Journal of Mathematical Extension
Vol. 15, SI-NTFCA, (2021) (3)1-23
URL: https://doi.org/10.30495/JME.SI.2021.2045
ISSN: 1735-8299
Original Research Paper

Concave Multifunctions and the Hammerstein Integral inclusion Problem

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Abstract. This paper presents sufficient conditions for the existence of positive solutions for the nonlinear Hammerstein and quadratic integral inclusion. In this way, we use concave multifunctions and fixed point theory for obtaining results. In fact, for solving the Hammerstein integral inclusion problem, we use fixed point of some concave multifunctions.

AMS Subject Classification: 34A60; 47H10; 47H07.
Keywords and Phrases: Concave multifunction, Dedekind complete, Hammerstein integral inclusion, Quadratic integral inclusion.

1 Introduction

A large number of real life problems can be framed as linear or nonlinear differential equations (see [3], [6], [7], [15], [21], [25], [27]) or differential inclusions(see [1], [9], [14], [24]). Using fixed point theorem of mappings and multifunctions is one of oldest and most well-known techniques to prove the existence of solutions for differential equations and inclusions. Picard ([26]) was the first to prove the existence of a positive solution

Received: June 2021; Published: July 2021
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for differential and integral equations by using concave operators. Later, u_0-concavity, ordered concavity(convexity), and $\alpha-$concavity(convexity) were introduced by Krasnoselskii([16],[17]), Amann ([5]), and Potter ([23]), respectively. In [20] the equivalent conditions to the existence of fixed point for u_0-concave operator were obtained. In [8], [19], [33], [28], and [29] fixed point theorems for some other types of concave operators were proved.

Upper and lower solutions([22]), compactness and continuity assumptions play important roles in order to prove the existence of positive solutions for nonlinear differential and integral equations and inclusions, which is difficult to verify for concrete nonlinear operators. One of the interesting and important techniques to remove or weaken these conditions is using monotone concave operators. See [12], [18], [31], [30], [32], and [34], [35] for some applications which include the existence of positive solutions.

The paper is organized as follows: In Section 2, we introduce some of the preliminaries needed for the next sections. In this section, we prove two lemmas; the first is essential to prove the main results and the second is an important equivalence between two types of concave multifunctions. In Section 3, we establish the existence of results for u_0-concave and $\alpha(t)-$concave multifunctions. Furthermore, we provide some examples that satisfy main results. In last section, we provide our main result about the existence of positive solutions for the Hammerstein and quadratic integral inclusions.

2 Preliminaries

Throughout this paper, we assume that E is a real Banach space which is partially ordered by a cone $P \subseteq E$, i.e., $x \leq y$(or $y \geq x$) iff $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$(or $y > x$). Note that the nonempty closed convex set $P \subset E$ is a cone if it satisfies $x \in P,\ \lambda \geq 0 \Rightarrow \lambda x \in P$ and $x \in P,\ -x \in P \Rightarrow x = \theta$. A cone P is said to be normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $||x|| \leq N||y||$, where $x, y \in E$ and θ denotes the zero element of E. It is easy to prove that the following assumptions are equivalent([11]):

(1) A cone P is normal,

(2) If $x_n \leq z_n \leq y_n$ (for all $n \in \mathbb{N}$), $||x_n - x|| \to 0$ and $||y_n - x|| \to 0$ (as $n \to \infty$), then $||z_n - x|| \to 0$ (as $n \to \infty$).
For $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E : x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. Given $h > \theta$, let P_h be the set

$$P_h = \{x \mid x \in E,\ \exists \lambda(x), \mu(x) > 0,\ s.t.\ \lambda(x)h \leq x \leq \mu(x)h\}.$$

It is easy to see that $P_h \subset P$.

Definition 2.1. ([13]) For two subsets X and Y of E, we write $X \leq Y$ if for any $x \in X$ there exists some $y \in Y$ such that $x \leq y$.

Also for subsets X, Y and Z of E we write $X \leq Y \leq Z$ if $X \leq Y$ and $Y \leq Z$. For $\{x\} \leq Y$ or $Y \leq \{x\}$ we write $x \leq Y$ or $Y \leq x$, respectively. Obviously if $Y \leq x$, then for all $y \in Y$ we have $y \leq x$.

Definition 2.2. ([13]) Given a nonempty subset D of E, we say that $A : D \to 2^E$ is increasing (decreasing) on D if for $x, y \in D$ such that $x \leq y$, we have $Ax \leq Ay (Ax \geq Ay)$.

Definition 2.3. A multifunction $A : P \to 2^P$ is said to be a u_0-concave multifunction ($u_0 > \theta$) if A satisfies the following two conditions:
(i) for any $x > \theta$, $Ax \subseteq P_{u_0}$;
(ii) for $x \in P$ and $t \in (0, 1)$ there exists a positive number $\eta = \eta(t, x)$ such that $A(tx) \geq t(1 + \eta)Ax$.

Definition 2.4. A multifunction $A : P_h \to 2^{P_h}$ ($h > 0$) is said to be an $\alpha(t, x)$-concave multifunction if there exists $\alpha : (0, 1) \times P_h \longrightarrow (0, 1)$ such that $\alpha(t, x) > t$ and $A(tx) \geq \alpha(t, x)Ax$ for all $x \in P_h$ and $t \in (0, 1)$.

Definition 2.5. A multifunction $A : P_h \to 2^{P_h}$ ($h > 0$) is said to be a $\alpha(t)$-concave multifunction if there exists $\alpha : (0, 1) \longrightarrow (0, 1)$ such that $\alpha(t) > t$ and $A(tx) \geq \alpha(t)Ax$ for all $t \in (0, 1)$ and $x \in P_h$.

A subset X of a E is called order bounded ([4]) if there is a $u \in E$ such that $x \leq u$ for all $x \in X$. In this case, u is called the upper bound of X.

Definition 2.6. ([4]) E is called Dedekind complete (or order complete) if every nonempty subset, which is order bounded, has a supremum in E.

Definition 2.7. The subset X of E is order closed if $\sup X$ belongs to X.

If $X \subseteq E$, then we denote by $B(X)$ the class of nonempty and order bounded subsets of X, by $OB(X)$ the class of nonempty, order bounded and order closed subsets of X, and by $OCB(X)$ the class of nonempty, norm closed, order bounded and order closed subsets of X.

Lemma 2.8. *Let E be Dedekind complete, $P \subset E$ be a normal cone and $A : P \to OB(P)$ be an increasing u_0-concave multifunction (for some $u_0 > \theta$). Suppose that there exist $w_0 \in P$ ($w_0 \neq \theta$), $v_0 \in P_{u_0}$, and $0 < \lambda_0 < 1$, s.t. $\lambda_0 v_0 \leq w_0 \leq v_0$. Then there exist two sequences $\{v_n\}$ and $\{w_n\}$ in P, and a real sequence λ_n such that for all $n \in \mathbb{N}$ we have*

$$\lambda_n v_n \leq w_n \leq v_n$$

in which $0 < \lambda_n = \lambda_0(1+\eta_n)^n < 1$ where, for any $n \in \mathbb{N}$, η_n is a positive real number.

Proof. Since A is increasing, then

$$A(\lambda_0 v_0) \leq A(w_0) \leq A(v_0). \tag{1}$$

In addition

$$A(\lambda_0 v_0) \geq \lambda_1 A(v_0) \tag{2}$$

where $0 < \lambda_1 = \lambda_0(1 + \eta_1) < 1$. By (1), (2) we get

$$\lambda_1 A(v_0) \leq A(w_0). \tag{3}$$

Let $v_1 = \sup A(v_0)$ and $w_1 = \sup A(w_0)$. Since E is Dedekind complete and $A(v_0)$, $A(w_0) \in OB(P)$, then $v_1 \in A(v_0)$ and $w_1 \in A(w_0)$. By (3), for some $w_1' \in A(w_0)$, $\lambda_1 v_1 \leq w_1'$ which implies that $w_1' \leq w_1$. Therefore, we have $\lambda_1 v_1 \leq w_1$. By $Aw_0 \leq Av_0$, there exists $v_1' \in Av_0$ such that $w_1 \leq v_1'$. Since $v_1 = A(v_0)$, then $w_1 \leq v_1$. Finally, we have

$$\lambda_1 v_1 \leq w_1 \leq v_1.$$

Hence

$$A(\lambda_1 v_1) \leq A(w_1) \leq A(v_1). \tag{4}$$

For some real number η_2', we have

$$A(\lambda_1 v_1) \geq \lambda_1(1+\eta_2')A(v_1),$$

where $0 < \lambda_1(1+\eta_2') < 1$. Let $\eta_2 = \min\{\eta_1, \eta_2'\}$. So

$$A(\lambda_1 v_1) \geq \lambda_1(1+\eta_2)A(v_1),$$

which implies

$$A(\lambda_1 v_1) \geq \lambda_0(1+\eta_1)(1+\eta_2)A(v_1).$$

Since $\eta_2 \leq \eta_1$, then

$$A(\lambda_1 v_1) \geq \lambda_0(1+\eta_2)^2 A(v_1).$$

If $\lambda_2 = \lambda_0(1+\eta_2)^2$, then

$$A(\lambda_1 v_1) \geq \lambda_2 A(v_1). \tag{5}$$

Similarly, by putting $v_2 = \sup A(v_1)$, $w_2 = \sup A(w_1)$, and using (4) and (5), we can get

$$\lambda_2 v_2 \leq w_2 \leq v_2.$$

By repeating this process, we obtain sequences $\{v_n\}$ and $\{w_n\}$ such that $v_n = \sup A(v_{n-1})$, $w_n = \sup A(w_{n-1})$ and a real sequence $0 < \lambda_n = \lambda_0(1+\eta_n) < 1$, in which $\eta_n = \min\{\eta_{n-1}, \eta_{n-1}'\}$ that is satisfying

$$\lambda_n v_n \leq w_n \leq v_n.$$

A real sequence η_n is called an adjoint sequence of A in w_0, v_0, and λ_0. □

The next example shows that Dedekind complete assumption in Lemma (2.8) is the sufficient condition. However, it is not necessary.

Example 2.9. Consider Banach space $(E, ||.||_\infty)$ where $E = C[0,1]$ is the vector space of a continuous function on $[0,1]$ and $||.||_\infty$ is the usual supremum norm ([4]). Assume that $P \subseteq E$ is the cone of non-negative continuous functions on $[0,1]$. Let us define multifunction $A : P \to OB(P)$ with $Ax = [\sqrt[3]{x}, \sqrt[3]{x}+1]$. Obviously A is increasing. For $u_0 = 1$, it is easy to notice that $P_{u_0} = \{x \in E \mid x > 0\}$. Thus for any $x \in P_{u_0}$, we have $Ax \subseteq P_{u_0}$. Also, for any $t \in (0,1)$, we have

$$\begin{aligned} A(tx) &= [\sqrt[3]{tx}, \sqrt[3]{tx}+1] = [\sqrt[3]{t}\sqrt[3]{x}, \sqrt[3]{t}\sqrt[3]{x}+1] \\ &\geq \sqrt{t}(1+\eta)[\sqrt[3]{x}, \sqrt[3]{x}+1] = \sqrt{t}(1+\eta)A(x) \end{aligned}$$

where $0 < \eta < \frac{1}{\sqrt[6]{t}} - 1$. Therefore, A is a u_0-concave multifunction. Now let $v_0 = 3$ and $w_0 = 2$. Since $\sup_{x\in[0,1]}[\sqrt[3]{x}, \sqrt[3]{x}+1] = \sqrt[3]{x}+1$, then we can make sequences $\{v_n\}$, and $\{w_n\}$ similar to Lemma 2.8. However $(C[0,1], ||.||_1)$ is not Dedekind complete ([4]).

Lemma 2.10. *Let $u_0 > \theta$ and*
(a), $A : P \to 2^P$ is a u_0-concave multifunction. Then $A|P_{u_0}$(A confined to P_{u_0}) is an $\alpha(t,x)$-concave multifunction.
(b), $A : P_{u_0} \to 2^{P_{u_0}}$ is an $\alpha(t,x)$-concave multifunction. Then A is a u_0-concave multifunction.

Proof. (a): By using (i) of Definition 2.3 for all $x \in P_{u_0}$, we have $Ax \subset P_{u_0}$. Set $\alpha(t,x) = t(1+\eta(t,x))$. Thus for all $x \in P_{u_0}$ and $t \in (0,1)$ we have $\alpha(t,x) > t$. Hence $A|P_{u_0} : P_{u_0} \to 2^{P_{u_0}}$ is an $\alpha(t,x)$-concave multifunction.
(b): Since $A : P_{u_0} \to 2^{P_{u_0}}$, then A satisfies condition (i) of Definition 2.3. Set $\eta(t,x) = \frac{\alpha(t,x)}{t} - 1$ (for all $x \in P_{u_0}$ and $t \in (0,1)$). Since for all $x \in P_{u_0}$ we have $A(tx) \geq \alpha(t,x)A(x)$, then

$$t(1+\eta)A(x) = t(1+\frac{\alpha(t,x)}{t} - 1)A(x) = \alpha(t,x)A(x) \leq A(tx).$$

Therefore, A satisfies condition (ii) of Definition 2.3. $\square$

We note that every $\alpha(t)$-concave multifunction is a u_0-concave multifunction.

3 Main Results

Now the main results could be stated and proved.

Theorem 3.1. *Let $P \subseteq E$ be a normal cone, and $A : P \to OCB(P)$ be an increasing u_0-concave multifunction (for some $u_0 > \theta$). Also, assume that*
(i) $\{v_n\}$ and $\{w_n\}$ are the defined sequences in Lemma 2.8 such that $v_0 \geq v_1$ and $w_0 \leq w_1$;
(ii) for the sequence η_n in Lemma 2.8, we have

$$\lim_{n\to\infty} n\eta_n = \ln(\frac{1}{\lambda_0}),$$

Then A has at least one fixed point.

Proof. Since A is increasing and $w_0 \leq w_1$, then we have $Aw_0 \leq Aw_1$. Also, since $w_2 = \sup A(w_1)$ and $w_1 \in Aw_0$, we get $w_1 \leq w_2$. Therefore $Aw_1 \leq Aw_2$. By repeating this process, we get

$$w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots . \tag{6}$$

Similarly, we get

$$v_0 \geq v_1 \geq \cdots \geq v_n \geq \cdots . \tag{7}$$

By Lemma 2.8, for each $n \in \mathbb{N}$, we have $w_n \geq \lambda_0(1+\eta_n)^n v_n$. Then

$$\theta \leq v_n - w_n \leq v_n - \lambda_0(1+\eta_n)^n v_n \leq (1-\lambda_0(1+\eta_n)^n)v_n.$$

In addition

$$\theta \leq w_{n+p} - w_n \leq v_{n+p} - w_n \leq v_n - w_n \leq (1-\lambda_0(1+\eta_n)^n)v_0,$$
$$\theta \leq v_n - v_{n+p} \leq v_n - w_{n+p} \leq v_n - w_n \leq (1-\lambda_0(1+\eta_n)^n)v_0.$$

Normality of P implies that

$$||w_{n+p} - w_n|| \leq N|1-\lambda_0(1+\eta_n)^n|||v_0||,$$

$$||v_{n+p} - v_n|| \leq N|1-\lambda_0(1+\eta_n)^n|||v_0||.$$

Because of the normality of P and that $(1-\lambda_0(1+\eta_n)^{(\frac{1}{\eta_n})^{n\eta_n}}) \to 0$ (as $n \to \infty$), $\{w_n\}$ and $\{v_n\}$ are Cauchy sequences. Since E is a Banach space and P is closed, then there exist $v^*, w^* \in P$ such that $w_n \to w^*$ and $v_n \to v^*$(as $n \to \infty$). So we have

$$\theta \leq v^* - w^* \leq v_n - w^* \leq v_n - w_n \leq (1-\lambda_0(1+\eta_n)^n)v_0.$$

Thus $||v^* - w^*|| \leq N||v_n - w_n||$. Now if $n \to \infty$, we have $v^* = w^*$. Let $x^* = v^* = w^*$. By (6) and (7), we get that $w_n \leq x^* \leq v_n$ and since A is increasing, we have $Aw_n \leq Ax^* \leq Av_n$. Also, since for each $n \in \mathbb{N}$, $w_{n+1} \in Aw_n$, then there exists $x_{n+1} \in Ax^*$ such that $w_{n+1} \leq x_{n+1}$. Since $v_{n+1} = \sup Av_n$, then $w_{n+1} \leq x_{n+1} \leq v_{n+1}$. Hence

$$\theta \leq v_{n+1} - x_{n+1} \leq v_{n+1} - w_{n+1}$$

By the normality of P we get

$$||v_{n+1} - x_{n+1}|| \leq N||v_{n+1} - w_{n+1}||.$$

Then $x_{n+1} \to x^*$(as $n \to \infty$). Since Ax^* is closed, then $x^* \in Ax^*$. □

Remark 3.2. As a result of Lemma 2.10, Theorem 3.1 holds true if we assume that $A : P_{u_0} \to OCB(P_{u_0})$ (for some $u_0 > \theta$) is $\alpha(t,x)$-concave multifunction.

In Example 3.3, using Lemma 2.8, we construct sequences $\{w_n\}$, $\{v_n\}$, and η_n that satisfy conditions (i) and (ii) of theorem 3.1.

Example 3.3. Let $E = \mathbb{R}$ and $P = [0, \infty)$. For any $u_0 > 0$, we have $P_{u_0} = (0, \infty)$. Define $A : [0, \infty) \to (0, \infty)$ as $A(x) = \sqrt{2 + x}$. It is easy to see that for any $u_0 > 0$, A is a u_0-concave function and it is increasing. Let $w_0 = A(0) = \sqrt{2}$, $w_1 = A(w_0) = \sqrt{2 + \sqrt{2}}$,

$$w_2 = A(w_1) = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$

and $w_n = A(w_{n-1}) = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}$ for all n. Also, put $v_0 = 3$, $v_1 = A(v_0) = \sqrt{5}$, $v_2 = A(v_1) = \sqrt{2 + \sqrt{5}}$ and

$$v_n = A(v_{n-1}) = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{5}}}}$$

for all n. We have $w_0 < v_0$, $v_0 \geq v_1$ and $w_0 \leq w_1$. Also for $\lambda_0 = \frac{1}{3}$ we have $\lambda_0 v_0 < w_0$. For any $n \in \mathbb{N}$ define $\lambda_n = \frac{w_n}{v_n}$. It can easily be noticed that for any $n \in \mathbb{N}$, we have $0 < \lambda_n < 1$ and $\lambda_n v_n \leq w_n$. Since $w_n \to 2$

and $v_n \to 2$ (as $n \to \infty$), we get $\lambda_n \to 1$ (as $n \to \infty$). Now let us define sequence $\eta_n = (\frac{\lambda_n}{\lambda_0})^{\frac{1}{n}} - 1$ (for any $n \in \mathbb{N}$). Since for any $n \in \mathbb{N}$ we have $0 < \lambda_n < 1$ and $\lambda_n \to 1$ ($n \to \infty$), then $\eta_n \to 0$ (as $n \to \infty$) and (for any $n \in \mathbb{N}$) we have $\eta_n > 0$. In addition

$$n\eta_n = \frac{\ln(\frac{\lambda_n}{\lambda_0})}{\ln(1+\eta_n)^{\frac{1}{\eta_n}}},$$

for each $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} n\eta_n = \frac{\ln(\frac{1}{\lambda_0})}{\ln(e)} = \ln(\frac{1}{\lambda_0}).$$

Then $\{\eta_n\}$ is an adjoint sequence of A in w_0, v_0 and λ_0. Also, we have proved that A satisfies in the all assumptions of Theorem 3.1.

Next we assume that A is $\alpha(t)$-concave multifunction and then we obtain a new result.

Lemma 3.4. *Let $A : P \to B(P)$ be an $\alpha(t)$-concave increasing multifunction and $A(h) \cap P_h \neq \emptyset$ (for some $h > \theta$). Then there are $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 \leq v_0$, $u_0 \leq Au_0 \leq Av_0 \leq v_0$.*

Proof. For some $h > \theta$ we have $Ah \cap P_h \neq \emptyset$. Let $a \in Ah \cap P_h$. Then there exist positive real $\mu(a)$ and $\lambda(a)$ such that $\lambda(a)h \leq a \leq \mu(a)h$. Let us choose $t_0 \in (0,1)$ such that

$$t_0 h \leq a \leq \frac{1}{t_0} h. \tag{8}$$

Since $\alpha(t_0) \in (t_0, 1]$, we can take positive integer k such that

$$(\frac{\alpha(t_0)}{t_0})^k \geq \frac{1}{t_0}. \tag{9}$$

Put $u_0 = t_0^k h$ and $v_0 = \frac{1}{t_0^k} h$. Clearly $u_0, v_0 \in P_h$. In addition

$$u_0 = t_0^k h = t_0^k t_0^k \frac{1}{t_0^k} h = t_0^{2k} v_0 < v_0.$$

If we take $r \in (0, t_0^{2k}] \subseteq (0,1)$, we have $rv_0 \le t_0^{2k} v_0 = u_0$. Thus $rv_0 \le u_0$. By the monotonicity of A, $Au_0 \le Av_0$. Further, considering conditions A is an $\alpha(t)$-concave multifunction with (8) and (9) we have

$$\begin{aligned} Au_0 &= A(t_0^k h) = A(t_0 t_0^{k-1} h) \ge \alpha(t_0) A(t_0^{k-1} h) \\ &\ge \cdots \ge (\alpha(t_0))^k Ah \ge (\alpha(t_0))^k t_0 h \ge t_0^k h = u_0. \end{aligned}$$

Thus $u_0 \le Au_0$. Now we set $x := \frac{x}{t}$. Since A is an $\alpha(t)$-concave multifunction, then

$$A(x) = A(t\frac{x}{t}) \ge \alpha(t) A(\frac{x}{t}),$$

so

$$A(\frac{x}{t}) \le \frac{1}{\alpha(t)} A(x). \tag{10}$$

By (10)

$$\begin{aligned} Av_0 &= A(\frac{1}{t_0^k} h) = A(\frac{1}{t_0} \frac{1}{t_0^{k-1}} h) \\ &\le \frac{1}{\alpha(t_0)} A(\frac{1}{t_0^{k-1}} h) = \frac{1}{\alpha(t_0)} A(\frac{1}{t_0} \frac{1}{t_0^{k-2}} h) \\ &\le \frac{1}{\alpha(t_0)} \frac{1}{\alpha(t_0)} A(\frac{1}{t_0^{k-2}} h) \\ &\le \cdots \le (\frac{1}{\alpha(t_0)})^k Ah \le \frac{h}{t_0 (\alpha(t_0))^k}. \end{aligned}$$

Application of (9) implies that

$$Av_0 \le \frac{h}{t_0(\alpha(t_0)^k)} \le \frac{h}{t_0^k} = v_0.$$

Thus we have $u_0 \le Au_0 \le Av_0 \le v_0$. $\square$

Theorem 3.5. *Let E be Dedekind complete, $P \subseteq E$ be a normal cone, and $A : P \to OCB(P)$ be an increasing $\alpha(t)$-concave multifunction. In addition, $x_0 \in P$ satisfies $(A(h) + x_0) \cap P_h \neq \emptyset$ (for some $h > \theta$). Then $A(x) + x_0$ has at least one fixed point in P_h.*

Proof. Let us define multifunction C on P by $Cx = Ax + x_0$. Thus $C : P \to B(P)$ for each $x \in P$. Also for each $x \in P$ and $t \in (0, 1)$, we have

$$C(tx) = A(tx) + x_0 \geq \alpha(t)A(x) + x_0 \geq \alpha(t)A(x) + \alpha(t)x_0 = \alpha(t)C(x).$$

Hence, C is an increasing and $\alpha(t)$-concave multifunction. In addition, for some $h > \theta$ we have

$$C(h) \cap P_h = (A(h) + x_0) \cap P_h \neq \emptyset.$$

Lemma 3.4 implies that there are $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that

$$rv_0 \leq u_0 \leq v_0, \qquad u_0 \leq Cu_0 \leq Cv_0 \leq v_0.$$

By $u_0 \leq Cu_0$, there exists $u_1' \in Cu_0$ such that $u_0 \leq u_1'$. Assume that $u_1 = \sup Cu_0$. Then $u_0 \leq u_1$. Let $v_1 = \sup Cv_0$. Since $Cv_0 \leq v_0$, we have $v_1 \leq v_0$. Also $Cu_0 \leq Cv_0$ and $u_1 \in Cu_0$. Since $v_1 = \sup Cv_0$, then $u_1 \leq v_1$. Hence $u_0 \leq u_1 \leq v_1 \leq v_0$. Since C is an increasing multifunctin, then $Cu_0 \leq Cu_1$, $Cu_1 \leq Cv_1$ and $Cv_1 \leq Cv_0$. Similarly if we take $u_2 = \sup Cu_1$ and $v_2 = \sup Cv_1$ we get

$$u_0 \leq u_1 \leq u_2 \leq v_2 \leq v_1 \leq v_0$$

By repeating this process we obtain sequences $\{u_n\}$ and $\{v_n\}$ satisfying

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0. \tag{11}$$

By $rv_0 \leq u_0$, we have

$$u_n \geq u_0 \geq rv_0 \geq rv_n$$

For each $n \in \mathbb{N}$, set $t_n = \sup\{t > 0 \mid u_n \geq tv_n\}$. Thus for each $n \in \mathbb{N}$, we have $u_n \geq t_n v_n$. In addition, for each $n \in \mathbb{N}$

$$u_{n+1} \geq u_n \geq t_n v_n \geq t_n v_{n+1}. \tag{12}$$

However, we know that $t_{n+1}v_{n+1} \leq u_{n+1}$. Thus by (12) and the definition of t_{n+1} we get $t_n \leq t_{n+1}$(for each $n \in \mathbb{N}$). Therefore $\{t_n\}$ is an increasing sequence such that $\{t_n\} \subset (0, 1]$. Suppose that $t_n \to t^*$ (as

$n \to \infty$). We are going to show that $t^* = 1$. Otherwise $0 < t^* < 1$. We distinguish two cases:
Case one: There exists an integer N such that $t_N = t^*$. In this case, it should be $t_n = t^*$ for all $n \geq N$. So for any $n \geq N$ we have

$$\alpha(t^*)Cv_n \leq C(t^*v_n) \leq Cu_n.$$

Since $v_{n+1} \in Cv_n$, $v_{n+1} = \sup Cv_n$ and $u_{n+1} = \sup Cu_n$, it follows that $\alpha(t^*)v_{n+1} \leq u_{n+1}$. By the definition of t_{n+1}, we have $\alpha(t^*) \leq t_{n+1}$. Then for $n \geq N$ we have $t_{n+1} = t^* \geq \alpha(t^*) > t^*$ which is a contradiction.
Case two: For all integer n, $t_n < t^*$. So we obtain

$$\begin{aligned} Cu_n \geq C(t_n v_n) = C(\frac{t_n}{t^*}t^*v_n) \geq \alpha(\frac{t_n}{t^*})C(t^*v_n) \\ \geq \alpha(\frac{t_n}{t^*})\alpha(t^*)C(v_n) \geq \frac{t_n}{t^*}\alpha(t^*)C(v_n). \end{aligned}$$

Thus $\frac{t_n}{t^*}\alpha(t^*)C(v_n) \leq Cu_n$. Since $u_{n+1} = \sup Cu_n$ and $v_{n+1} = \sup Cv_n$, we have $\frac{t_n}{t^*}\alpha(t^*)v_{n+1} \leq u_{n+1}$. By definition of t_{n+1} we get $\frac{t_n}{t^*}\alpha(t^*)v_{n+1} \leq t_{n+1}$. By letting $n \to \infty$, $t^* \geq \alpha(t^*) > t^*$, which is a contradiction. Thus, $\lim_{n\to\infty} t_n = 1$. For any natural number m, we have

$$\theta \leq u_{n+m} - u_n \leq v_n - u_m \leq v_n - t_n v_n = (1 - t_n)v_n \leq (1 - t_n)v_0,$$

$$\theta \leq v_n - v_{n+m} \leq v_n - u_n \leq v_n - t_n v_n = (1 - t_n)v_n \leq (1 - t_n)v_0.$$

Since P is normal, then

$$||u_{n+m} - u_n|| \leq N(1 - t_n)||v_0||,$$

$$||v_n - v_{n+m}|| \leq N(1 - t_n)||v_0||.$$

So $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Because E is complete, there exist u^* and v^* such that $u_n \to u^*$ and $v_n \to v^*$ (as $n \to \infty$). By (11), it follows that $u_n \leq u^* \leq v^* \leq v_n$, $u^*, v^* \in P_h$. Also

$$\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - t_n)v_0.$$

Furthermore

$$||v^* - u^*|| \leq N(1 - t_n)||v_0||.$$

By letting $n \to \infty$, we get $u^* = v^*$. Let $x^* = u^* = v^*$. Then we have

$$Cu_n \leq Cx^* \leq Cv_n$$

Since for any $n \in \mathbb{N}$, $u_{n+1} = \sup Cu_n$ and $v_{n+1} = \sup Cv_n$, for each n, there exists $x_{n+1} \in Cx^*$ such that $u_{n+1} \leq x_{n+1} \leq v_{n+1}$. Thus

$$\theta \leq x_{n+1} - u_{n+1} \leq v_{n+1} - u_{n+1} \leq (1 - t_{n+1})v_0,$$

so

$$||x_{n+1} - u_{n+1}|| \leq N(1 - t_{n+1})||v_0||.$$

Since $u_n \to x^*$, $x_{n+1} \to x^*$(as $n \to \infty$). Since Cx^* is closed, $x^* \in Cx^*$. $\square$

Example 3.6 shows that there exists an $\alpha(t)$-concave multifunction such that all the conditions of Theorem 3.5 are satisfied.

Example 3.6. Let $E = C[0,1]$ and $P = \{x \in E \mid x(t) \geq 0 \ for\ all\ t \in [0,1]\}$. Define $A : P \to OCB([0,1])$ as $A(x) = [\sqrt{x}, \sqrt{x}+1]$. It is easy to seen that A is increasing and $\alpha(t)$-concave multifunction for $\alpha(t) = \sqrt{t}$.

Example 3.7 shows that there exists a u_0-concave operator which is not $\alpha(t)$-concave.

Example 3.7. Let $E = \mathbb{R}$ and $P = [0, \infty)$. For any $u_0 > 0$, we have $P_{u_0} = (0, \infty)$. Let us define $f : [0, \infty) \to [0, \infty)$ as the following

$$f(x) = \begin{cases} \sin(x) & 0 \leq x < \frac{\pi}{2}, \\ 1 & x \geq \frac{\pi}{2} \end{cases}$$

Let $t \in (0,1)$ be fixed. It is easily noticed that for any $x \in (0, \frac{\pi}{2})$ we have $\frac{\sin(tx)}{t\sin(x)} > 1$. Then $f : [0, \infty) \to [0, \infty)$ is a u_0-concave operator(for any $u_0 > 0$). But f is not an $\alpha(t)$-concave operator. If f is an $\alpha(t)$-concave operator, then for any $t \in (0,1)$ and $x \in [0, \frac{\pi}{2})$, we should have $\frac{\sin(tx)}{\sin(x)} \geq \alpha(t) > t$ for some $\alpha : (0,1) \to (0,1)$. Let $t \in (0,1)$ be fixed. Since $\lim_{x \to 0^+} \frac{\sin(tx)}{\sin(x)} = t$, then there exists $x_0 \in (0,1)$ such that $\alpha(t) > \frac{\sin(tx_0)}{\sin(x_0)} > t$ which is a contradiction.

Remark 3.8. Comparing Theorem 3.1 with Theorem 3.5, we see that the conditions of Theorem 3.5 are greatly weaker than the conditions of Theorem 3.1. In Theorem 3.5, conditions "there exist $w_0, v_0 \in P$ and $0 < \lambda_0 < 1$ such that $\lambda_0 v_0 \leq w_0 \leq v_0$" , "$v_0 \geq v_1$" , "$w_0 \leq w_1$" and "there exist adjoint sequence of A in w_0, v_0 and $\lambda_0 \in (0,1)$ such that $\lim_{n\to\infty} n\eta_n = \ln(\frac{1}{\lambda_0})$" are removed.

4 Application

To illustrate the ideas involved in Theorem 3.5, we need to discuss the Hammerstein and quadratic integral inclusions. Consider Banach space $E = L^1[0,T]$ with the usual $||.||_1$ norm([4]). Let $0 < T < \infty$. Hammerstein integral inclusion is defined as

$$u(r) \in \int_0^T k(r,s)g(s,u(s))ds \quad on \quad [0,T], \tag{13}$$

such that k is a real single-valued function, while $g : [0,T] \times E \to 2^E$ is a multifunction with nonempty values. Also for any $r \in [0,T]$, quadratic integral inclusion is defined as

$$u(r) \in y(r,u(r)) \int_0^r k(r,s)g(s,u(s))ds \quad r \in [0,T], \tag{14}$$

such that k is a real single-valued function, while $g : [0,T] \times E \to 2^E$ and $y : [0,T] \times E \to 2^E$ are the multifunctions with nonempty values. If $y(r,u(r)) = 1$, the quadratic integral inclusion is called Volterra integral inclusion.

Theorem 4.1. *Consider Banach space $E = L^1[0,T]$ with the usual norm $||.||_1$ where $0 < T < \infty$. Also consider normal cone $P = \{x \in E \mid x(s) \geq 0 \text{ for all } s \in [0,T]\}$ in E. Suppose that*
(i) $k : [0,T] \times [0,T] \to (0,\infty)$ is a bounded function respect to each of its variables(which means there exists $M > 0$ such that $k(r,s) \leq M$, for any $r,s \in [0,T]$);
(ii) $g : [0,T] \times E \longrightarrow OCB(E)$ be an multifunction such that $g : [0,T] \times P \longrightarrow OCB(E)$ is an $\alpha(t)$-concave increasing multifunction;

(iii) for each $u \in P$, $g(s, u(s))$ is sequentially compact respect to pointwise convergence(which means for any sequence $\{w_n\} \subseteq g(s, u(s))$, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that w_{n_k} is a pointwise convergence to w (as $k \to \infty$) for some $w \in g(s, u(s))$);
(iv) there exists $h > \theta$ such that $g(s, h(s)) \cap P_h \neq \emptyset$ and $\int_0^T k(r, s)h(s)ds \in P_h$.
Then Hammerstein integral inclusions (13) have at least one positive solution in P_h.

Proof. For each $w \in P$, let us define $f(w) = \int_0^T k(r, s)w(s)ds$. By (i) and Fubini theorem we have

$$\begin{aligned}\int_0^T |\int_0^T k(r,s)w(s)ds|dr &= \int_0^T \int_0^T k(r,s)w(s)drds \\ &= \int_0^T w(s)(\int_0^T k(r,s)dr)ds < \infty\end{aligned}$$

Then $f : P \to P$ is an operator. Also for each $u \in P$, let us define $G(u) = \{w \in P : w \in g(s, u(s))\}$. Moreover for each $u \in P$ define $A(u) = \{f(w) \ : \ w \in G(u)\}$. By (ii), for any $u \in P$ we have $A(u) \neq \emptyset$ and since $f : P \to P$, we have $A : P \to 2^P$. Assume that $\{f(w_n)\}$ is a sequence in $A(u)$ such that

$$f(w_n) \xrightarrow{L^1} v \quad (as \ \ n \to \infty)$$

for some $v \in P$. Then $\{w_n\} \subseteq G(u)$ and by (iii) there exists a subsequence $\{n_k\}$ of $\{n\}$ such that w_{n_k} is a pointwise convergence to w (as $k \to \infty$) for some $w \in G(u)$. Since $G(u)$ is bounded, by Lebesgue's dominated convergence theorem we have

$$f(w_n) \xrightarrow{L^1} f(w) \ \ (as \ \ n \to \infty)$$

Thus $v = f(w)$. Therefore for any $u \in P$, $A(u)$ is a closed subset of P (respect to the L^1 norm). For any $u \in P$, $G(u)$ is order bounded and E is Dedekind complete space. So there exists $z \in G(u)$ such that $z = \sup G(u)$. Thus for any $w \in G(u)$, we have $f(w) \leq f(z)$ and it follows that $A(u)$ is bounded. Since E is Dedekind complete, there exists $y \in E$ such that $y = \sup A(u)$. Since $f(z)$ is an upper bound of

$A(u)$, $y \leq f(z)$. But since $z \in G(u)$, we should have $f(z) \leq y$. Therefore $y = f(z)$ and we have $A : P \to OCB(P)$. Now we prove that A is an $\alpha(t)$-concave multifunction. Let $u \in P$, $w \in G(u)$ and $t \in (0,1)$. By (ii) there exists $c \in G(tu)$ such that $c \geq \alpha(t)w$. So we have

$$\begin{aligned}\alpha(t)f(w) &= \alpha(t)\int_0^T k(r,s)w(s)ds \\ &= \int_0^T k(r,s)\alpha(t)w(s)ds \leq \int_0^T k(r,s)c(s)ds = f(c).\end{aligned}$$

Since $f(c) \in A(tu)$ and $f(w) \in A(u)$, then A is an $\alpha(t)$-concave multifunction. At the end of the proof we show that there exists $h > \theta$ such that $A(h) \cap P_h \neq \emptyset$. By (iv), there exists $h > \theta$ such that $G(h) \cap P_h \neq \emptyset$. If $w \in G(h) \cap P_h$, there exists $\lambda, \mu > 0$ such that

$$\lambda h \leq f(w) = \int_0^T k(r,s)w(s)ds \leq \mu h.$$

Now if we put $x_0 = \theta$, by theorem 3.5, Hammerstein integral inclusions (13) have at least one positive solution in P_h. $\square$

Example 4.2. Let $E = L^1[0,T]$ with the usual norm $||.||_1$ where $0 < T < \infty$, and $P = \{x \in E \mid x(s) \geq 0 \ for\ all\ s \in [0,T]\}$. Also, assume that g is defined by $g(u(s),s) = [\sqrt[3]{u(s)}, \sqrt[3]{u(s)}+1]$ for any $u \in E$, $h = 1$ is a constant function on $[0,T]$ and $k(r,s) = \frac{e^{rs}}{1+rs}$. Then by Theorem 4.1 there exists $u \in P_h$ such that

$$u(r) \in \int_0^T k(r,s)g(s,u(s))ds.$$

Theorem 4.3. *Consider Banach space $E = L^1[0,T]$ with the usual norm $||.||_1$ where $0 < T < \infty$. Also consider the normal cone $P = \{x \in E \mid x(s) \geq 0 \ for\ all\ s \in [0,T]\}$ in E. Suppose that*
(i) $k : [0,T] \times [0,T] \to (0,\infty)$ is a bounded function respect to each of its variables (which means there exists $M > 0$ such that $k(r,s) \leq M$, for any $r,s \in [0,T]$);
(ii) $g : [0,T] \times E \longrightarrow OCB(E)$ is an multifunction such that $g : [0,T] \times P \longrightarrow OCB(E)$ is an $\alpha(t)$-concave increasing multifunction;

(iii) for each $u \in P$, $g(s, u(s))$ is sequentially compact respect to pointwise convergence(which means for any sequence $\{w_n\} \subseteq g(s, u(s))$, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that w_{n_k} is a pointwise convergence to w (as $k \to \infty$) for some $w \in g(s, u(s))$);
(iv) $y : [0, T] \times P \longrightarrow P$ is a multifunction such that y is bounded above (which means there exist $L > 0$ such that for any $r \in [0, T]$, $y(r, u(r)) < L$). Also y is sequentially continuous respect to u(which means for any sequence $\{u_n\} \subseteq P$ such that u_n is a pointwise convergence to u, $y(r, u_n(r))$ is a pointwise convergence to $y(r, u(r))$ as $n \to \infty$);
(v) there exists $h > \theta$ such that $g(s, h(s)) \cap P_h \neq \emptyset$, $\int_0^T k(r, s)h(s)ds \in P_h$ and $y(r, h(r)) \in P_h$.
Then quadratic integral inclusions (14) have at least one positive solution in P_h.

Proof. For each $w \in P$, let us define $f(w)(r) = y(r, u(r)) \int_0^r k(r, s)w(s)ds$. By (i) we have

$$\begin{aligned}
\int_0^T |\int_0^r f(w(s))(r)ds|dr &= \int_0^T |y(r, u(r)) \int_0^r k(r, s)w(s)ds|dr \\
&= \int_0^T |y(r, u(r))|| \int_0^r k(r, s)w(s)ds|dr \\
&\leq \int_0^T |y(r, u(r))| \int_0^r |k(r, s)w(s)|dsdr \\
&\leq \int_0^T L \int_0^r (M|w(s)|)dsdr < \infty.
\end{aligned}$$

Then $f : P \to P$ is an operator. Also for each $u \in P$, let us define $G(u) = \{w \in P : w \in g(s, u(s))\}$. Moreover for each $u \in P$, define $A(u) = \{f(w) \ : \ w \in G(u)\}$. By (ii), for any $u \in P$, we have $A(u) \neq \emptyset$ and since $f : P \to P$, we have $A : P \to 2^P$. Assume that $\{f(w_n)\}$ is a sequence in $A(u)$, such that

$$f(w_n) \xrightarrow{L^1} v \quad (as \ \ n \to \infty)$$

for some $v \in P$. Then $\{w_n\} \subseteq G(u)$ and by (iii) there exists a subsequence $\{n_k\}$ of $\{n\}$ such that w_{n_k} is pointwise convergence to w (as

$k \to \infty$) for some $w \in G(u)$. Since $G(u)$ is bounded, by Lebesgue's dominated convergence theorem, for any $r \in [0, T]$, we have

$$\int_0^r k(r,s)w_{n_k}ds \to \int_0^r k(r,s)w(s)ds \quad (as \quad k \to \infty).$$

Also by (iv), $y(r, w_{n_k})$ is a pointwise convergence to $y(r, w)$(as $k \to \infty$) and we can deduce that

$$f(w_n) \xrightarrow{L^1} f(w) \quad (as \quad n \to \infty).$$

Thus $v = f(w)$. Therefore for any $u \in P$, $A(u)$ is closed subset of P (respect to the L^1 norm). For any $u \in P$, $G(u)$ is order bounded and E is Dedekind complete space. So, there exists $z \in G(u)$ such that $z = \sup G(u)$. Thus for any $w \in G(u)$, we have

$$f(w)(r) = y(r, u(r)) \int_0^r k(r,s)w(s)ds < LM \int_0^r w(s)ds < \infty.$$

Hence $A(u)$ is bounded. Since E is Dedekind complete, there exists $y \in E$ such that $y = \sup A(u)$. Since $f(z)$ is upper bound of $A(u)$, $y \leq f(z)$. But since $z \in G(u)$, we should have $f(z) \leq y$. Therefore $y = f(z)$ and we have $A : P \to OCB(P)$. Now we show that A is an $\alpha(t)$-concave multifunction. Let $u \in P$, $w \in G(u)$ and $t \in (0, 1)$. By (ii) there exists $c \in G(tu)$ such that $c \geq \alpha(t)w$. So we have

$$\begin{aligned}\alpha(t)f(w) &= \alpha(t)y(r, u(r)) \int_0^r k(r,s)w(s)ds \\ &= y(r, u(r)) \int_0^r \alpha(t)w(s)ds \leq \int_0^T k(r,s)c(s)ds = f(c).\end{aligned}$$

Since $f(c) \in A(tu)$ and $f(w) \in A(u)$, then A is an $\alpha(t)$-concave multifunction. At the end of the proof we show that there exists $h > \theta$ such that $A(h) \cap P_h \neq \emptyset$. By (v), there exists $h > \theta$ such that $G(h) \cap P_h \neq \emptyset$. If $w \in G(h) \cap P_h$, there exists $\lambda, \mu > 0$ such that

$$\lambda h \leq f(w) = y(r, u(r)) \int_0^r k(r,s)w(s)ds \leq \mu h.$$

Now if we put $x_0 = \theta$, by theorem 3.5, quadratic integral inclusions (14) have at least one positive solution in P_h. $\square$

Example 4.4. Let $E = L^1[0,T]$ with the usual norm $||.||_1$ where $0 < T < \infty$, and $P = \{x \in E \mid x(s) \geq 0 \; for \; all \; s \in [0,T]\}$. Also, assume that for any $u \in E$, $g(u(s),s) = [\frac{u(s)}{1+u(s)}, \frac{u(s)}{1+u(s)} + 1]$, $y(r,u(r)) = e^{-u(r)} + 1$, $h = 1$ is a constant function on $[0,T]$ and $k(r,s) = \ln(1 + rs)$. Then by Theorem 4.3 there exists $u \in P_h$ such that

$$u(r) \in y(r,u(r)) \int_0^r k(r,s)g(s,u(s))ds.$$

5 Conclusion

In this paper, we generalized concave operators to multifunction version. Also, we extended the fixed point theorems of concave operators ([33],[20]) to multifunction version on Dedekind complete spaces. In addition, we provided an example for adjoint sequence of u_0-concave operators (example 3.3) and an example which shows that there exists a u_0-concave operator such that it is not an $\alpha(t)$-concave(example 3.7). Before this article, these two types of mappings had not been compared. Main part of this work relates to investigation of the existence of solutions for the Hammerstein and quadratic integral inclusions by using some new ideas.

Remark 5.1. It is suggested that our results be extended to fractional differential equations and inclusions([7], [15], [27]). Another useful extension of our main theorems can be proving in metric like spaces similar to the generalized metric space([2]) or $G-$metric spaces([10]). Furthermore, it seems interesting to prove Theorems 3.1 and 3.5, without assuming Dedekind complete. Example 2.9 shows that in both theorems, Dedekind complete assumption is sufficient but not necessary.

Acknowledgements
The authors were supported by Payame Noor University. The author express their gratitude to dear unknown referees for their helpful suggestions which improved the final version of this paper.

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