g(N,\xi_1)=c$ and $\epsilon_X=-\epsilon_1=\epsilon_2$, we get the following equation from (17)

$$\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(\frac{eL'(\theta)}{\lambda}\pm\epsilon_2\kappa\frac{\lambda}{c_1+d_1}\sqrt{\epsilon_X\epsilon_2-\epsilon_1\epsilon_2L^2(\theta)}\right)+\frac{e\,|\xi_1|\,L'(\theta)}{\tilde{\kappa}}=c.$$

and

$$\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(\frac{eL'(\theta)}{\lambda}\pm\kappa\frac{\lambda}{c_1+d_1}\sqrt{1+L^2(\theta)}\right)+\frac{e\,|\xi_1|\,L'(\theta)}{\tilde{\kappa}}=c.$$

If $\tilde{T}$ and ξ_1 satisfy the relation (10 (iii)), we have

$$L(\theta)=\sinh\theta,\ L'(\theta)=\cosh\theta,$$

and so,

$$\begin{aligned}
\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\left(\frac{e\cosh\theta}{\lambda}\pm\frac{\lambda}{c_1+d_1}\kappa\cosh\theta\right)+e\left|\xi_1\right|\cosh\theta &= c\tilde{\kappa} \\
\cosh\theta\left(e(\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}+|\xi_1|)\pm\frac{\lambda\kappa}{c_1+d_1}\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}\right) &= c\tilde{\kappa} \\
\arg\cosh\frac{c\tilde{\kappa}}{e(\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}+|\xi_1|)\pm\frac{\lambda\kappa}{c_1+d_1}\frac{c_2(\sigma(s)-1)}{2\alpha^2\sqrt{c_1+d_1}}} &= \theta.
\end{aligned}$$

Thus, we prove the last proposition of the paper. □

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