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On Rings of Real-Valued R_{cl}-supercontinuous Functions

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Abstract. The notion of $\mathrm{R_{cl}}$-supercontinuity, a strong variant of continuity, is considered. The ring $\mathrm{R_{cl}(X)}$ consists of all real-valued $\mathrm{R_{cl}}$-supercontinuous functions on a topological space X is studied. It is shown that $\mathrm{R_{cl}(X)} \cong \mathrm{C(Y)}$, where Y is an ultra-Hausdorff $\mathrm{r_{cl}}$-quotient of X and it turns out that whenever X is $\mathrm{r_{cl}}$-compact, then Y is zero-dimensional. The maximal ideals of $\mathrm{R_{cl}(X)}$ are specified. The spaces X are determined for which every maximal ideal in $\mathrm{R_{cl}(X)}$ is fixed. Finally, $\mathrm{P_{r_{cl}}}$-spaces and almost $\mathrm{P_{r_{cl}}}$-spaces are defined and characterized both algebraically and topologically.

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1 Introduction

In 2007, Singh introduced the concept of a *cl-open* set in the study of clopen continuous maps. A set A in a topological space X is called *cl-open* if A is a union of clopen sets. The complement of a *cl*-open set

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is called *cl-closed*. In 2013, Tyagi et al. introduced the concept of an r_{cl}-*open* set as follows. An open set U in a topological space X is said to be r_{cl}-*open* if U is a union of *cl*-closed sets. The complement of an r_{cl}-open set is called an r_{cl}-*closed* set. For a set A in a topological space X, the set of all $x \in A$ such that A contains an $\mathrm{r_{cl}}$-open set containing x is called r_{cl}-*interior* of A and it is denoted by $\mathrm{int}_{r_{\mathrm{cl}}} A$. Clearly, a subset B of X is $\mathrm{r_{cl}}$-open if and only if $B = \mathrm{int}_{r_{\mathrm{cl}}} B$. The r_{cl}-*closure* of a set A in a topological space X, denoted by $\mathrm{cl}_{r_{\mathrm{cl}}} A$, is the set of all $x \in X$ such that each $\mathrm{r_{cl}}$-open set containing x intersects A nontrivially. Clearly, a subset B of X is $\mathrm{r_{cl}}$-closed if and only if $B = \mathrm{cl}_{r_{\mathrm{cl}}} B$. According to [13], a topological space X is called an R_{cl}-*space* if each open set in X is r_{cl}-open. If X and Y are topological spaces, then a function $f : X \to Y$ is said to be R_{cl}-*supercontinuous* if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an r_{cl}-*open* set U in X containing x such that $f(U) \subseteq V$. A bijection $\sigma : X \to Y$ is said to be an R_{cl}-*homeomorphism* if both σ and σ^{-1} are R_{cl}-supercontinuous. In this case, X and Y are said to be R_{cl}-*homeomorphic* and it is written as $X \cong_{r_{cl}} Y$.

Let $R_{cl}(X)$ be the set of all real-valued R_{cl}-supercontinuous functions on X. It is easily seen that $R_{cl}(X)$ is a subring and sublattice of $C(X)$ where $C(X)$ is the ring of all real-valued continuous functions on a space X. Throughout this paper, for $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of f. The set-theoretic complement of $Z(f)$ is denoted by $coz(f)$ and is called the cozero-set of f. We denote by $Z(X)$ the set of all zero-sets in X and $\mathrm{Z_{r_{cl}}(X)}$ denotes the set of all zero-sets $Z(f)$ in X, where $f \in \mathrm{R_{cl}(X)}$. We refer the reader to [6] for undefined terms and notations.

2 $R_{cl}(X)$ Is a $C(Y)$

In this section for any topological space X, an ultra-Hausdorff space Y is established such that $R_{cl}(X)$ and $C(Y)$ are isomorphic. First, let us recall some definitions and facts. A T_1-space is called *zero-dimensional* if it has a base consisting of clopen sets. According to [11], a topological space X is called *ultra-Hausdorff* if every pair of distinct points in X are contained in disjoint clopen sets.

Remark 2.1. The following implications hold.

zero-dimensional space $\Rightarrow$ ultra-Hausdorff space $\Rightarrow$ R_{cl}-space.

However, none of the above implications is reversible. For example, the space of strong ultrafilter topology [12, Example 113] is an ultra-Hausdorff space which is not zero-dimensional. A nondegenerate indiscrete space is an R_{cl}-space, but it is not ultra-Hausdorff, see [9]. We cite the following result from [13].

Theorem 2.2. ([13], Theorem 8.2) *For a topological space (X, τ) the following statements are equivalent.*

1. *(X, τ) is an R_{cl}-space.*
2. *Every continuous function from (X, τ) into a space (Y, ϖ) is R_{cl}-supercontinuous*

Now, let us make the following observation.

Lemma 2.3. *If X is an R_{cl}-space, then $C(X) = R_{cl}(X)$ and whenever X is completely regular, the converse is also true.*

Proof. Using [13, Theorem 8.2], the first implication is immediate. Now, suppose that X is a completely regular space and $C(X) = R_{cl}(X)$. By [6, Theorem 3.2] the collection $\beta = \{coz(f) : f \in C(X)\}$ is a base for open subsets of X. Since $C(X) = R_{cl}(X)$, for each $f \in C(X)$ and every $x \in coz(f)$, we infer that there is an r_{cl}-open set U in X containing x such that $U \subseteq coz(f)$. This shows that X is an R_{cl}-space. □

In the following, we give some properties of R_{cl}-supercontinuous functions. Before stating our results, recall that for a point x in a topological space X, the maximal connected subset of X containing x, denoted by C_x, is called the *component of x*. A space is called *totally disconnected* if the only nonempty components are one-point sets. For a point x in a topological space X, the intersection of all clopen subsets of X containing x, denoted by Q_x, is called the *quasi-component of x*. The collection of all components (resp., quasi-components) of a topological space X constitutes a decomposition of X into pairwise disjoint closed sets. By [5, Theorem 6.1.22], $C_x \subseteq Q_x$ for each $x \in X$ and the inclusion may be proper as it is shown by [5, Example 6.1.24]. It is easily seen that an open (resp., closed) set in a topological space X is $\mathrm{r_{cl}}$-open (resp., $\mathrm{r_{cl}}$-closed) if and only if it is a union of quasi-components of X. By [13, Theorem 4.1], every $Z \in \mathrm{Z_{r_{cl}}(X)}$ is $\mathrm{r_{cl}}$-closed. Hence, each $Z \in \mathrm{Z_{r_{cl}}(X)}$ is a union of quasi-components in X. However, an $\mathrm{r_{cl}}$-closed set need not be a zero-set in $\mathrm{Z_{r_{cl}}(X)}$, see [6, 4N].

Proposition 2.4. *For an R_{cl}-supercontinuous function $f : X \to Y$, the following statements are true.*

1. *$f[Q_x] \subseteq Q_{f(x)}$ for each $x \in X$.*
2. *If Y is a T_1-space, then $f[Q_x] = \{f(x)\}$ for each $x \in X$.*
3. *If f is injective and Y is a T_1-space, then X is totally disconnected.*
4. *If f is an R_{cl}-homeomorphism, then $f[Q_x] = Q_{f(x)}$ for each $x \in X$. Furthermore, if X or Y is a T_1-space, then both X and Y are totally disconnected.*

Proof. (1) Let $x \in X$, $p \in Q_x$ and $f(p) \notin Q_{f(x)}$. Then there is a clopen set V in Y containing $f(p)$ such that $f(x) \notin V$. Since f is R_{cl}-supercontinuous, $f[Q_x] \subseteq V$. So $f(x) \in V$ which is a contradiction.
(2) Let $x \in X$, $p \in Q_x$ and $f(p) \neq f(x)$. Then there is an open set V in Y containing $f(x)$ such that $f(p) \notin V$. Since f is R_{cl}-supercontinuous, $f[Q_x] \subseteq V$ which yields that $f(p) \in V$, and it is a contradiction.
(3) By part (2), $f[Q_x] = \{f(x)\}$ for every $x \in X$, this implies that $Q_x = \{x\}$. Since f is injective we infer that X is totally disconnected.
(4) It is an immediate consequence of parts (1) and (3). $\square$

Corollary 2.5. *Let X be a topological space. If $f \in R_{cl}(X)$, then $f[Q_x] = \{f(x)\}$ for each $x \in X$.*

Definition 2.6. Let X be a space and Y be a set and let $p : X \to Y$ be a surjection. The collection $\tau_p = \{U \subseteq Y : p^{-1}(U)\,\text{is}\, r_{cl}\text{-open in}\, X\}$ of subsets of Y is called the *r_{cl}-quotient topology* on Y induced by p, see [13]. Moreover, (Y, τ_p) is called an *r_{cl}-quotient space of* X.

Next, we state the main result of this section.

Theorem 2.7. *For any topological space X, there is an ultra-Hausdorff space X_u which is an r_{cl}-quotient space of X and $R_{cl}(X) \cong C(X_u)$.*

Proof. Let $X_u = \{Q_x : x \in X\}$ and define $p : X \longrightarrow X_u$ by $p(x) = Q_x$, for all $x \in X$. Let τ_p be the r_{cl}-quotient topology on X_u induced by p. Then (X_u, τ_p) is ultra-Hausdorff. In fact, if Q_x and Q_y are distinct points in X_u, then $x \notin Q_y$ and hence there is a clopen set V in X such that $x \in V$ and $y \notin V$. Now, let $H = \{Q_z : z \in V\}$. Then H is a clopen set in X_u, for $V = p^{-1}(H) = \bigcup_{z \in V} Q_z$ is a clopen set in X. Clearly, $Q_x \in H$ and $Q_y \notin H$ which shows that X_u is ultra-Hausdorff.
To complete the proof, we show that $R_{cl}(X) \cong C(X_u)$. To this end, define $\theta : R_{cl}(X) \to C(X_u)$ by $\theta(f) = f_u$, for each $f \in R_{cl}(X)$, where $f_u : X_u \to \mathbb{R}$ be defined as $f_u(Q_x) = f(x)$, for all $x \in X$. By Corollary 2.5, f_u and θ are well-defined. To show that $f_u \in C(X_u)$, let $x \in X$, $Q_x \in X_u$ and let $f_u(Q_x) = f(x) = a$. Then for each $\epsilon > 0$, there is an r_{cl}-open set U in X containing x such that $f(U) \subseteq (a - \epsilon, a + \epsilon)$. Now $G = \{Q_z : Q_z \subseteq U\}$ is an open set in X_u containing Q_x such that $f_u(G) \subseteq (a - \epsilon, a + \epsilon)$. It is easily seen that θ is a one to one homomorphism. Finally, we show that θ is onto. To this end, let $g \in C(X_u)$ and define $f : X \to \mathbb{R}$ by $f(x) = g(Q_x)$, for all $x \in X$. To see that $f \in R_{cl}(X)$, let $x \in X$ and let $f(x) = a$. Since $g \in C(X_u)$, there is an open set G in X_u containing Q_x such that $g(G) \subseteq (a - \epsilon, a + \epsilon)$ for every $\epsilon > 0$. So $U = \bigcup_{Q_z \in G} Q_z$ is an r_{cl}-open set in X containing x such that $f(U) \subseteq (a - \epsilon, a + \epsilon)$ and this shows that f is R_{cl}-supercontinuous. Also we have $\theta(f) = g$. □

Remark 2.8. From now on, for each topological space X, we consider X_u and the isomorphism $\theta : R_{cl}(X) \to C(X_u)$ as defined in the proof of Theorem 2.7.

Definition 2.9. A topological space X is said to be *r_{cl}-compact* if every r_{cl}-open cover of X has a finite subcover.

Note that every compact space is r_{cl}-compact, but not conversely. For instance, $\mathbb{R}$ is an r_{cl}-compact space which is not compact. Clearly, a space X is r_{cl}-compact if and only if X_u is compact.

Corollary 2.10. *If X is r_{cl}-compact, then $R_{cl}(X) \cong C(Y)$ for a zero-dimensional space Y.*

Proof. By Theorem 2.7, X_u is ultra-Hausdorff and since X_u is compact, X_u is zero-dimensional. Now, let $Y = X_u$, so by Theorem 2.7, $R_{cl}(X) \cong C(Y)$. □

According to [7], a topological space X is called *sum connected* if each component in X is open. Obviously, if X is sum connected, then $C_x = Q_x$ for every $x \in X$.

Corollary 2.11. *If X is sum connected, then $R_{cl}(X) \cong C(Y)$ for a discrete space Y.*

Proof. Since X is sum connected, Q_x is r_{cl}-open for every $x \in X$. So every one point set $\{Q_x\}$ is open in X_u which yields that X_u is discrete. Now, let $Y = X_u$, so by Theorem 2.7, $R_{cl}(X) \cong C(Y)$. □

We conclude this section by the following proposition.

Proposition 2.12. *Let X and Y be two topological spaces. If $X \cong_{r_{cl}} Y$, then $X_u \cong Y_u$.*

Proof. Let $\varphi : X \to Y$ be an R_{cl}-homeomorphism. Define $\tau : X_u \longrightarrow Y_u$ by $\tau(Q) = \varphi(Q)$, for all $Q \in X_u$. Clearly, τ is one to one. To see that τ is onto, let $y \in Y$ and let $Q_y \in Y_u$. Then there is $x \in X$ such that $\varphi(x) = y$ and hence $\tau(Q_x) = \varphi(Q_x) = Q_y$ by part (4) of Proposition 2.4. Now, we show that τ and τ^{-1} are continuous. To this end, let $x \in X$, $Q_x \in X_u$ and let H be an open set in Y_u containing $\varphi(Q_x)$. Then $V = \bigcup_{Q_z \in H} Q_z$ is an open set in Y containing $\varphi(x)$. So by R_{cl}-supercontinuity of φ, there is an r_{cl}-open set U in X containing x such that $\varphi(U) \subseteq V$. Therefore, $G = \{Q_x | x \in U\}$ is an open set in X_u containing Q_x such that $\tau(G) \subseteq H$. Similarly, τ^{-1} is continuous. □

We remind the reader that the converse of Proposition 2.12 is not true in general. For instance, let $X = \{a\}$ and let $Y = \mathbb{R}$. Then X_u is homeomorphic to Y_u, but X and Y are not R_{cl}-homeomorphic.

3 Maximal Ideals of $R_{cl}(X)$

In this section, we turn our attention to the maximal ideals in the rings $R_{cl}(X)$. First, let us recall that an ideal I of $R_{cl}(X)$ is called a fixed ideal if $\bigcap_{f\in I} Z(f) \neq \emptyset$, otherwise I is called a *free ideal*. We begin with the following easy lemma, its proof is left to the reader.

Lemma 3.1. *An ideal I in $R_{cl}(X)$ is fixed if and only if $\theta(I)$ is fixed in $C(X_u)$.*

Now, we completely characterize the fixed maximal ideals of a ring $R_{cl}(X)$.

Theorem 3.2. *For a topological space X, any fixed maximal ideal in $R_{cl}(X)$ is in the form of*

$$M_{Q_x} = \{f \in R_{cl}(X) : Q_x \subseteq Z(f)\}, \quad x \in X.$$

The ideals M_{Q_x} are distinct for distinct Q_x. Furthermore, $\frac{R_{cl}(X)}{M_{Q_x}} \cong \mathbb{R}$ for every $x \in X$.

Proof. By [6, Theorem 4.6] and Lemma 3.1, M is a fixed maximal ideal in $R_{cl}(X)$ if and only if $M = \theta^{-1}(M_y)$ for some $y \in X_u$. So $M = M_{Q_x}$ for some $x \in X$. Now, suppose that $Q_x \neq Q_y$ for $x, y \in X$. Then $Q_x \cap Q_y = \emptyset$, so there is a clopen set C_x in X containing x such that $Q_x \subseteq C_x$ and $C_x \cap Q_y = \emptyset$. Let $f : X \to \mathbb{R}$ be defined as $f(x) = 0$, if $x \in C_x$ and $f(x) = 1$, if $x \notin C_x$. Then $f \in R_{cl}(X)$, $f \in M_{Q_x} \setminus M_{Q_y}$ which shows that $M_{Q_x} \neq M_{Q_y}$. For the last assertion, let $x \in X$ and define $\varphi : R_{cl}(X) \to \mathbb{R}$ by $\varphi(f) = f(x)$, for all $f \in R_{cl}(X)$. Then φ is a homomorphism and Ker$\varphi = M_{Q_x}$. Consequently, $\frac{R_{cl}(X)}{M_{Q_x}} \cong \mathbb{R}$. □

Lemma 3.3. *If X is r_{cl}-compact, then each ideal in $R_{cl}(X)$ is fixed.*

Proof. If X is r_{cl}-compact, then X_u is zero-dimensional by Corollary 2.10. So in view of [6, Theorem 4.11], every ideal in $C(X_u)$ is fixed. Consequently, each ideal in $R_{cl}(X)$ is fixed by Lemma 3.1. □

Definition 3.4. A topological space X is called *s_{cl}-completely regular* if for each r_{cl}-closed set A and each $x \notin A$, there exists an R_{cl}-supercontinuous function $f : X \to \mathbb{R}$ such that $f[A] = \{0\}$ and $f[Q_x] = \{1\}$.

Clearly, a space X is s_{cl}-completely regular if and only if X_u is completely regular.

Theorem 3.5. *For a s_{cl}-completely regular space X, the following statements are equivalent.*

1. *X is r_{cl}-compact.*
2. *X_u is compact.*
3. *Every ideal in $R_{cl}(X)$ is fixed.*
4. *Every maximal ideal in $R_{cl}(X)$ is fixed.*

Proof. Clearly, parts (1) and (2) are equivalent.
(2) $\Rightarrow$ (3). It is immediate by Lemma 3.3.
(3) $\Rightarrow$ (4). It is immediate.
(4) $\Rightarrow$ (2). Since X is s_{cl}-completely regular, X_u is completely regular and by Lemma 3.1, each maximal ideal in $C(X_u)$ is fixed. These follow by [6, Theorem 4.11], that X_u is compact. □

Theorem 3.6. *Let X and Y be two topological spaces. If $X_u \cong Y_u$, then $R_{cl}(X) \cong R_{cl}(Y)$ and whenever X and Y are r_{cl}-compact, then the converse is also true.*

Proof. If $X_u \cong Y_u$, then $C(X_u) \cong C(Y_u)$. So $R_{cl}(X) \cong R_{cl}(Y)$ by Theorem 2.7. Now, suppose that X and Y are r_{cl}-compact. Then X_u and Y_u are compact zero-dimensional spaces by Theorem 3.5 and Corollary 2.10. If $R_{cl}(X) \cong R_{cl}(Y)$, then $C(X_u) \cong C(Y_u)$ by Theorem 2.7. This implies that $X_u \cong Y_u$, see [6, Theorem 4.9]. □

Corollary 3.7. *For two topological spaces X and Y, if $X \cong_{r_{cl}} Y$, then $R_{cl}(X) \cong R_{cl}(Y)$.*

Proof. It follows immediately from Proposition 2.12 and Theorem 3.6. □

Remark 3.8. For two r_{cl}-compact spaces X and Y, the rings $R_{cl}(X)$ and $R_{cl}(Y)$ may be isomorphic, while X and Y may not be homeomorphic. To see that, we utilize the example of [1, Remark 4.7]. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $Y = \bigcup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n}) \cup \{0\}$ as subspaces of $\mathbb{R}$. Since X and Y are compact, we infer that X and Y are r_{cl}-compact. Using a proof similar to [1, Remark 4.7], we can show that $R_{cl}(X) \cong R_{cl}(Y)$, but X and Y are not homeomorphic.

Theorem 3.9. *For a s_{cl}-completely regular space X, the maximal ideals of $R_{cl}(X)$ are precisely the sets*

$$M^p = \{f \in R_{cl}(X) : p \in cl_{\beta X_u} Z(f_u)\}, \quad p \in \beta X_u.$$

Proof. Using Theorem 2.11 and in view of [6, Theorem 7.3], it is evident. □

Remark 3.10. Let X be a space and let M be a maximal ideal of $R_{cl}(X)$. Similar to $C(X)$, we define

$$O_M = \{f \in R_{cl}(X) : fg = 0 \text{ for some } g \notin M_{Q_x}\},$$

see the discussion preceding [2, Theorem 2.12] .For each $x \in X$, let $O_{Q_x} = \{f \in R_{cl}(X) : Q_x \subseteq int_{r_{cl}} Z(f)\}$. If X is s_{cl}-completely regular and M is a fixed maximal ideal of $R_{cl}(X)$, then $O_M = O_{Q_x}$ for some $x \in X$. In fact, if M is a fixed maximal ideal in $R_{cl}(X)$, then by Theorem 3.2, $M = M_{Q_x}$ for some $x \in X$. Now we show that $O_M = O_{Q_x}$. To this end, let $f \in O_M$. Then $fg = 0$ for some $g \notin M_{Q_x}$ and hence $Q_x \subseteq X \setminus Z(g) \subseteq Z(f)$. Since $X \setminus Z(g)$ is r_{cl}-open, then $Q_x \subseteq int_{r_{cl}} Z(f)$ which follows that $f \in O_{Q_x}$. Now, let $f \in O_{Q_x}$. Then $Q_x \subseteq int_{r_{cl}} Z(f)$, so there is an r_{cl}-open set U in X such that $Q_x \subseteq U \subseteq Z(f)$. Since X is s_{cl}-completely regular, there exists $g \in R_{cl}(X)$ such that $g[X \setminus U] = \{0\}$ and $g[Q_x] = \{1\}$. Therefore $g \notin M_{Q_x}$ and $fg = 0$ which shows that $f \in O_M$.

Theorem 3.11. *Let X be an r_{cl}-compact space. Then for every $x \in X$, the ideal O_{Q_x} in $R_{cl}(X)$ is generated by a set of idempotents.*

Proof. By Corollary 2.10, X_u is zero-dimensional. In view of [4, Theorem 2.4], a space X is zero-dimensional if and only if for each $x \in X$, the ideal O_x in $C(X)$ is generated by a set of idempotents. Using this fact, for each $x \in X$, the ideal O_{Q_x} in $C(X_u)$ is generated by a set of idempotents. This follows by Theorem 2.7, that the ideal O_{Q_x} in $R_{cl}(X)$ is generated by a set of idempotents. □

4 $P_{r_{cl}}$-Spaces and Almost $P_{r_{cl}}$-Spaces

In this section the counterparts of P-spaces and almost P-spaces are defined and characterized both algebraically and topologically. We recall that a completely regular Hausdorff space X is called a P-*space* if $Z(f)$ is open for each $f \in C(X)$, equivalently, $C(X)$ is a von Neumann regular ring, see [6, 4J]. Recall that a ring R is called *von Neumann regular* if for every $a \in R$ there is $x \in R$ for which $axa = a$. We observe trivially that if $C(X)$ is von Neumann regular, then so is $R_{cl}(X)$, but the converse is not true in general. For example, the space $R_{cl}(\mathbb{R})$ is von Neumann regular but $C(\mathbb{R})$ is not von Neumann regular. Motivated by this, we offer the following definition.

Definition 4.1. A s_{cl}-completely regular space X is called $P_{r_{cl}}$-space if $Z(f)$ is open for each $f \in R_{cl}(X)$.

Lemma 4.2. *Every P-space is a $P_{r_{cl}}$-space.*

Proof. Suppose that X is a P-space. Then X is zero-dimensional and hence X is s_{cl}-completely regular. Also each $Z \in Z_{r_{cl}}(X)$ is open, for $Z_{r_{cl}}(X) \subseteq Z(X)$. □

Theorem 4.3. *Let X be a s_{cl}-completely regular space X. The following statements are equivalent.*

1. *X is a $P_{r_{cl}}$-space.*
2. *X_u is a P-space.*
3. *$M_{Q_x} = O_{Q_x}$ for every $x \in X$.*
4. *Each countable intersection of r_{cl}-open sets is r_{cl}-open.*
5. *$R_{cl}(X)$ is von Neumann regular.*

Proof. (1) ⇔ (2) If $f \in R_{cl}(X)$, then $Z(f) = \bigcup_{Q_x \in Z(f_u)} Q_x$. So $Z(f)$ is open in X if and only if $Z(f_u)$ is open in X_u. This implies that X is a $P_{r_{cl}}$-space if and only if X_u is a P-space.
Using [6, 4J], parts (2), (3) and (4) are equivalent.
(2) ⇔ (5) By [6, 4J], X_u is a P-space if and only if $C(X_u)$ is a von Neumann regular ring. This implies by Theorem 2.7, that X_u is a P-space if and only if $R_{cl}(X)$ is a von Neumann regular ring. □

Proposition 4.4. *A Hausdorff space X is a P-space if and only if X is both an R_{cl}-space and a $P_{r_{cl}}$-space.*

Proof. If X is a P-space, then X is zero-dimensional and hence X is an R_{cl}-space. Furthermore X is a $P_{r_{cl}}$-space by Lemma 4.2. Conversely, suppose that X is an R_{cl}-space and a $P_{r_{cl}}$-space. Since every s_{cl}-completely regular R_{cl}-space is completely regular, we infer that X is completely regular. X is a $P_{r_{cl}}$-space and by Theorem 4.3 $R_{cl}(X)$ is a von Neumann regular ring which implies that $C(X)$ is von Neumann regular, for $R_{cl}(X) = C(X)$ by Lemma 2.2. □

We recall that a completely regular Hausdorff space X is an *almost P-space* if every nonempty zero-set in $Z(X)$ has non-empty interior. Motivated by this, we offer the following definition.

Definition 4.5. A s_{cl}-completely regular space X is called an *almost $P_{r_{cl}}$-space* if every non-empty zero-set in $Z_{r_{cl}}(X)$ has non-empty r_{cl}-interior.

The following shows that the classes of almost P-spaces and almost $P_{r_{cl}}$-spaces are independent of each other.

Example 4.6. The space $\mathbb{R}$ is an almost $P_{r_{cl}}$-space but it is not an almost P-space. Now, let $Y = X \cup \mathbb{N}$, where X is a connected almost P-space, see [1, Example 5.3]. Then Y is an almost P-space. To see that Y is not an almost $P_{r_{cl}}$-space, let $f : Y \to \mathbb{R}$ be defined as $f(n) = \frac{1}{n}$, if $n \in \mathbb{N}$ and $f(x) = 0$, if $x \in X$. Then $f \in R_{cl}(Y)$ and $Z(f) = X$, but $int_{r_{cl}} Z(f) = \emptyset$, for if $x \in int_{r_{cl}} Z(f)$, then there is an open set U in Y such that $X = Q_x \subseteq U \subseteq Z(f) = X$ which shows that X is open in Y, a contradiction.

We conclude this section by the following theorem which characterizes almost $P_{r_{cl}}$-space. We call a set A in a space X is *r_{cl}-dense* in X if every r_{cl}-open set in X intersects A nontrivially.

Theorem 4.7. *The following statements are equivalent for a s_{cl}-completely regular space X.*

1. *X is an almost $P_{r_{cl}}$-space.*
2. *X_u is an almost P-space.*
3. *Every non-unit element in $R_{cl}(X)$ is a zero-divisor.*

4. $\bigcup_{x\in X} O_{Q_x} = \bigcup_{x\in X} M_{Q_x}$.

5. *If* $G = \bigcap_{i\in\mathbb{N}} G_i$*, where each* G_i *is* r_{cl}*-open, then* $int_{r_{cl}}G$ *is* r_{cl}*-dense in* G*.*

6. *Every non-empty zero-set in* $Z_{r_{cl}}(X)$ *has non-empty* r_{cl}*-interior.*

Proof. Using [3, Theorem 2.2] and by Theorem 2.7, the proof is easy. □

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