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Original Research Paper

A New Technique to Solve Generalized Caputo-type Fractional Differential Equations with the Example of Computer Virus Model

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Abstract. In this research work, we propose a new fractional numerical algorithm to obtain the numerical or exact solutions of generalized-Caputo type fractional-order differential equations of order $0 < \theta \leq 1$. For finding the numerical or exact solutions by the proposed technique, we use the solutions of integer-order differential equations. Generalization of the proposed scheme to finite systems is also introduced. At the last, we derive the numerical simulations of some specific equations along with the solution of a computer virus model to illustrate the applications of our results. The proposed scheme is very effective and easy to apply on different kind of systems.

AMS Subject Classification: 26A33; 65D05; 65D30; 65L07
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1 Introduction

Current days, Fractional Calculus, also known as non-integer calculus, is the mostly used phenomena in the different fields of science and engineering. Day by day so many different types of fractional order derivatives have been proposed or investigated by researchers along which numerical methods to solve non-integer order (fractional) differential equations (FDEs). So many techniques to find the numerical solution of FDEs have been studied in previous works [15, 26]. In most of the cases, in such methods, either the solutions of a classical differential equation is the translation of the given non-integer order differential equations or the series diffusions in the neighbourhood of the initial constraints are utilized. These numerical methods are playing a very important role in the study of complex dynamics. Also, fractional (non-integer) order differential equations (FDEs) are very useful in mathematical modelling. So many non-integer order derivatives have been used in various parts of engineering and science [11, 15, 24, 26, 13]. Caputo, Atangana-Baleanu-Caputo (AB), and Caputo-Fabrizio (CF) derivatives are the well known fractional derivative operators. Caputo derivative has derived with singular type non-local kernel (or power law type), Caputo-Fabrizio with exponentially decay-type (or non-singular kernel) and AB derivative has given with Mittag-Leffler kernel memory. Recently *Odibat et al.* [23] has introduced a new modified-version of Caputo-type variable-order derivative with the modified P-C scheme. The existence proof of a unique solution of the generalized Caputo type FDEs is proved by *katugampola et al.* in [14]. Currently so many applications of different fractional derivatives have been come in epidemiology [7, 11, 16, 17, 18, 22]. In [12], the authors proposed a study on tight-bounds for the path-factors existence in the parameter settings of network vulnerability. In [20], authors simulated a non-linear model of COVID-9 for studying the disease outbreaks in Japan and Iran. A new fractional-order mathematical model of human liver is given in [2]. A structure of a non-linear dynamical system is given in [3]. A fractional-order structure of a linear triatomic molecule is simulated in [4]. A study on p-Laplacian non-periodic non-linear boundary value problem in the sense of modified Caputo derivative is proposed in [21]. *Rezapour et al.* in [27] has proposed a study on the existence analysis of quantum integro-difference fractional-order

boundary value problem. *Etemad et al.* in [8] has investigated some aspects on neutral non-classical differential inclusions in the sense of Katugampola derivative. Recently, a number of mathematicians have analysed some important new algorithms for finding the solutions of non-linear FDEs. In [10], a new scheme to simulate Atangana-Baleanu type non-linear volterra integro-differential equations is produced. In [28], one more method for the FDEs based on Genocchi polynomials has mentioned. *Ganji et al.* in [9] suggested a new method to solve AB-type multi-variable order differential equations. The complex dynamics which can not be studied by classical derivatives, can be studied by these non-classical derivatives more clearly. Still, there are many drawbacks in the non-classical calculus. In several cases, the solutions existence for many FDEs can not be smoothly described. Lately, an existence analysis for infinite-coefficient symmetric integro-DEs in Caputo-Fabrizio form is given by *Baleanu* in [1].
In this paper, we generalize the numerical technique introduced by *Demirci et al.* in [6]. We utilize a transformation in the equivalent non-integer order Volterra integral equation (VIE) of given FDE and enlist its exact solution in the form of the solution of an classical-order differential equation in the form of generalized Caputo type non-classical derivatives. We explained some examples to show the applications of the given scheme clearly. The paper is distributed as follows. In Section 2, we remind some specific definitions of variable-order derivatives. In Section 3, we review the original method. Section 4 is devoted to the description of the main results. Some examples with the solution of a computer virus epidemic model to show the applications of the given method are given in section 5. A conclusion finishes the paper.

2 Preliminaries

Here, we recall some necessary definitions of the fractional (or non-integer order) derivatives.

Definition 2.1. [26] *The Riemann and Liouville (R-L) non-classical derivative of order $\theta > 0$ of a mapping $X : (0, \infty) \to \mathbb{R}$ is formulated by*

$$D_\eta^\theta X(\eta) = \left(\frac{d}{d\eta}\right)^n \frac{1}{\Gamma(n-\theta)} \int_0^\eta (\eta - \xi)^{n-\theta-1} X(\xi) d\xi,$$

where $n = [\theta] + 1$ *and* $[\theta]$ *is the integer-part of* θ.

Definition 2.2. [26] *The Caputo-type non-integer order derivative of order* $\theta > 0$ *of a mapping* $X : (0, \infty) \to \mathbb{R}$ *is described by*

$$D_{\eta}^{\theta} X(\eta) = \frac{1}{\Gamma(k-\theta)} \int_0^{\eta} (\eta - \xi)^{k-\theta-1} X^k(\xi) d\xi,$$

where $k = [\theta] + 1$ *and* $[\theta]$ *is the integer-part of* θ.

Definition 2.3. [14] *The generalized Riemann-type non-classical derivative operator,* ${}^{R}D_{c+}^{\theta,\rho}$, *of order* $\theta > 0$ *is given as:*

$$({}^{R}D_{c+}^{\theta,\rho} X)(\eta) = \frac{\rho^{\theta-n+1}}{\Gamma(n-\theta)} \left(\eta^{1-\rho} \frac{d}{d\eta}\right)^n \int_c^{\eta} s^{\rho-1} (\eta^{\rho} - s^{\rho})^{n-\theta-1} X(s) ds, \ \ \eta > c,$$

where $c \geq 0, \ \rho > 0, \ \& \ n - 1 < \theta \leq n$.

Definition 2.4. [14] *The generalized-version of Caputo-type non-classical derivative,* ${}^{C}D_{c+}^{\theta,\rho}$, *of order* $\theta > 0$ *is given as:*

$$({}^{C}D_{c+}^{\theta,\rho} X)(\eta) = \left({}^{R}D_{c+}^{\theta,\rho}\left[X(x) - \sum_{k=0}^{n-1} \frac{X^{(k)}(c)}{k!}(x-c)^k\right]\right)(\eta), \quad \eta > c,$$

where $c \geq 0, \ \rho > 0, \ \& \ n = \lceil \theta \rceil$.

Definition 2.5. [23] *The version of new modified generalized Caputo non-integer order derivative,* $D_{c+}^{\theta,\rho}$, *of order* $\theta > 0$ *is described as:*

$$(D_{c+}^{\theta,\rho} X)(\eta) = \frac{\rho^{\theta-n+1}}{\Gamma(n-\theta)} \int_c^{\eta} s^{\rho-1} (\eta^{\rho} - s^{\rho})^{n-\theta-1} \left(s^{1-\rho} \frac{d}{ds}\right)^n X(s) ds, \ \ \eta > c,$$

where $c \geq 0, \ \rho > 0, \ \& \ n - 1 < \theta \leq n$.

3 The Solution Method In Caputo Sense

In this portion of the study, we review the numerical algorithm proposed in [6]. This technique is rooted on converting the classical order systems to a non-classical (fractional-order) systems and finding the solution of

the fractional-order system in to the form of the solution of a classical systems
Let us remind the initial value problem (IVP)

$$\begin{aligned} &{}^{C}D_t^{\theta}\zeta(t) = \mathcal{G}(t, \zeta(t)), \\ &\zeta(0) = \zeta_0, \end{aligned} \tag{1}$$

where $\mathcal{G} \in C([0,T] \times \mathbb{R}, \mathbb{R}),\ 0 < \theta < 1$. ${}^{C}D_t^{\theta}$ is the Caputo derivative operator mentioned in Def. (2.2).

Since $\mathcal{G}$ is considered as a continuous mapping or function, so each solution of the IVP (1) is also satisfy the given Volterra fractional integral equation (VFIE):

$$\zeta(t) = \zeta_0 + \frac{1}{\Gamma(\theta)} \int_0^t (t-\chi)^{\theta-1} \mathcal{G}(\chi, \zeta(\chi)) d\chi, \quad t \in [0,T]. \tag{2}$$

Beyond it, every solution of (2) is also satisfy the IVP (1). We find that IVP (1) is similar to the IVP

$$\begin{aligned} &{}^{C}D_t^{\theta}(\zeta(t) - \zeta_0) = \mathcal{G}(t, \zeta(t)), \\ &\zeta(0) = \zeta_0. \end{aligned}$$

Theorem 3.1. *[19](existence) Assume that* $\mathcal{G} \in C[R_0, R]$ *where* $R_0 = \{(t, \zeta) : 0 \leq t \leq m \text{ and } |\zeta - \zeta_0| \leq b\}$ *and fix* $|\mathcal{G}(t,\zeta)| \leq M$ *on* R_0. *Then at least one solution for the IVP* (1) *exists on* $0 \leq t \leq \delta$, *where* $\delta = \min\big(m, \big[\frac{b}{M}\Gamma(\theta+1)\big]^{\frac{1}{\theta}}\big), 0 < \theta < 1$.

Theorem 3.2. *Consider the IVP proposed by* (1). *Let*

$$\phi(\nu, \zeta_*(\nu)) = \mathcal{G}(t - (t^{\theta} - \nu\Gamma(\theta+1))^{\frac{1}{\theta}}, \zeta(t - (t^{\theta} - \nu\Gamma(\theta+1))^{\frac{1}{\theta}})),$$

and suppose that the Theorem 3.1 hold. Then, a solution of (1), $\zeta(t)$, *is established by*

$$\zeta(t) = \zeta_*(t^{\theta}/\Gamma(\theta+1)),$$

where $\zeta_*(\nu)$ *is a solution of the integer-order differential equations*

$$\frac{d\zeta_*(\nu)}{d\nu} = \phi(\nu, \zeta_*(\nu)), \tag{3}$$

with the initial conditions

$$\zeta_*(0) = \zeta_0. \tag{4}$$

Proof. According to the Theorem 3.1, the solution of the (1) exists. If $\zeta(t)$ is a solution of (1) then, it will also satisfy the (2). Let $\tau = t - (t^\theta - \nu\Gamma(\theta+1))^{1/\theta}$. So, VFIE (2) can be defined as

$$\zeta(t) = \zeta_0 + \int_0^{t^\theta/\Gamma(\theta+1)} \mathcal{G}(t - (t^\theta - \nu\Gamma(\theta+1))^{1/\theta}, \zeta(t - (t^\theta - \nu\Gamma(\theta+1))^{1/\theta}))d\nu$$
$$= \zeta_0 + \int_0^{t^\theta/\Gamma(\theta+1)} \phi(\nu, \zeta_*(\nu))d\nu. \tag{5}$$

Also every solution of (3)-(4) is a solution of the VFIE written below and vice versa.

$$\zeta_*(\nu) = \zeta_0 + \int_0^\nu \phi(s, \zeta_*(s))ds, \quad 0 \le \nu \le a^\theta/\Gamma(\theta+1).$$

Since $0 \le t^\theta/\Gamma(\theta+1) \le a^\theta/\Gamma(\theta+1)$, the right-hand part of equation (5) is equal to $\zeta_*(t^\theta/\Gamma(\theta+1))$. □

4 Main Results in new Generalized Caputo Sense

After successfully reviewed the above method in the Caputo sense, now we do our main simulations with the generalize form of Caputo differential operator. Let us adopt the IVP

$${}^C D_t^{\theta,\rho}\Lambda(t) = \mathcal{G}(t, \Lambda(t)), \tag{6}$$

with the initial condition

$$\Lambda(0) = \Lambda_0, \tag{7}$$

where $\mathcal{G} \in C([0,T] \times \mathbb{R}, \mathbb{R}), 0 < \theta \le 1, \rho > 0$ and ${}^C D_t^{\theta,\rho}$ is the new generalised fractional derivative operator mentioned in Def. (2.5).

Since, $\mathcal{G}$ is considered as a continuous mapping so every solution of the IVP (6)-(7) is also satisfy the given Volterra integral equation (VIE):

$$\Lambda(t) = \Lambda_0 + \frac{\rho^{1-\theta}}{\Gamma(\theta)} \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\theta-1}\mathcal{G}(\tau, \Lambda(\tau))d\tau, \quad t \in [0,T]. \tag{8}$$

Also, the IVP (6) is equivalent to the IVP

$$\begin{aligned}&{}^{C}D_t^{\theta,\rho}(\Lambda(t)-\Lambda_0)=\mathcal{G}(t,\Lambda(t)),\\&\Lambda(0)=\Lambda_0.\end{aligned}$$

Now, first we mention the solution existence of the given IVP by the following theorem.

Theorem 4.1. *[7, 14] (Existence analysis). Let* $0<\theta\leq 1$, $\Lambda_0\in\mathbb{R}$, $\eta>0$ *and* $a>0$. *Let* $R_0:=\{(t,\Lambda):\ t\in[0,a],\ |\Lambda-\Lambda_0|\leq\eta\}$ *and assume that the mapping* $\mathcal{G}:R_0\to\mathbb{R}$ *be continuous. Next, allocate* $S:=\sup_{(t,\Lambda)\in R_0}|\mathcal{G}(t,\Lambda)|$ *and*

$$T=\begin{cases} a, & if S=0,\\ min\{a,\left(\dfrac{\eta\Gamma(\theta+1)\rho^{\theta}}{S}\right)^{\dfrac{1}{\theta}}\} & else.\end{cases}$$

Then, here a function $\Lambda\in\mathcal{C}[0,T]$ *exists that satisfy the IVP* (6) *and* (7).

Theorem 4.2. *Consider the initial value problem proposed by* (6)-(7). *Let*

$$f(\nu,\Lambda_*(\nu))=\mathcal{G}(t^{\rho}-(t^{\theta}-\nu\Gamma(\theta+1)\rho^{\theta})^{\frac{1}{\theta}},\Lambda(t^{\rho}-(t^{\theta}-\nu\Gamma(\theta+1)\rho^{\theta})^{\frac{1}{\theta}})),$$

with the assumption of existence of Theorem 4.1. Then, a solution of (6), $\Lambda(t)$, *is established by*

$$\Lambda(t)=\Lambda_*(t^{\theta}\rho^{-\theta}/\Gamma(\theta+1)),$$

where $\Lambda_*(\nu)$ *is a solution of integer-order differential equation*

$$\frac{d\Lambda_*(\nu)}{d\nu}=f(\nu,\Lambda_*(\nu)), \tag{9}$$

and

$$\Lambda_*(0)=\Lambda_0. \tag{10}$$

Proof. The solution of the (6)-(7) is existed from the result of Theorem 4.1. If $\Lambda(t)$ is a solution of (6)-(7) then, it also satisfies (8). Let $\tau^\rho = t^\rho - (t^\theta - \nu\Gamma(\theta+1)\rho^\theta)^{1/\theta}$. So, VFIE (8) can be established as

$$\begin{aligned}\Lambda(t) &= \Lambda_0 + \int_0^{t^\theta\rho^{-\theta}/\Gamma(\theta+1)} [\mathcal{G}(t^\rho - (t^\theta - \nu\Gamma(\theta+1)\rho^\theta)^{1/\theta},\\ &\Lambda(t^\rho - (t^\theta - \nu\Gamma(\theta+1)\rho^\theta)^{1/\theta}))]d\nu \\ &= \Lambda_0 + \int_0^{t^\theta\rho^{-\theta}/\Gamma(\theta+1)} f(\nu, \Lambda_*(\nu))d\nu.\end{aligned} \tag{11}$$

Also, every solution of (9)-(10) satisfies the VIE given below and vice versa.

$$\Lambda_*(\nu) = \Lambda_0 + \int_0^\nu f(s, \Lambda_*(s))ds, \quad 0 \le \nu \le a^\theta\rho^{-\theta}/\Gamma(\theta+1).$$

Since $0 \le t^\theta\rho^{-\theta}/\Gamma(\theta+1) \le a^\theta\rho^{-\theta}/\Gamma(\theta+1)$, the right-hand part of equation (11) is equal to $\Lambda_*(t^\theta\rho^{-\theta}/\Gamma(\theta+1))$. □

A simplification of Theorem 4.1 and 4.2 for n-dimensional system is as follows:

Theorem 4.3. *Let $\|.\|$ denotes any convenient norm on R^n. Let $\mathcal{A} \in [R_1, R^n]$, where $R_1 = \{(t,\Lambda) : 0 \le t \le a$ and $|\Lambda - \Lambda_0| \le K\}, \mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n)^T, \Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_n)^T$, and let $\|\mathcal{A}(t,\Lambda)\| \le M$, on R_1. Then, atleast one solution for the given system of FDE's is exists and defined by*

$$^C D^{\theta,\rho}\Lambda(t) = \mathcal{A}(t, \Lambda(t)), \tag{12}$$

with initial conditions

$$\Lambda(0) = \Lambda_0, \tag{13}$$

on $0 \le t \le T^*$, where $T^* = \min\big(a, [\frac{K}{M}\Gamma(\theta+1)\rho^\theta]^{\frac{1}{\theta}}\big),\ 0 < \theta \le 1,\ \rho > 0.$

Theorem 4.4. *Consider the IVP demonstrated by (12)-(13) of order $\theta, 0 < \theta \le 1, \rho > 0$. Assume*

$$f(\nu, \Lambda_*(\nu)) = \mathcal{A}(t^\rho - (t^\theta - \nu\rho^\theta\Gamma(\theta+1))^{1/\theta}, \Lambda(t^\rho - (t^\theta - \nu\rho^\theta\Gamma(\theta+1))^{1/\theta})),$$

then when the Theorem 4.3 hold, a solution of (6)-(7), $\Lambda(t)$,*can be expressed by*

$$\Lambda(t) = \Lambda_*(t^\theta \rho^{-\theta}/\Gamma(\theta+1)),$$

where $\Lambda_*(\nu)$ *is a solution of the model of classical differential equations*

$$\frac{d\Lambda_*(\nu)}{d\nu} = f(\nu, \Lambda_*(\nu)),$$

alongwith the constraints

$$\Lambda_*(0) = \Lambda_0.$$

5 Important Examples

Here we give some important examples which are interpreted by utilizing Mathematica 10 software.

Example 5.1 Consider the following linear non-homogeneous non-integer order equation

$$\begin{aligned} &\mathcal{D}_{0+}^{\theta,\rho} y(t) = t, \ \ \rho > 0, \ \ 0 < \theta \leq 1, \\ &y(0) = y_0. \end{aligned} \tag{14}$$

For this example,

$$g(\nu) = \frac{\rho^{-\theta}\left(\theta\big((\theta+1)t^\theta - \nu\rho^\theta\Gamma(\theta+2)\big)\big(t^\theta - \nu\rho^\theta\Gamma(\theta+1)\big)^{\frac{1}{\theta}} + (\theta+1)\left(\rho^\theta\Gamma(\theta+2)(\nu t^\rho + y_0) - \theta\big(t^\theta\big)^{\frac{1}{\theta}+1}\right)\right)}{(\theta+1)\Gamma(\theta+2)}.$$

The solution of the related integer order problem as mentioned in Theorem 4 is

$$y_1(\nu) = \frac{\rho^{-\theta}t^\theta\left((\theta+1)t^\rho - \theta\big(t^\theta\big)^{\frac{1}{\theta}}\right)}{\Gamma(\theta+2)} + y_0.$$

So, the solution of the given non-integer order problem is

$$y(t) = y_1\left(\frac{\rho^{-\theta}t^\theta}{\Gamma(\theta+1)}\right) = \frac{\rho^{-\theta}t^\theta\left((\theta+1)t^\rho - \theta\left(t^\theta\right)^{\frac{1}{\theta}}\right)}{\Gamma(\theta+2)} + y_0. \tag{15}$$

Indeed, it can be illustrated that (15) is a solution of (14) by using the generalized non-integer order derivative. it should be note that solution (15) is the same as solution [p.2757, Eq.(11)] obtained in [6] for $\theta = 1/2$ and $\rho = 1$.

Example 5.2 Let us remind the fractional order Riccati differential equation given in [23]:

$$\begin{aligned} &\mathcal{D}_{0+}^{\theta,\rho} y(t) = 2y(t) - y^2(t) + 1, \;\; t, \;\rho > 0, \;\; 0 < \theta \leq 1, \\ &y(0) = 0. \end{aligned} \tag{16}$$

The exact solution of (16), at $\rho = 1$ and $\theta = 1$ is

$$y(t) = 1 + \sqrt{2}\tanh\left[\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right].$$

In Fig. 1, we give the comparison of the approximate solution plots of Eq.(16) by the new method and the adaptive predictor-corrector algorithm [23] with the exact solution of Eq.(16). Here we checked that the approximate solutions are in contract with exact solution, graphically. The absolute errors in the proposed techniques are given in Figure 2 where solid-type and dashed-type lines demonstrate errors in the new method and the adaptive predictor-corrector algorithm, respectively.

Example 5.3 Now we choose the following non-classical order differential equation system given in [5]:

$$\begin{aligned} &\mathcal{D}_{0+}^{\theta,\rho} x(t) = wx(t) - y^2(t), \\ &\mathcal{D}_{0+}^{\theta,\rho} y(t) = \mu(z(t) - y(t)), \\ &\mathcal{D}_{0+}^{\theta,\rho} z(t) = ay(t) - bz(t) + x(t)y(t), \end{aligned} \tag{17}$$

where $t, \;\rho > 0, \;\; 0 < \theta \leq 1$, and $a, \; b, \; w, \; \mu$ are constant quantities. Moreover, $w = -2.667, \;\; a = 27.3, \;\; b = 1, \;\; \mu = 10$ (time step $h = 0.02$, initial conditions are $(0, 10, 10)$).

Figures 3, 4 and 5 show the solutions $x(t), \; y(t)$ and $z(t)$ of the system (17) for ($\theta = 0.89, \;\; \rho = 1.2$) whereas 6, 7 and 8 show phase portrait of the system (17) for the same values of θ and ρ. The CPU time, needed to get the solution for the system (17) by the new method is just 0.265625

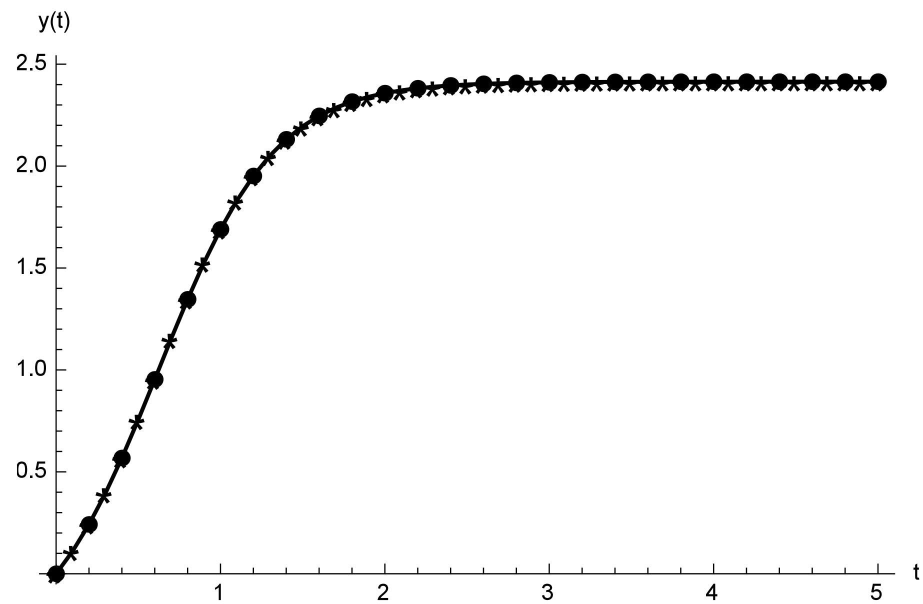

Figure 1: Solutions of equation (16)

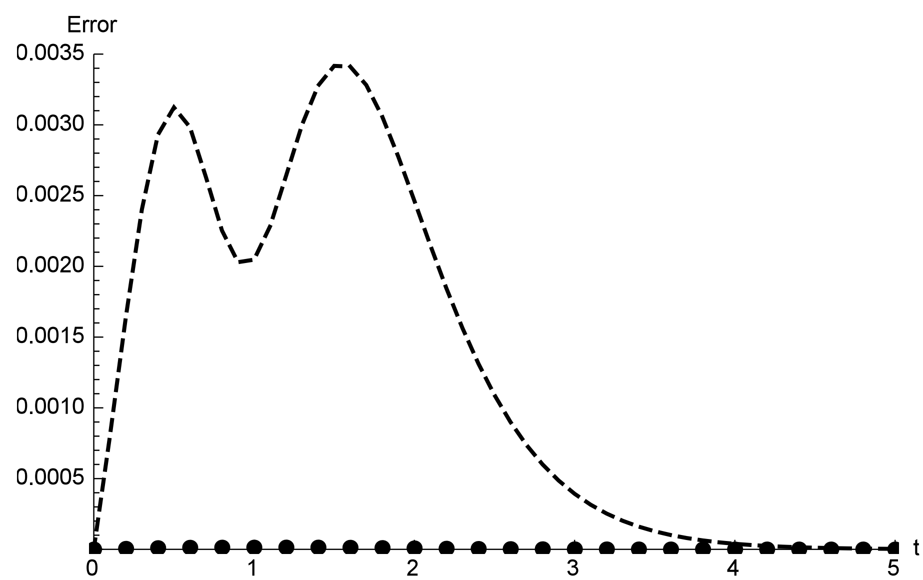

Figure 2: Error in approximate solutions of (16)

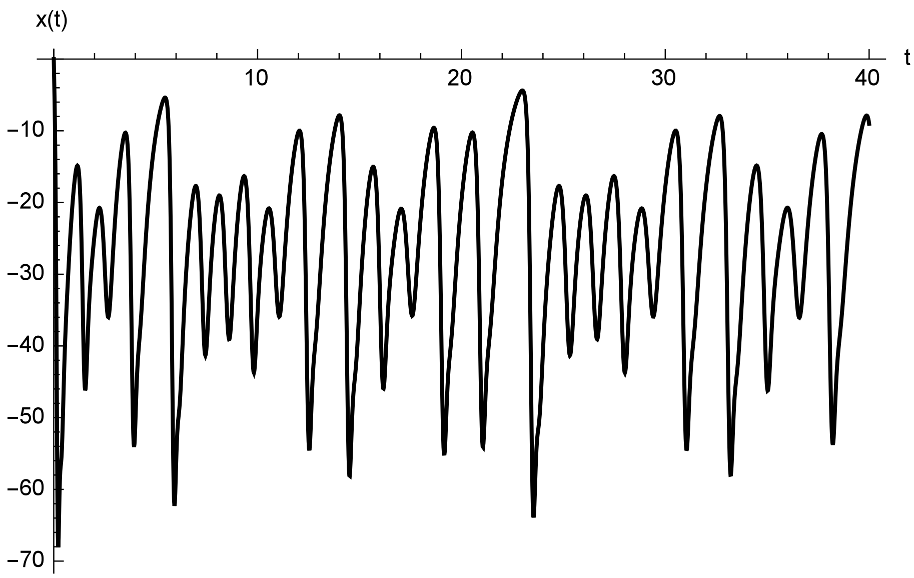

Figure 3: Outputs of Eqn. (17) for in the (t, x)-plane.

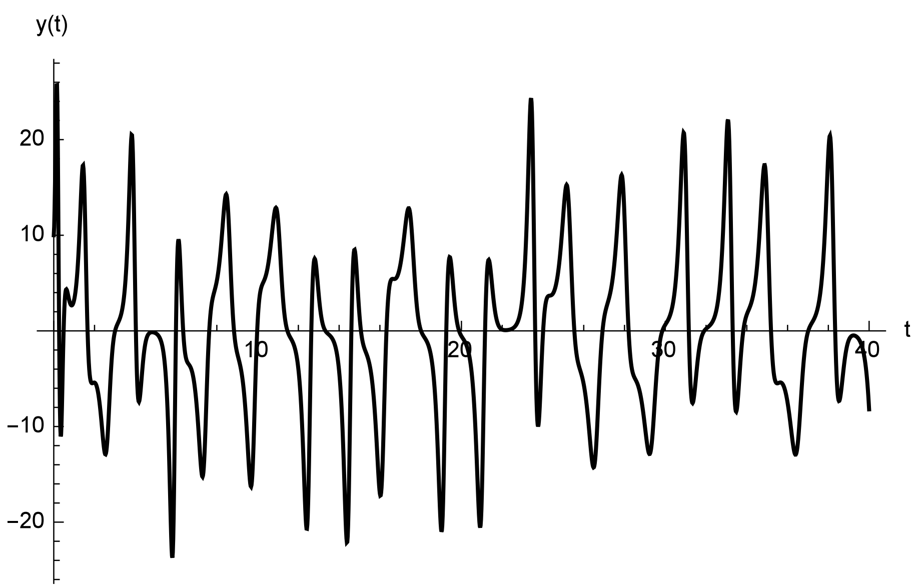

Figure 4: Outputs of Eqn. (17) for in the (t, y)-plane.

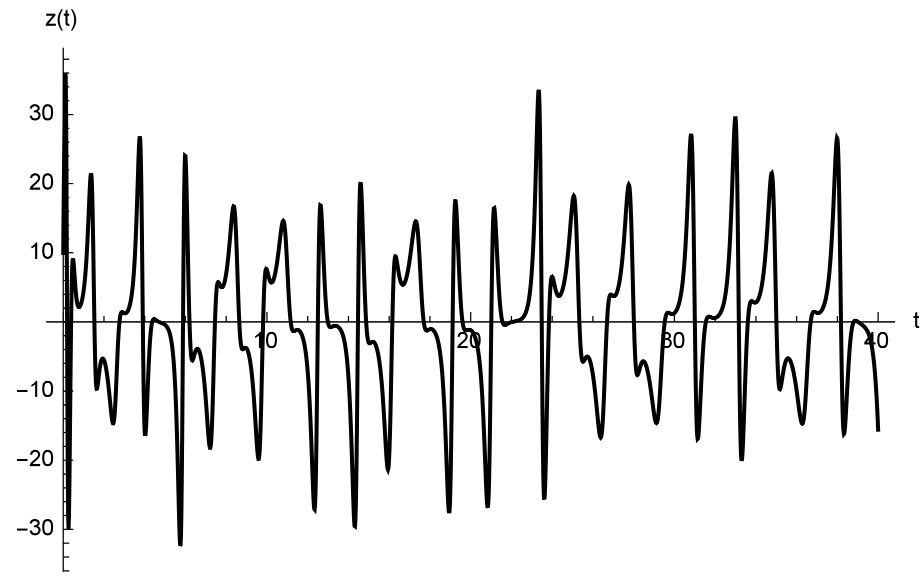

Figure 5: Outputs of Eqn. (17) for in the (t, z)-plane.

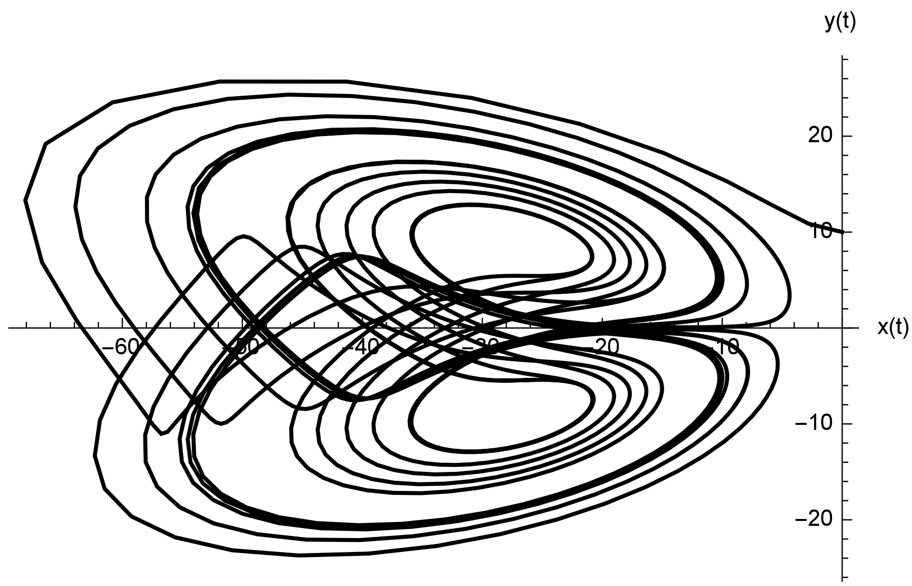

Figure 6: Outputs of Eqn. (17) in the xy-phase plane.

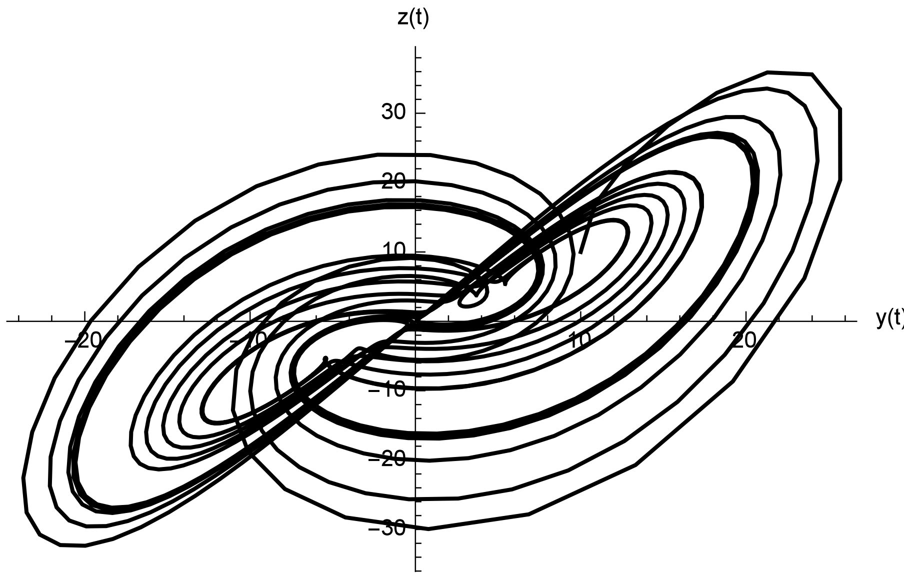

Figure 7: Solutions of equation (17) in the yz-phase plane.

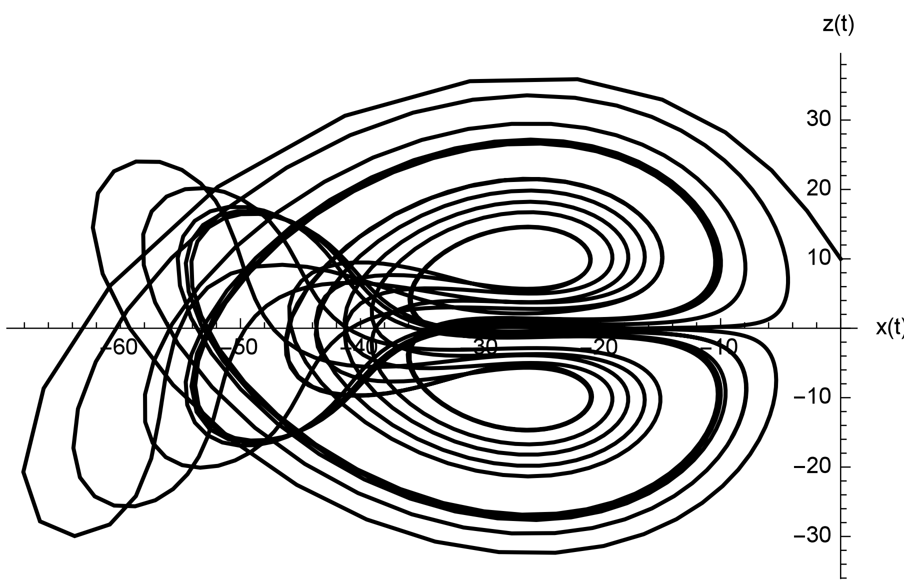

Figure 8: Solutions of equation (17) in the xz-phase plane.

in seconds. It may be observed from Figures 6, 7 and 8 that the system show chaotic behaviour for $(\theta = 0.89,\ \rho = 1.2)$.

Example 5.4 As a last example, we consider a computer virus model proposed by *Piqueira et al.* [25] in integer order sense, which is as follows:

$$\begin{cases} \dfrac{dS}{dt} = -\beta_{SA}SA - \alpha SI + \eta R, \\ \dfrac{dI}{dt} = -\beta_{IA}IA + \alpha SI - \zeta I, \\ \dfrac{dR}{dt} = -\eta R + \zeta I, \\ \dfrac{dA}{dt} = \beta_{SA}SA + \beta_{IA}IA, \end{cases} \tag{18}$$

where $S(t)$ is for susceptible computers subjected to possible infection, $A(t)$ for non-infected computers furnished with anti-virus, $I(t)$ denotes virus-infected computer systems and $R(t)$ define removed computers due to infection or not. Parameter β_{SA} denotes the conversion rate of susceptible into antidotal, α denotes the transmission rate of susceptible into infection, β_{IA} is the transmission rate of infected computers into antidotal, ζ is the removed rate and η is the rate of removed computers which can be restored and varied into susceptible.

Now for numerical simulations we use two different values of given parameters with initial conditions for endemic equilibrium (EE) and disease-free equilibrium (DFE) conditions, respectively as given in [25] .

Numerical values for DFE case;

$\alpha = 0.1,\ \zeta = 20,\ \beta_{SA} = 0.025,\ \beta_{IA} = 0.25,\ \eta = 0.8$ with initial constraints $S(0) = 74,\ I(0) = 25,\ R(0) = 0,\ A(0) = 1$.

Numerical values for EE case;

$\alpha = 0.1,\ \zeta = 9,\ \beta_{SA} = 0.025,\ \beta_{IA} = 0.25,\ \eta = 0.8$ with initial restrictions $S(0) = 3,\ I(0) = 95,\ R(0) = 1,\ A(0) = 1$.

Now the generalization of the above system (18) in the new modified Caputo-type non-classical derivative sense is as follows:

$$\begin{cases} {}^{C}D_t^{\theta,\rho}S = -\beta_{SA}SA - \alpha SI + \eta R, \\ {}^{C}D_t^{\theta,\rho}I = -\beta_{IA}IA + \alpha SI - \zeta I, \\ {}^{C}D_t^{\theta,\rho}R = -\eta R + \zeta I, \\ {}^{C}D_t^{\theta,\rho}A = \beta_{SA}SA + \beta_{IA}IA, \end{cases} \tag{19}$$

The related integer order system given in Theorem 4.2 is

$$\begin{cases} \dfrac{dS^*}{dt} = -\beta_{SA} S^* A^* - \alpha S^* I^* + \eta R^*, \\ \dfrac{dI^*}{dt} = -\beta_{IA} I^* A^* + \alpha S^* I^* - \zeta I^*, \\ \dfrac{dR^*}{dt} = -\eta R^* + \zeta I^*, \\ \dfrac{dA^*}{dt} = \beta_{SA} S^* A^* + \beta_{IA} I^* A^*, \end{cases}$$

If $(S_*(\nu),\ I_*(\nu),\ R_*(\nu),\ A_*(\nu)$, is the solution of this classical model then the solution of the system (19) is $(S_*(t^\theta \rho^{-\theta}/\Gamma(\theta+1)),\ I_*(t^\theta \rho^{-\theta}/\Gamma(\theta+1)),\ R_*(t^\theta \rho^{-\theta}/\Gamma(\theta+1)),\ A_*(t^\theta \rho^{-\theta}/\Gamma(\theta+1)))$. So the numerical solution of the system (19) for the given numerical values is calculated using the method of Theorem 4.2. From the above graphical simulations, we studied the nature of $S(t)$ susceptible, non-infected $A(t)$, infected $I(t)$ and removed $R(t)$ computers in Figures 9, 10, 11 and 12 respectively. In the given group of Figure 13 (fig 9, 10, 11 and 12), we settled the parameter values for DFE case. To perform the simulations for EE case, we studied the nature of $S(t)$ susceptible, non-infected $A(t)$, infected $I(t)$ and removed $R(t)$ computers in Figures 14, 15, 16 and 17 respectively. In the given group of Figure 18 (fig 14, 15, 16 and 17), we settled the parameter values for EE case. From the all above graphical calculations, we concluded that the given numerical technique works well to frame the structures of non-linear epidemic models.

6 Conclusion

In this article, we have proposed an generalised numerical algorithm for finding the solutions of non-linear new generalized Caputo-type non-classical differential equations of order $0 < \theta \leq 1$. We have proposed the propinquity between the solutions of classical and fractional-order systems. A generalized form of the given technique to finite systems is also presented. By the given scheme, we can find the numerical or exact solutions of the important FDEs in the terms of classical differential equations solution, which is the main benefit of the proposed scheme. This technique is precious in the applications of FDEs in various fields. There are so many various types of numerical techniques

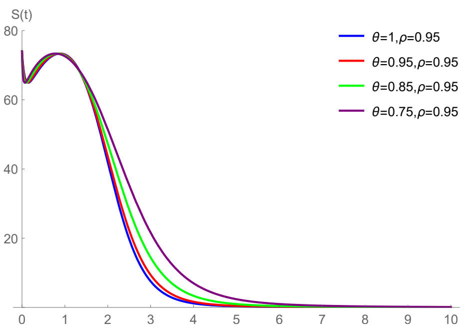

Figure 9: Nature of $S(t)$ for DFE case

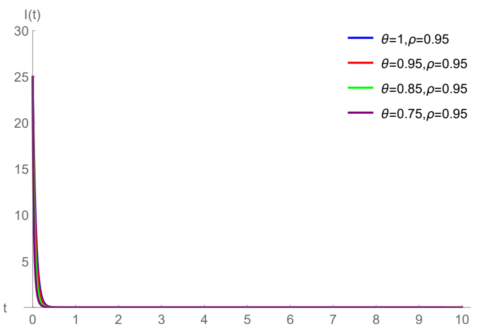

Figure 10: Nature of $I(t)$ for DFE case

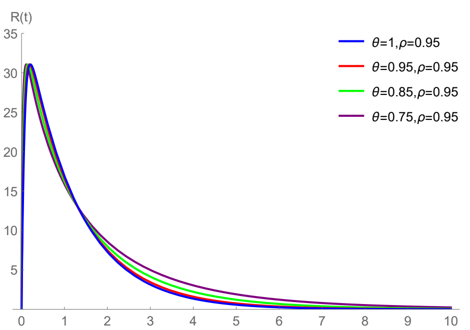

Figure 11: Nature of $R(t)$ for DFE case

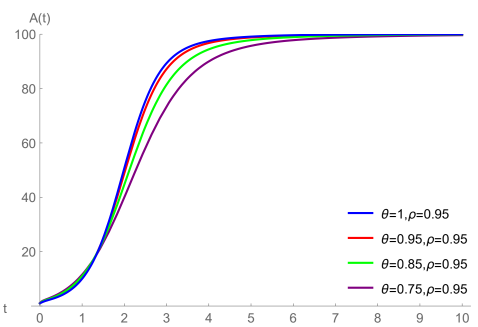

Figure 12: Nature of $A(t)$ for DFE case

Figure 13: Nature of all given classes for DFE case

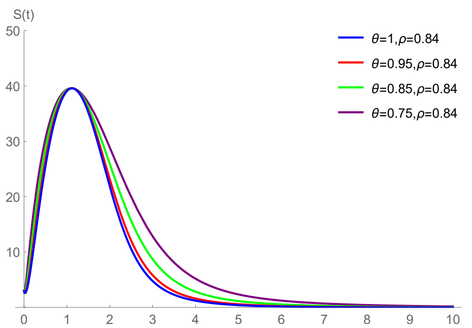

Figure 14: Nature of $S(t)$ for EE case

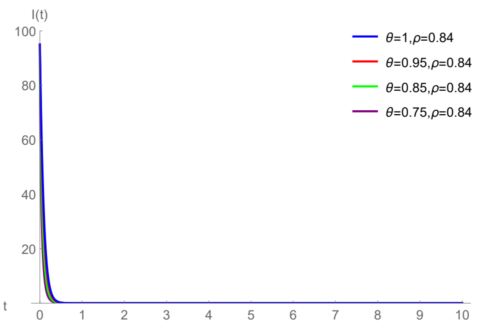

Figure 15: Nature of $I(t)$ for EE case

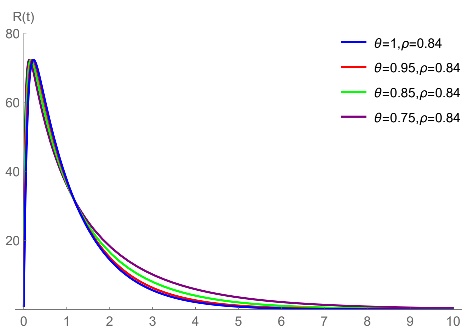

Figure 16: Nature of $R(t)$ for EE case

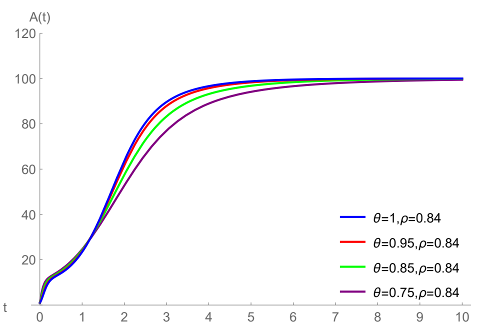

Figure 17: Nature of $A(t)$ for EE case

Figure 18: Nature of all given classes for EE case

are available to find the solutions of applied fractional order problems but the techniques for integer-order equations are much stronger in the view of chastity and convergence rate. By this scheme, we can use the numerical techniques of classical differential equations for the numerical solutions of non-integer order differential equations. Some important examples are explained with the comparison of their exact solutions to the numerical solutions founded by some other techniques. Solution of a computer virus epidemic model is also given to prove the availability of the proposed method in mathematical epidemiology. In the future, the given technique will be very useful to solve different kinds of important FDEs and also will be easy in the implementations.

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