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Original Research Paper

New Type Caputo Fractional Inequalities

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Abstract. The aim of inequalities is to develop tools for analyzing the problems in pure and applied mathematics. Our primary objective in this research is to introduce some new type inequalities connected with Caputo fractional derivative. In the light of the operator, we extend and generalize some important inequalities to this fractional calculus dealing with synchronous functions.

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1 Introduction

The theory of inequalities plays an important role in mathematical analysis. Because of its importance, many authors has studied to extend and generalize the inequalities. Since L'Hospital and Leibniz asked about the notion of the derivative of order $n = 1/2$, fractional calculus was developed in the seventeenth century. Fractional calculus theory is not only the subject of mathematics but also physics, engineering, economics and others. Over the last decade, fractional inequalities have gained considerable importance and popularity. The most powerful and effective ways

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to generalize the classical inequalities are the fractional inequalities. The results obtained by using fractional operators are more general than the classical results in the literature. For instance, Khan et al. [15] considered a class of n decreasing positive functions and used Saigo fractional integral operator to produce some inequalities. For a detailed information about fractional calculus can be found in [10, 12, 16, 19].

In the literature, we use the fractional inequalities in scientific problems. In [7], the authors investigated a new version for the mathematical model of HIV. Baleanu et al. [5] studied the existence of solutions for a fractional hybrid integro-differential equation. Aydogan et al. [3] introduced new higher order derivatives and studied the existence of solutions for two such type higher order fractional equations. For a review of the applications of this topic, we direct the reader to [1, 2, 4, 6, 8, 9, 11, 13, 14, 17, 18, 20, 22].

In [21], Ucar and Hatipoglu obtained some new inequalities using beta- fractional operator for synchronous functions. In this paper, we motivate by the two differentiable functions u and v which are synchronous on $[a, b]$. $(i.e.\ (u(x) - u(y))(v(x) - v(y)) \geq 0\ for\ any\ x, y)$ We use more useful fractional Caputo operator which is a modified concept of the Riemann Liouville fractional derivative to obtain inequalities. We start with the following definition.

Definition 1.1. *Assume that $\beta > 0,\ t > a,\ \beta, a, t \in \mathbb{R}$. Then the Caputo fractional differential operator of order β is defined as*

$$ {}^{C}D^{\beta} f(t) := \begin{cases} \frac{1}{\Gamma(n-\beta)} \int\limits_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, & n-1 < \beta < n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \beta = n \in \mathbb{N} \end{cases} .$$

We call this operator Caputo fractional derivative. Italian mathematician Caputo introduced this operator in 1967.

2 Main Results

In this section, we first prove the following useful inequality using Caputo derivative and extend this inequality for this interesting calculus.

Theorem 2.1. *Assume that the functions $f^{(n)}$ and $g^{(n)}$ are two synchronous functions on $[0,\infty)$. Then the following inequality holds*

$$\frac{{}^{C}D_{t}^{\alpha}\left(fg\left(t\right)\right)t^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)}\geq\ {}^{C}D_{t}^{\alpha}\left(f\left(t\right)\right)^{C}D_{t}^{\alpha}\left(g\left(t\right)\right) \tag{1}$$

for all $\alpha>0$, $t\geq 0$.

Proof. Since the functions $f^{(n)}$ and $g^{(n)}$ are two synchronous functions on $[0,\infty)$, then we can write

$$\left(f^{(n)}\left(\tau\right)-f^{(n)}\left(\rho\right)\right)\left(g^{(n)}\left(\tau\right)-g^{(n)}\left(\rho\right)\right)\ \geq\ 0$$

and

$$f^{(n)}\left(\tau\right)g^{(n)}\left(\tau\right)+f^{(n)}\left(\rho\right)g^{(n)}\left(\rho\right)\geq f^{(n)}\left(\tau\right)g^{n}\left(\rho\right)+f^{(n)}\left(\rho\right)g^{(n)}\left(\tau\right). \tag{2}$$

If we multiply both sides of the inequality (2) by $\frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)}$ and integrate with respect to τ from 0 to t, we have

$$\begin{aligned}
&\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{(n-\alpha-1)}f^{(n)}\left(\tau\right)g^{(n)}\left(\tau\right)d\tau\\
&\quad+\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{(n-\alpha-1)}f^{(n)}\left(\rho\right)g^{(n)}\left(\rho\right)d\tau\\
&\geq\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{(n-\alpha-1)}f^{(n)}\left(\tau\right)g^{(n)}\left(\rho\right)d\tau\\
&\quad+\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{(n-\alpha-1)}f^{(n)}\left(\rho\right)g^{(n)}\left(\tau\right)d\tau
\end{aligned}$$

$$\begin{aligned}
{}^{C}D_{t}^{\alpha}\left(fg\left(t\right)\right)+\frac{f^{(n)}\left(\rho\right)g^{(n)}\left(\rho\right)}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{(n-\alpha-1)}d\tau &\geq g^{(n)}\left(\rho\right)^{C}D_{t}^{\alpha}\left(f\left(t\right)\right)\\
&\quad+f^{(n)}\left(\rho\right)^{C}D_{t}^{\alpha}\left(g\left(t\right)\right).
\end{aligned}$$

Then we get

$$
\begin{aligned}
{}^{C}D_{t}^{\alpha}\left(fg\left(t\right)\right)+f^{(n)}\left(\rho\right)g^{(n)}\left(\rho\right)\frac{t^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)} &\geq g^{(n)}\left(\rho\right)^{C}D_{t}^{\alpha}\left(f\left(t\right)\right) \\
&+f^{(n)}\left(\rho\right)^{C}D_{t}^{\alpha}\left(g\left(t\right)\right).
\end{aligned} \tag{3}
$$

Multiplying both sides of the inequality (3) by $\frac{(t-\rho)^{n-\alpha-1}}{\Gamma(n-\alpha)}$ we have

$$
\begin{aligned}
&\frac{\left(t-\rho\right)^{n-\alpha-1}}{\Gamma\left(n-\alpha\right)}\,{}^{C}D_{t}^{\alpha}\left(fg\left(t\right)\right) \\
&+\frac{\left(t-\rho\right)^{n-\alpha-1}}{\Gamma\left(n-\alpha\right)}f^{(n)}\left(\rho\right)g^{(n)}\left(\rho\right)\frac{t^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)} \\
\geq&\frac{\left(t-\rho\right)^{n-\alpha-1}}{\Gamma\left(n-\alpha\right)}g^{(n)}\left(\rho\right)^{C}D_{t}^{\alpha}\left(f\left(t\right)\right)+\frac{\left(t-\rho\right)^{n-\alpha-1}}{\Gamma\left(n-\alpha\right)}f^{(n)}\left(\rho\right)^{C}D_{t}^{\alpha}\left(g\left(t\right)\right).
\end{aligned} \tag{4}
$$

If we integrate the inequality (4) with respect to ρ from 0 to t, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}{}^{C}D_{t}^{\alpha}\left(fg\left(t\right)\right)\left(t-\rho\right)^{(n-\alpha-1)}d\rho \\
&+\frac{\left(t-\rho\right)^{n-\alpha-1}}{\Gamma\left(n-\alpha\right)\Gamma\left(n-\alpha+1\right)}\int_{0}^{t}f^{(n)}\left(\rho\right)g^{(n)}\left(\rho\right)\left(t-\rho\right)^{(n-\alpha-1)}d\rho \\
&\geq\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}{}^{C}D_{t}^{\alpha}\left(f\left(t\right)\right)\;g^{(n)}\left(\rho\right)\left(t-\rho\right)^{(n-\alpha-1)}d\rho \\
&+\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}{}^{C}D_{t}^{\alpha}\left(g\left(t\right)\right)f^{(n)}\left(\rho\right)\left(t-\rho\right)^{(n-\alpha-1)}d\rho
\end{aligned}
$$

and then we can write

$$
\begin{aligned}
\frac{{}^{C}D_{t}^{\alpha}\left(fg\left(t\right)\right)\;t^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)}+\frac{{}^{C}D_{t}^{\alpha}\left(fg\left(t\right)\right)\;t^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)} &\geq{}^{C}D_{t}^{\alpha}\left(f\left(t\right)\right)^{C}D_{t}^{\alpha}\left(g\left(t\right)\right) \\
&+{}^{C}D_{t}^{\alpha}\left(g\left(t\right)\right)^{C}D_{t}^{\alpha}\left(f\left(t\right)\right) \\
{}^{C}D_{t}^{\alpha}\left(fg\left(t\right)\right)\frac{t^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)} &\geq\;{}^{C}D_{t}^{\alpha}\left(f\left(t\right)\right)^{C}D_{t}^{\alpha}\left(g\left(t\right)\right).
\end{aligned}
$$

So that we prove the inequality. □

Theorem 2.2. *Let the functions $f^{(n)}$ and $g^{(n)}$ be as in Theorem 1. Then for all $t \geq 0,\ \alpha > 0,\ \beta > 0$ we have*

$$\frac{t^{n-\beta}}{\Gamma(n-\beta+1)}\ {}^{C}D_t^{\alpha}\left(fg(t)\right) + \frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}\ {}^{C}D_t^{\beta}\left(fg(t)\right)$$
$$\geq\ {}^{C}D_t^{\alpha}\left(f(t)\right){}^{C}D_t^{\beta}\left(g(t)\right) + {}^{C}D_t^{\alpha}\left(g(t)\right){}^{C}D_t^{\beta}\left(f(t)\right). \tag{5}$$

Proof. If we multiply both sides of the inequality (3) by $\frac{(t-\rho)^{n-\beta-1}}{\Gamma(n-\beta)}$ we have

$$\frac{(t-\rho)^{n-\beta-1}}{\Gamma(n-\beta)}\ {}^{C}D_t^{\alpha}\left(fg(t)\right) + \frac{(t-\rho)^{n-\beta-1}}{\Gamma(n-\beta)} f^{(n)}(\rho)\, g^{(n)}(\rho)\, \frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}$$
$$\geq \frac{(t-\rho)^{n-\beta-1}}{\Gamma(n-\beta)}\ {}^{C}D_t^{\alpha}\left(f(t)\right) g^{(n)}(\rho) + \frac{(t-\rho)^{n-\beta-1}}{\Gamma(n-\beta)} f^{(n)}(\rho){}^{C}D_t^{\alpha}\left(g(t)\right)$$

and integrating this inequality with respect to ρ from 0 to t we obtain

$$\frac{{}^{C}D_t^{\alpha}\left(fg(t)\right)}{\Gamma(n-\beta)}\int_0^t (t-\rho)^{n-\beta-1}\, d\rho$$
$$+\frac{t^{n-\alpha}}{\Gamma(n-\beta)\,\Gamma(n-\alpha+1)}\int_0^t (t-\rho)^{n-\beta-1} f^{(n)}(\rho)\, g^{(n)}(\rho)\, d\rho$$
$$\geq \frac{{}^{C}D_t^{\alpha}\left(f(t)\right)}{\Gamma(n-\beta)}\int_0^t (t-\rho)^{n-\beta-1} g^{(n)}(\rho)\, d\rho$$
$$+\frac{{}^{C}D_t^{\alpha}\left(g(t)\right)}{\Gamma(n-\beta)}\int_0^t (t-\rho)^{n-\beta-1} f^{(n)}(\rho)\, d\rho.$$

Then we have

$$\frac{t^{n-\beta}}{\Gamma(n-\beta+1)}\ {}^{C}D_t^{\alpha}\left(fg(t)\right) + \frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}\ {}^{C}D_t^{\beta}\left(fg(t)\right)$$
$$\geq\ {}^{C}D_t^{\alpha}\left(f(t)\right){}^{C}D_t^{\beta}\left(g(t)\right) +\ {}^{C}D_t^{\alpha}\left(g(t)\right){}^{C}D_t^{\beta}\left(f(t)\right).$$

Hence the proof is completed. □

Theorem 2.3. *We assume that the functions* $(f_i)^{(n)}$ $(i = 1, 2, ..., m)$ *are* m *positive increasing functions on* $[0, \infty)$. *Then we have the following inequality*

$$^{C}D_t^{\alpha}\left(\prod_{i=1}^{m} f_i(t)\right) \geq \left(\frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\right)^{m-1} \prod_{i=1}^{m}\left({}^{C}D_t^{\alpha}(f_i(t))\right) \qquad (6)$$

holds for any $t \geq 0, \alpha > 0$.

Proof. Using induction method we can write

$$^{C}D_t^{\alpha}(f_1(t)) \geq {}^{C}D_t^{\alpha}(f_1(t))$$

for $m = 1$ and for all $t \geq 0, \alpha > 0$. We have from (1),

$$^{C}D_t^{\alpha}(f_1 f_2(t)) \geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\, {}^{C}D_t^{\alpha}(f_1(t))^{C}D_t^{\alpha}(f_2(t))$$

for $m = 2$ and for all $t \geq 0,\ \alpha > 0$. We assume that the inequality

$$^{C}D_t^{\alpha}\left(\prod_{i=1}^{m-1} f_i(t)\right) \geq \left(\frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\right)^{m-2}\left(\prod_{i=1}^{m-1}\left({}^{C}D_t^{\alpha}(f_i(t))\right)\right) \qquad (7)$$

holds for all $t \geq 0,\ \alpha > 0$. Since the functions $(f_i)_{i=1,2,...,m}$ are positive increasing functions then $\prod_{i=1}^{m-1} f_i$ is an increasing function. Applying Theorem (2.1) for $\prod_{i=1}^{m-1} f_i := g(t)$ and $f_m(t) = f(t)$, we write

$$\begin{aligned} {}^{C}D_t^{\alpha}\left(\left(\prod_{i=1}^{m-1} f_i(t)\right) f_m(t)\right) &= {}^{C}D_t^{\alpha}(gf(t)) \\ &\geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\, {}^{C}D_t^{\alpha}(g(t))^{C}D_t^{\alpha}(f(t)). \end{aligned}$$

Using the inequality (7) we prove the inequality (6).

$$^{C}D_t^{\alpha}\left(\left(\prod_{i=1}^{m-1} f_i(t)\right) f_m(t)\right) \geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\, {}^{C}D_t^{\alpha}\prod_{i=1}^{m-1}(f_i(t))\left(D_t^{\alpha}(f_m(t))\right)$$

$$^{C}D_t^{\alpha}\left(\prod_{i=1}^{m} f_i(t)\right) \geq \left(\frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\right)^{m-1} \prod_{i=1}^{m}\left({}^{C}D_t^{\alpha}(f_i(t))\right).$$

This completes the proof. □

Theorem 2.4. *We assume that the functions f and g are defined on $[0,\infty)$ and f is increasing, g is Caputo differentiable. Then the inequality*

$$\begin{aligned} {}^{C}D_t^{\alpha}(gf)(t) \geq{}& \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\Big[{}^{C}D_t^{\alpha}(g(t))^{C}D_t^{\alpha}(f(t)) \\ &-m\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\,{}^{C}D_t^{\alpha}(f(t))\Big]+m^{C}D_t^{\alpha}(tf(t)) \end{aligned} \tag{8}$$

holds for all $t\geq 0, \alpha>0$ where $m:=\inf_{t\geq 0}\,{}^{C}D_t^{\alpha}(g(t))$ is a real number.

Proof. We define the function $h(t):=g(t)-mt$. Since $g^{(n)}$ is Caputo differentiable, the function h is also Caputo differentiable and it is increasing on $[0,\infty)$. Using the inequality (1), we obtain

$$\begin{aligned} {}^{C}D_t^{\alpha}(h(t)f(t)) &\geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\,{}^{C}D_t^{\alpha}(h(t))^{C}D_t^{\alpha}(f(t)) \\ {}^{C}D_t^{\alpha}(g(t)f(t))-m^{C}D_t^{\alpha}(tf(t)) &\geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\Big[{}^{C}D_t^{\alpha}(g(t))^{C}D_t^{\alpha}(f(t)) \\ &\quad -m\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\,{}^{C}D_t^{\alpha}(f(t))\Big] \\ {}^{C}D_t^{\alpha}(gf(t)) &\geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\Big[{}^{C}D_t^{\alpha}(g(t))^{C}D_t^{\alpha}(f(t)) \\ &\quad -m\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\,{}^{C}D_t^{\alpha}(f(t))\Big] \\ &\quad +m^{C}D_t^{\alpha}(tf(t)). \end{aligned}$$

We proved (8) for all $t\geq 0, \alpha>0$. □

Theorem 2.5. *Let the functions f and g are defined $[0,\infty)$. We assume the function f is decreasing and the function g is Caputo differentiable. If there exists a real number $M:=\sup_{t\geq 0}\,{}^{C}D_t^{\alpha}(g(t))$, then we have*

$$\begin{aligned} {}^{C}D_t^{\alpha}(gf)(t) \geq{}& \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\Big[{}^{C}D_t^{\alpha}(g(t))^{C}D_t^{\alpha}(f(t)) \\ &-M\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\,{}^{C}D_t^{\alpha}(f(t))\Big] \\ &+M^{C}D_t^{\alpha}(tf(t)) \end{aligned}$$

for all $t\geq 0,\ \alpha>0$.

Proof. We defined the function $G(t) := g(t) - Mt$. Since the function G is Caputo differentiable and decreasing on $[0, \infty)$, one can easily prove the inequality as in the proof of Theorem (2.4) □

Theorem 2.6. *Assume that the functions f and g are Caputo differentiable and there exists $m_1 := \inf_{t \geq 0} {}^{C}D_t^{\alpha}(f(t))$, $m_2 := \inf_{t \geq 0} {}^{C}D_t^{\alpha}(g(t))$. Then we have*

$$\begin{aligned}
&{}^{C}D_t^{\alpha}[(f(t) - m_1 t)(g(t) - m_2 t)] \\
&\geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}} \Big\{ {}^{C}D_t^{\alpha}(f(t)) {}^{C}D_t^{\alpha}(g(t)) \\
&\quad - m_2 \, {}^{C}D_t^{\alpha}(f(t)) \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} \\
&\quad - m_1 \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} \, {}^{C}D_t^{\alpha}(g(t)) \\
&\quad + m_1 m_2 \frac{\Gamma(2)\Gamma(2)}{\Gamma(2-\alpha)\Gamma(2-\alpha)} \Big\}
\end{aligned}$$

Proof. We define the functions $F(t) := f(t) - m_1 t$ and $G(t) := g(t) - m_2 t$. It is clear that the functions $F(t)$ and $G(t)$ are increasing on $[0, \infty)$. Using inequality (1), we calculate

$$\begin{aligned}
{}^{C}D_t^{\alpha}(F(t)G(t)) &= {}^{C}D_t^{\alpha}[(f(t) - m_1 t)(g(t) - m_2 t)] \\
&\geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}} \, {}^{C}D_t^{\alpha}(f(t) - m_1 t) {}^{C}D_t^{\alpha}(g(t) - m_2 t)
\end{aligned}$$

$$\begin{aligned}
&{}^{C}D_t^{\alpha}\left[f(t)g(t) - m_2 f(t)t - m_1 g(t)t + m_1 m_2 t^2\right] \\
&\geq \frac{\Gamma(n-\alpha+1)}{t^{n-\alpha}} \Big\{ {}^{C}D_t^{\alpha}(f(t)) {}^{C}D_t^{\alpha}(g(t)) - m_2 \, {}^{C}D_t^{\alpha}(f(t)) {}^{C}D_t^{\alpha}(t) \\
&\quad - m_1 \, {}^{C}D_t^{\alpha}(t) \, {}^{C}D_t^{\alpha}(g(t)) + m_1 m_2 \, {}^{C}D_t^{\alpha}(t) {}^{C}D_t^{\alpha}(t) \Big\}
\end{aligned}$$

$$\begin{aligned} & {}^{C}D_t^{\alpha}\left(fg\left(t\right)\right)-m_2^{C}D_t^{\alpha}\left(f\left(t\right)t\right)-^{C}D_t^{\alpha}\left(g\left(t\right)t\right)+m_1m_2\ {}^{C}D_t^{\alpha}\left(t^2\right) \\ \geq\ & \frac{\Gamma\left(n-\alpha+1\right)}{t^{n-\alpha}}\left\{{}^{C}D_t^{\alpha}\left(f\left(t\right)\right)^{C}D_t^{\alpha}\left(g\left(t\right)\right)\right. \\ & -m_2\ {}^{C}D_t^{\alpha}\left(f\left(t\right)\right)\frac{\Gamma\left(2\right)}{\Gamma\left(2-\alpha\right)}t^{1-\alpha}-m_1\ \frac{\Gamma\left(2\right)}{\Gamma\left(2-\alpha\right)}t^{1-\alpha}\ {}^{C}D_t^{\alpha}\left(g\left(t\right)\right) \\ & \left.+m_1m_2\ \frac{\Gamma\left(2\right)\Gamma\left(2\right)}{\Gamma\left(2-\alpha\right)\Gamma\left(2-\alpha\right)}\right\} \end{aligned}$$

$$\begin{aligned} & {}^{C}D_t^{\alpha}\left(fg\left(t\right)\right)-m_2\ t^{C}D_t^{\alpha}\left(f\left(t\right)\right)-m_1\ t^{C}D_t^{\alpha}\left(g\left(t\right)\right) \\ & \qquad +m_1m_2\ \tfrac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha} \\ \geq\ & \tfrac{\Gamma(n-\alpha+1)}{t^{n-\alpha}}\left\{{}^{C}D_t^{\alpha}\left(f\left(t\right)\right)^{C}D_t^{\alpha}\left(g\left(t\right)\right)-m_2\ {}^{C}D_t^{\alpha}\left(f\left(t\right)\right)\tfrac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\right. \\ & \left.-m_1\ \tfrac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}\ {}^{C}D_t^{\alpha}\left(g\left(t\right)\right)+m_1m_2\ \tfrac{\Gamma(2)\Gamma(2)}{\Gamma(2-\alpha)\Gamma(2-\alpha)}\right\}. \end{aligned}$$

Hence the proof is completed. □

Remark 2.7. If the functions are asynchronous on $[0,\infty]$, then the inequalities (1) and (5) are reversed.

Example 2.8. Using the definition of synchronous functions, let the functions f and g are defined on $[0,\infty]$ and differentiable. If we define any functions f and g which are increasing, (or decreasing) one can easily see that the functions satisfy the results of the theorems.

Conclusion 2.9. *In this paper, we have introducedsome new type inequalities to a modified concept of a fractional calculus. The main results improve previously results and this presents an approach to new versions of inequalities using some different definitions of differential and integral operators. As a result, we predict that the new versions of well known inequalities may be obtained.*

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