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Operator Arithmetic-Harmonic Mean Inequality on Krein Spaces

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Abstract. We prove an operator arithmetic-harmonic mean type inequality in Krein space setting, by using some block matrix techniques of indefinite type. We also give an example which shows that the operator arithmetic-geometric-harmonic mean inequality for two invertible selfadjoint operators on Krein spaces is not valid, in general.

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1. Introduction and Preliminaries

A Krein space is a triple $(\mathscr{H}, \langle\cdot,\cdot\rangle, J)$ such that $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $J : \mathscr{H} \to \mathscr{H}$ is a selfadjoint involution i.e., $J = J^* = J^{-1}$ which defines an indefinite inner product on $\mathscr{H}$, given by

$$[x, y] := \langle Jx, y\rangle \quad (x, y \in \mathscr{H}).$$

Note that the indefinite inner product space is not assumed to be positive, that is, $[x, x]$ may be negative for some $x \in \mathscr{H}$. We denote this Krein space by $(\mathscr{H}, J)$.

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Let $(\mathscr{H}, J)$ be a Krein space and let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ with the identity I. An operator $T \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Tx, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. We denote by $\mathbb{B}^+(\mathscr{H})$ the subspace of all positive operators on $\mathscr{H}$. If T is a positive invertible operator we write $T > 0$. For bounded selfadjoint operators T and S on $\mathscr{H}$, we say $T \leqslant S$ if $S - T \geqslant 0$.
The J-adjoint operator of $A \in \mathbb{B}(\mathscr{H})$ is defined by

$$[Ax, y] = [x, A^\sharp y], \qquad (x, y \in \mathscr{H}),$$

which is equivalent to say that $A^\sharp = JA^*J$. An operator $A \in \mathbb{B}(\mathscr{H})$ is said to be J-selfadjoint if $A^\sharp = A$, or equivalently, $A = JA^*J$.
For J-selfadjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ the J-order, denoted by $A \leqslant^J B$, is defined by

$$[Ax, x] \leqslant [Bx, x], \qquad (x \in \mathscr{H}).$$

Clearly $A \leqslant^J B$ if and only if $JA \leqslant JB$ $(AJ \leqslant BJ)$. The J-selfadjoint operator $A \in \mathbb{B}(\mathscr{H})$ is said to be J-positive if $A \geqslant^J 0$. For a complete exposition on the subject see [2, 6, 11].
The theory of matrix and operator means started from the presence of the notion of parallel sum in engineering by Anderson and Duffin [1]. An axiomatic theory of matrix means was developed in [10] by Kubo and Ando. Three classical means, namely, arithmetic mean, harmonic mean and geometric mean for matrices and operators are considered in [4, 5].
A binary operation $\cdot : \mathbb{B}^+(\mathscr{H}) \times \mathbb{B}^+(\mathscr{H}) \to \mathbb{B}^+(\mathscr{H})$, $(A, B) \mapsto A\tau B$ is called an operator mean if the following conditions are satisfied:
(i) $A \leqslant C$, $B \leqslant D$ imply $A\tau B \leqslant C\tau D$.
(ii) $A_n \searrow A, B_n \searrow B$ imply $A_n\tau B_n \searrow A\tau B$.
(iii) $T^*(A\tau B)T \leqslant (T^*AT)\tau(T^*BT)$ for all $T \in \mathbb{B}(\mathscr{H})$.
(iv) $I\tau I = I$ cf. [9, Chapter 5].
Let A and B be positive operators on a Hilbert space $\mathscr{H}$. Then their arithmetic mean is defined by

$$A\nabla_\lambda B = \lambda A + (1-\lambda)B \quad (\lambda \in [0,1]). \tag{1}$$

If $A > 0$ and $B > 0$, then the harmonic mean $A!_\lambda B$ is defined by

$$A!_\lambda B = \left(\lambda A^{-1} + (1-\lambda)B^{-1}\right)^{-1} \ \ (\lambda \in [0,1]) \tag{2}$$

and the geometric mean $A\sharp B$ between A and B is defined as follows:

$$A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

We denote $A\nabla B := A\nabla_{\frac{1}{2}}B$ and $A!B := A!_{\frac{1}{2}}B$.
The following arithmetic-geometric-harmonic mean inequality hold, see [4].

$$A!_\lambda B \leqslant A\sharp B \leqslant A\nabla_\lambda B. \tag{3}$$

Matrix and operator inequalities in the setting of Krein spaces is a fascinating subject of operator theory. For instance, operator monotone functions in finite dimensional Krein spaces (specially, the Lowner inequality) has been studied in [3]. In addition, a notion of operator convexity in Krein spaces was studied recently, by Moslehian and Dehghani [11].
In this Note, we consider the notions of arithmetic and harmonic mean of two J-positive operators on a Krein space $(\mathscr{H}, J)$. We will prove the operator arithmetic-harmonic mean inequality on Krein spaces, by using some block matrix techniques of indefinite type. We describe appropriate conditions to define the notion of power mean for two invertible J-selfadjoint operators on a Krein space $(\mathscr{H}, J)$. Also we give an example which shows that the inequality (3) is not correct for operators on Krein spaces, in general.

2. Main Results

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces. It is well-know that an operator $\mathbf{A} \in \mathbb{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ is uniquely determined by the bounded linear operators $A_{ij} : \mathscr{H}_j \to \mathscr{H}_i$ $(1 \leqslant i, j \leqslant 2)$. We write $\mathbf{A}$ by the block matrix

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{4}$$

The diagonal block matrix $\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$ is denoted by $A_{11} \oplus A_{22}$.
Let $(\mathscr{H}, J)$ be a Krein space. We consider the selfadjoint involution

$\tilde{\mathbf{J}} = J \oplus J$ on the Hilbert space $\mathscr{H} \oplus \mathscr{H}$. Therefore $(\mathscr{H} \oplus \mathscr{H}, \tilde{\mathbf{J}})$ is a Krein space. Let $\mathbf{A} \in \mathbb{B}(\mathscr{H} \oplus \mathscr{H})$ be the block matrix introduced in (4). Note that

$$\mathbf{A}^\sharp = \tilde{\mathbf{J}}\mathbf{A}^*\tilde{\mathbf{J}} = \begin{pmatrix} JA_{11}^*J & JA_{21}^*J \\ JA_{12}^*J & JA_{22}^*J \end{pmatrix}.$$

Therefore $\mathbf{A}$ is $\tilde{\mathbf{J}}$-selfadjoint if and only if $\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\sharp & A_{22} \end{pmatrix}$ in which A_{11} and A_{22} are J-selfadjoint cf. [8]

We need the following lemma, which is a consequence of [8, Theorem 8] in the setting of Krein spaces.

Lemma 2.1. *Let $(\mathscr{H}, J)$ be a Krein space. Suppose that A and B are J-selfadjoint operators. If A is invertible, then the operator $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive if and only if $A \geqslant^J 0$ and $X^\sharp A^{-1} X \leqslant^J B$.*

Corollary 2.2. *Let $(\mathscr{H}, J)$ be a Krein space. If C is an invertible J-positive operator on $\mathscr{H}$, then the operator $\begin{pmatrix} C & I \\ I & C^{-1} \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive.*

Proof. Let $A = C$, $B = C^{-1}$ and $X = I$ in Lemma 2.1.

A real valued continuous function f on an interval $\mathcal{I}$ is said to be operator monotone if $A \leqslant B$ implies $f(A) \leqslant f(B)$ for all selfadjoint operators A and B on a Hilbert space $\mathscr{H}$ whose spectra are contained in $\mathcal{I}$, where $f(A)$ is defined by the usual functional calculus for a selfadjoint operator [9, Chapter 1]. $\square$

Lemma 2.3. [9, Example 1.6] *The function $f(t) = -\frac{1}{t}$ is operator monotone on $(0, \infty)$.*

Operator means for Krein space operators is naturally defined as follows:

Definition 2.4. *Let $(\mathscr{H}, J)$ be a Krein space and let $\mathbb{B}_J^+(\mathscr{H})$ be the space of all J-positive operators on $\mathscr{H}$. A binary operation $\cdot : \mathbb{B}_J^+(\mathscr{H}) \times \mathbb{B}_J^+(\mathscr{H}) \to \mathbb{B}_J^+(\mathscr{H})$, $(A, B) \mapsto A\tau B$ is called an operator mean if the following conditions are satisfied:*

(i) $A \leqslant^J C$, $B \leqslant^J D$ imply $A\tau B \leqslant^J C\tau D$.

(ii) $A_n \searrow A, B_n \searrow B$ *imply* $A_n \tau B_n \searrow A\tau B$.
(iii) $T^\sharp(A\tau B)T \leqslant^J (T^\sharp AT)\tau(T^\sharp BT)$ *for all* $T \in \mathbb{B}(\mathscr{H})$.
(iv) $I\tau I = I$.
The arithmetic and harmonic means of two J-positive operators are defined by (1) and (2), respectively. Indeed, suppose that A and B are J-positive operators on a Krein space $(\mathscr{H}, J)$. Clearly $A\nabla_\lambda B \geqslant^J 0$. If A and B are invertible, then the J-positivity of A^{-1} and B^{-1} implies that $A!_\lambda B \geqslant^J 0$. It is easy to see that other properties of an operator mean (properties (i)-(iv) of Definition 2.4) are satisfied by replacing $\leqslant$ and $*$ by $\leqslant^J$ and $\sharp$, respectively. Therefore $A\nabla_\lambda B$ and $A!_\lambda B$ can be regarded as means of two J-positive operators.
One may immediately say that if A and B are invertible J-positive operators, then $JA > 0$ and $JB > 0$. It follows from the usual operator arithmetic-harmonic mean inequality that $JA!_\lambda JB \leqslant JA\nabla_\lambda JB$. Therefore $A!_\lambda B \leqslant^J A\nabla_\lambda B$. In the following theorem a direct proof of this inequality (without using the usual operator arithmetic-harmonic mean inequality) is provided.

Theorem 2.5. *Let* $(\mathscr{H}, J)$ *be a Krein space. If* A *and* B *are invertible* J*-positive operators on* $\mathscr{H}$*, then*

$$A!_\lambda B \leqslant^J A\nabla_\lambda B.$$

Proof. Let $\tilde{\mathbf{A}} = \begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix}$ and $\tilde{\mathbf{B}} = \begin{pmatrix} B & I \\ I & B^{-1} \end{pmatrix}$. Then Corollary 2.2 implies that $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are $\tilde{\mathbf{J}}$-positive. Therefore,

$$\lambda\tilde{\mathbf{A}} + (1-\lambda)\tilde{\mathbf{B}} = \begin{pmatrix} \lambda A + (1-\lambda)B & I \\ I & \lambda A^{-1} + (1-\lambda)B^{-1} \end{pmatrix}$$

is $\tilde{\mathbf{J}}$-positive for all $\lambda \in [0,1]$. Lemma 2.1 implies that

$$\Big(\lambda A + (1-\lambda)B\Big)^{-1} \leqslant^J \lambda A^{-1} + (1-\lambda)B^{-1}.$$

By the definition, we have

$$J\Big(\lambda A + (1-\lambda)B\Big)^{-1} \leqslant J\Big(\lambda A^{-1} + (1-\lambda)B^{-1}\Big)$$

which is equivalent to

$$\Big(\lambda AJ + (1-\lambda)BJ\Big)^{-1} \leqslant \lambda(AJ)^{-1} + (1-\lambda)(BJ)^{-1}.$$

By the assumption, $AJ > 0$, $BJ > 0$ so is $\lambda AJ + (1-\lambda)BJ$. It follows from Lemma 2.3 that

$$\Big(\lambda(AJ)^{-1} + (1-\lambda)(BJ)^{-1}\Big)^{-1} \leqslant \lambda AJ + (1-\lambda)BJ.$$

So

$$\Big(\lambda A^{-1} + (1-\lambda)B^{-1}\Big)^{-1} J \leqslant \Big(\lambda A + (1-\lambda)B\Big)J.$$

Hence $A!_\lambda B \leqslant^J A\nabla_\lambda B$. $\square$

It is well-known that the spectrum of a J-positive operator on a Krein space $(\mathscr{H}, J)$ is real and it contains a non-negative number as well as a non-positive one; see [3, Theorem 2.1]. According to this fact, the square root of a J-positive operator can not be defined by usual functional calculus such as a positive operator. Let J be a selfadjoint involution on $\mathbb{C}^n$. For a J-selfadjoint matrix A with nonnegative eigenvalues on Krein space $(\mathbb{C}^n, J)$, the J-selfadjoint square root $A^{\frac{1}{2}}$ was defined by Ando [4, Lemma 5]. Moreover A^α was defined by Sano in [12] for all $0 < \alpha < 1$. By a similar argument, for the J-selfadjoint operator C on a Krein space $(\mathscr{H}, J)$ with positive spectrum, the J-selfadjoint square root of C is defined by the Riesz-Dunford integral as follows:

$$C^{\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} C(\lambda I + C)^{-1} d\lambda. \tag{5}$$

An operator $C \in \mathbb{B}(\mathscr{H})$ on a Krein space $(\mathscr{H}, J)$ is called a J-contraction if $C^\sharp C \leqslant^J I$. The operator C is called a J-bicontraction if both C and $C^\sharp$ are J-contractions. Note that in contrast to the setting of Hilbert spaces, not all J-contractions are J-bicontractions. As a result of Potapov-Ginzburg theorem [6, Chapter 2, Section 4] we have the following proposition; also see [3, Corollary 3.4.1].

Proposition 2.6. *Let $(\mathscr{H}, J)$ be a Krein space and let $C \in \mathbb{B}(\mathscr{H})$. Then C is a J-bicontraction if and only if $\sigma(C^\sharp C) \subseteq [0, \infty)$.*

Moreover The following proposition appropriate a condition for a J-contraction to being a J-bicontraction.

Proposition 2.7. [3, Corollary 3.3.3] *Let $(\mathscr{H}, J)$ be a Krein space. If $C \in \mathbb{B}(\mathscr{H})$ is an invertible J-contraction, then C is a J-bicontraction. The notion of α-power mean for two J-selfajoint matrices with non-negative eigenvalues was defined by Bebiano et al. in [7]. Now, we are going to construct power mean of two invertible J-selfadjoint operators on Krein spaces.*

Let A and B be invertible J-selfadjoint operators on a Krein space $(\mathscr{H}, J)$ with nonnegative spectrum such that $A \geqslant^J B$. Then $A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leqslant^J I$. By the definition, we have

$$\begin{aligned}
(B^{\frac{1}{2}} A^{-\frac{1}{2}})^{\sharp} B^{\frac{1}{2}} A^{-\frac{1}{2}} &= J(B^{\frac{1}{2}} A^{-\frac{1}{2}})^* J B^{\frac{1}{2}} A^{-\frac{1}{2}} \\
&= J(A^{-\frac{1}{2}})^* (B^{\frac{1}{2}})^* J B^{\frac{1}{2}} A^{-\frac{1}{2}} \\
&= A^{-\frac{1}{2}} J (B^{\frac{1}{2}})^* J B^{\frac{1}{2}} A^{-\frac{1}{2}} \qquad (\ A^{-\frac{1}{2}} \text{ is } J\text{-selfadjoint }) \\
&= A^{-\frac{1}{2}} J^2 B^{\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} \qquad (\ B^{\frac{1}{2}} \text{ is } J\text{-selfadjoint }) \\
&= A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leqslant^J I.
\end{aligned}$$

Therefore $B^{\frac{1}{2}} A^{-\frac{1}{2}}$ is an invertible J-contraction. It follows from Propositions 2.6 and 2.7 that $\sigma(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \subseteq (0, \infty)$. Then the operator $(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}}$ is well defined by Riesz-Dunford integral (5). Therefore, the power mean of A and B is well-defined as follows:

$$A \natural B := A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Since B is J-selfadjoint it is easy to see that $A \natural B$ is J-selfadjoint. Note that this notion is like the geometric mean of two positive operators, but in fact, it is not a mean. For instance it is not J-positive, in general; see Example 2.8.

Let $\mathbb{C}^n$ be the n-dimensional complex Hilbert space consisting of all column vectors $x = (x_1, x_2, \cdots, x_n)$ for which $x_j \in \mathbb{C}$ $(j = 1, 2, \cdots, n)$. The standard inner product in $\mathbb{C}^n$ is denoted by $\langle ., . \rangle$. The formula

$$[x, y] = \sum_{k=1}^{n-1} x_k \bar{y}_k - x_n \bar{y}_n \qquad (x, y \in \mathbb{C}^n)$$

determines an indefinite inner product on $\mathbb{C}^n$. It is clear that the selfadjoint involution corresponding to this indefinite inner product is $J_0 = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$, where I_{n-1} denotes the identity matrix of order $n-1$, and

$$[x, y] = \langle J_0 x, y \rangle \qquad (x, y \in \mathbb{C}^n).$$

The Krein space $(\mathbb{C}^n, J_0)$ is called the n-dimensional Minkowski space. The following example shows that the J-positivity of operators in Theorem 2.5 is an essential assumption. Also, it shows that the arithmetic-geometric-harmonic mean inequality for operators on Hilbert spaces is not true for operators on Krein spaces, in general.

Example 2.8. Consider the 2-dimensional Minkowski space $(\mathbb{C}^2, J_0)$ with $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Suppose that $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a 2×2 complex J_0-selfadjoint matrix. Since the J_0-selfadjoitness of A is equivalent to the usual selfadjointness of $J_0 A$, we have $A = \begin{pmatrix} a_{11} & a_{12} \\ -\overline{a_{12}} & a_{22} \end{pmatrix}$ in which a_{11} and a_{22} are real.
Let $A = \begin{pmatrix} 2 & \frac{1}{4} \\ -\frac{1}{4} & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & \frac{1}{3} \\ -\frac{1}{3} & 2 \end{pmatrix}$. Then A and B are J_0-selfadjoint with positive eigenvalues and

$$J_0(A - B) = \begin{pmatrix} 1 & 0.0833 \\ -0.0833 & 1 \end{pmatrix}$$

is positive. It follows that $A \geqslant^{J_0} B$. Some matrix calculation shows

$$A \natural B = \begin{pmatrix} 1.4208 & 0.2755 \\ -0.2755 & 1.4153 \end{pmatrix}$$

and

$$\frac{A + B}{2} - A \natural B = \begin{pmatrix} 0.0792 & 0.0162 \\ -0.0162 & 0.0847 \end{pmatrix}.$$

The matrix $J_0(\frac{A+B}{2} - A \natural B)$ has a negative eigenvalue. It follows that

$$A \natural B \nleqslant^{J_0} \frac{A + B}{2} = A \nabla B.$$

Moreover

$$A\natural B - 2(A^{-1} + B^{-1})^{-1} = \begin{pmatrix} 0.0750 & 0.0153 \\ -0.0153 & 0.0799 \end{pmatrix}.$$

The matrix $J_0\big(A\natural B - 2(A^{-1}+B^{-1})^{-1}\big)$ has a negative eigenvalue. Hence

$$A!B = 2(A^{-1} + B^{-1})^{-1} \nleq^{J_0} A\natural B.$$

Therefore

$$A!B \nleq^{J_0} A\natural B \nleq^{J_0} A\nabla B.$$

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