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A Predictor-Corrector Algorithm for Finding all Zeros of Nonlinear Equations

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Abstract. In this paper, an efficient algorithm is proposed to approximate all real solutions of a nonlinear equation. This algorithm is based on convergence conditions of Adomian decomposition method (ADM) for solving functional equations. The presented algorithm is well done, particularly, when we desire to obtain more than one solution of a nonlinear equation with using only one initial solution. The scheme is tested for some examples and the obtained results demonstrate reliability and efficiency of the proposed algorithm.

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1. Introduction

One of the oldest and most basic problems in mathematics is that of solving a nonlinear equations $f(x) = 0$. This problem has motivated many theoretical developments including the fact that solution formulas do not in general exist. Thus, the development of algorithms for finding solutions has historically been an important enterprise. Newton-Raphson method [1] is the most popular technique for solving nonlinear equations. Many topics related to Newton's method still attract attention from researchers. As is well known, a disadvantage of the

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method is that the initial approximation x_0, must be chosen sufficiently close to a true solution in order to guarantee their convergence. Finding a criterion for choosing x_0 is quite difficult and therefore effective and globally convergent algorithms are needed [2]. In recent years, the study of numerical methods has provided an attractive field for researchers of mathematical sciences which have risen to the appearance of different numerical computational methods and efficient algorithms to solve the nonlinear equations. In fact, there are several iterative methods have been developed to solve the nonlinear equations $f(x) = 0$, by using ADM, iterative method, and other techniques (for example see [3-15]). Theoretical treatment of the convergence of the decomposition series to the ADM has been considered in [16-21].
In this work, a new algorithm for solving nonlinear equations is presented. The proposed method can be found more than one zero of nonlinear equation $f(x) = 0$ (if exist), by using only one initial solution x_0. This method is based on ADM and the Banach's fixed point theorem [16]. The proposed algorithm is numerically performed through Maple programming. The obtained results show the advantage using this method. Note that, the authors in [9, 10] have applied convergence conditions of ADM to solve nonlinear equations and system of nonlinear equations. This paper is organized into following sections of which this introduction is the first. ADM is described in Section 2. Section 3 derives the method. Also, a Predictor-Corrector algorithm for finding all zeros of nonlinear equations is presented in Section 4. In Section 5 we present some numerical examples to illustrate the efficiency and reliability of the presented method. Finally, the paper is concluded with conclusion.

2. Adomian Decomposition Method Synthesis

Adomian decomposition method was presented by Adomian in 1981. This method and its modifications have a good usage in solving the differential, algebraic-differential, integral equations, etc [22]. The convergence of the ADM have investigated by many researchers. Here, the review of the standard ADM for solving nonlinear equations is presented. For this reason, consider the nonlinear equation,

$$f(x) = 0\,, \tag{1}$$

which can be transformed to,

$$x = F(x) + c\,, \tag{2}$$

where $F(x)$ is a nonlinear function and c is a real constant. The ADM decomposes the solution x by an infinite series of components,

$$x = \sum_{n=0}^{+\infty} x_n\,, \tag{3}$$

and the nonlinear term $F(x)$ by an infinite series,

$$F(x) = \sum_{n=0}^{+\infty} A_n\,, \tag{4}$$

where the components of A_n are the so-called Adomian polynomials [17], for each i, A_i depends on x_0, x_1, ..., x_i only. Substituting (3) and (4) into (2) yields,

$$\sum_{n=0}^{+\infty} x_n = c + \sum_{n=0}^{+\infty} A_n\,.$$

Now, we define

$$\begin{aligned} x_0 &= c\,, \\ x_{n+1} &= A_n, \qquad n \geqslant 0. \end{aligned} \tag{5}$$

If the series converges in a suitable way, then we can see that

$$x = \lim_{M\to+\infty} \Psi_M(x),$$

where $\Psi_M(x) = \sum_{i=0}^{M} x_i$. Now, we require on expression for the A_i. The Adomian polynomials A_i for the nonlinear term $F(x)$ can be evaluated by using the following expression,

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i} F\big(\sum_{j=0}^{i} \lambda^j y_j\big)_{\lambda=0}, \quad i = 0, 1, 2, \cdots \tag{6}$$

The general formula (6) can be simplified as follows:

$$\begin{aligned} A_0 &= F(x_0), \\ A_1 &= x_1 F'(x_0), \\ A_2 &= x_2 F'(x_0) + \frac{1}{2} x_1^2 F''(x_0), \\ A_3 &= x_3 F'(x_0) + x_1 x_2 F''(x_0) + \frac{1}{3!} x_1^3 F'''(x_0), \end{aligned} \tag{7}$$

$$\vdots$$

substituting (7) into (4) gives,

$$\begin{aligned} F(x) &= A_0 + A_1 + A_2 + \cdots = F(x_0) + (x_1 + x_2 + x_3 + \cdots)F'(x_0) \\ &+ \frac{1}{2!}(x_1^2 + 2x_1x_2 + 2x_1x_3 + x_2^2 + \cdots)\ F''(x_0) + \cdots \\ &= F(x_0) + (x - x_0)F'(x_0) + \frac{1}{2!}(x - x_0)^2 F''(x_0) + \cdots . \end{aligned}$$

Note that, The last expansion confirms a fact that the series in A_i polynomials is a Taylor series about a function x_0 and not about a point as is usually used. Here, we use the following Maple procedure to construct Adomian polynomials of nonlinear term $F(x)$:

```
Am:=proc(K,m) global k; k:=20;
   if m=0 then K(sum(lambda^j*(diff(U[j](tt, x), x)),j=0..k));
   else
   (diff(K(sum(lambda^j*(diff(U[j](tt,x),x)),j=0..k)),'$'(lambda, m)))/m!
   end if;
   subs(lambda=0,%);
end proc;
```

As a well-known powerful tool, for convergence of the ADM we have the Banach's Fixed Point Theorem as follow [16-21].

Theorem 2.1. ([23]) *Assume that B is a Banach space, and further, that $\mathcal{T} : B \to B$ is a contractive mapping with contractively constant α, $0 \leqslant \alpha < 1$. Then the following results hold.*

(1) *There exists a unique $u \in B$ such that $u = \mathcal{T}(u)$.*

(2) *For any $u_0 \in B$, the sequence $\{u_n\} \subset B$ defined by $u_{n+1} = \mathcal{T}(u_n)$, $n = 0, 1, \cdots$, converges to u.*

The theoretical treatment of the convergence of ADM has been considered in [16, 17]. As it was seen in [16], $\sum_{i=0}^{\infty} x_i$, which is obtained by (5), converges to the exact solution x, when,

$$\exists\ 0 \leqslant \alpha < 1, \quad |x_{k+1}| \leqslant \alpha\ |x_k|\ , \quad \forall k \in \mathrm{N} \cup \{0\}. \tag{8}$$

3. Description of the Method

In this section, we solve the nonlinear equation (1) by considering Convergence conditions of ADM (8) and construct an efficient algorithm. For this reason, we need of an initial approximation. This initial approximation must be guessed. But it can be computed if the nonlinear equation (1) has a real constant as below,

$$0 = F(x) + c\,, \tag{9}$$

where $F(x)$ is a nonlinear function and c is a real constant. In this case, we add $\beta\,x$ on both sides of (9). So, we have,

$$\beta x = F(x) + \beta x + c\,. \tag{10}$$

Here, β is unknown and nonzero real constant and it will be determined such that the convergence condition (8) will be hold. Equation (10) implies that,

$$x = \frac{F(x) + \beta x}{\beta} + \frac{c}{\beta}\,,$$

and by (5) and (7), we have,

$$\begin{cases} x_0 = \frac{c}{\beta}\,, \\ x_1 = \frac{F(x_0)+\beta x_0}{\beta}\,, \\ x_2 = \frac{x_1 F'(x_0)+\beta x_1}{\beta}\,. \end{cases} \tag{11}$$

Replacing $F(x)$ with two terms of Taylor's series of $F(x)$, at $x = 0$, becomes,

$$\begin{cases} x_0 = \frac{c}{\beta}\,, \\ x_1 \approx \frac{F(0)+x_0 F'(0)+\beta x_0}{\beta} = \frac{cF'(0)+\beta(F(0)+c)}{\beta^2}\,, \\ x_2 \approx \frac{x_1 F'(0)+\beta x_1}{\beta}\,. \end{cases} \tag{12}$$

For an arbitrary number α , $0 < \alpha < 1$, and by attention to (8), we consider two equations,

$$x_1 = \alpha x_0, \tag{13}$$

and

$$x_2 = \alpha x_1. \tag{14}$$

Substituting (12) into (13) and (14) yields, respectively,

$$\frac{cF'(0) + \beta(F(0) + c)}{\beta^2} = \alpha\frac{c}{\beta} \Rightarrow \beta = \frac{cF'(0)}{c(\alpha - 1) - F(0)} \; or \; \beta = \frac{-cF'(0)}{c + F(0)} \; (for\; \alpha = 0)\,,$$

and

$$\frac{x_1 F'(0) + \beta x_1}{\beta} = \alpha x_1 \Rightarrow \quad \beta = \frac{F'(0)}{\alpha - 1}\,.$$

Now, we able to compute a suitable initial approximation for nonlinear equation (9) by using predictor part of proposed algorithm.

Now, we have the nonlinear equation (1) and its initial approximation, $\bar{x}$, which is obtained by guessing or by using predictor part of proposed algorithm. To continue, we rewrite (1) as below,

$$x = \frac{f(x) + \beta x}{\beta} - \bar{x} + \bar{x}.$$

Here, β is an unknown real constant and it will be determined such that the convergence condition (8) will be hold. As it is seen in (11), we have,

$$\begin{cases} x_0 = \bar{x}\,, \\ x_1 = \frac{f(x_0)+\beta x_0}{\beta} - \bar{x} = \frac{f(\bar{x})}{\beta}\,, \\ x_2 = \frac{x_1 f'(x_0)+\beta x_1}{\beta} = \frac{f(\bar{x})f'(\bar{x})}{\beta^2} + \frac{f(\bar{x})}{\beta}\,, \\ x_3 = \frac{x_2 f'(x_0)+\frac{x_1^2 f''(x_0)}{2}+\beta x_2}{\beta}\,. \end{cases} \tag{15}$$

For an arbitrary number α , $0 < \alpha < 1$, and by attention to (8), we consider two equations,

$$x_2 = \alpha x_1, \tag{16}$$

and

$$x_3 = \alpha x_2. \tag{17}$$

Substituting (15) into (16) and (17) yields, respectively,

$$\beta = \frac{f'(\bar{x})}{\alpha - 1},$$

and

$$\beta = \frac{f'(\bar{x})\,(2-\alpha) \pm \sqrt{\alpha^2\, f'^2(\bar{x}) + 2\; f(\bar{x})\, f''(\bar{x})\,(\alpha-1)}}{2(\alpha-1)}\,.$$

Thus, we able to compute an approximated solution of nonlinear equation (1) by using corrector part of proposed algorithm.

4. An Efficient Algorithm

The above result is summarized in the following algorithm. The main idea of this algorithm is solve a nonlinear equation. First, a suitable initial approximation for nonlinear equation (9) is obtain by using predictor part of proposed algorithm. Then, an approximated solution of nonlinear equation (1) is compute by using corrector part of proposed algorithm.

I. The Predictor Part

Object: To compute an initial approximation of a nonlinear equation.
Input: Nonlinear equation $f(x) = F(x)+c = 0$, α $(0 < \alpha < 1)$ and m $(m \geqslant 1)$.
Output : Initial approximation, $\bar{x}$.

Step 1: Choose,

$$\beta = \tfrac{cF'(0)}{c(\alpha-1)-F(0)} \text{ or } \beta = \tfrac{-cF'(0)}{c+F(0)} \quad \text{or} \quad \beta = \tfrac{F'(0)}{\alpha-1} \ .$$

Step 2: Compute Adomian's polynomials A_i, for nonlinear term $\frac{F(x)+\beta x}{\beta}$ and put

$$\begin{cases} x_0 = \frac{c}{\beta} \, , \\ x_1 = A_0, \\ x_2 = A_1, \\ \vdots \\ x_{k+1} = A_k \, , \end{cases}$$

while $1 \leqslant k \leqslant m$ and the condition $|\, x_k \,| \leqslant |\, x_{k-1} \,| \leqslant ... \leqslant |\, x_0 \,|$ is hold. Now, put

$$\bar{x} = x_0 + x_1 + ... + x_k.$$

II. The Corrector Part

Object: To compute approximated solution of a nonlinear equation.
Input: Nonlinear equation$f(x) = 0$, Initial approximation $\bar{x}$, α $(0 < \alpha < 1)$, $\varepsilon > 0$ and m $(m \geqslant 1)$.
Output: Approximated solution $\tilde{x}$.
Step 1: If $|\, f(\bar{x}) \,| < \varepsilon$ then go to step 5.
Step 2: Choose,

$$\beta = \frac{f'(\bar{x})}{\alpha - 1} \, ,$$

or

$$\beta = \frac{f'(\bar{x})\,(2-\alpha) + \sqrt{\alpha^2\, f'^2(\bar{x}) + 2\ f(\bar{x})\, f''(\bar{x})\,(\alpha-1)}}{2(\alpha-1)} \, ,$$

or

$$\beta = \frac{f'(\bar{x})\,(2-\alpha) - \sqrt{\alpha^2\, f'^2(\bar{x}) + 2\ f(\bar{x})\, f''(\bar{x})\,(\alpha-1)}}{2(\alpha-1)} \, .$$

Step 3: Compute Adomian's polynomials B_i, for nonlinear term $\frac{f(x)+\beta x}{\beta}$ and put

$$\begin{cases} x_0 = \bar{x}\,, \\ x_1 = B_0 - \bar{x}, \\ x_2 = B_1, \\ \vdots \\ x_{k+1} = B_k\,, \end{cases}$$

while $1 \leqslant k \leqslant m$ and the condition $|\,x_k\,| \leqslant |\,x_{k-1}\,| \leqslant ... \leqslant |\,x_0\,|$ is hold. Now, put

$$\bar{x} = x_0 + x_1 + ... + x_k$$

Step 4: If $|\,f(\bar{x})\,| < \varepsilon$ then go to step 5 else go to step 2.
Step 5: Put $\tilde{x} = \bar{x}$ and stop.

Remark 4.1. *By using different obtained real constants β which are appeared in step 1 and step 2 of above algorithms, respectively. In fact, it is possible that the obtained approximated solutions converge to different solutions of equation* (1) *(see Examples 5.2 and 5.3).*

5. Numerical Examples

In this section, some examples are solved by the proposed algorithm of this paper. The obtained results show that the proposed algorithm can appropriately solve the nonlinear equation (1). In fact, the standard ADM and its modifications [3-8] can solve nonlinear equation (1) in specific cases whereas the proposed algorithm can solve equation (1) in more cases (see Example 5.3). In addition, we can obtain more than one zero of nonlinear equation $f(x) = 0$ (if exist), with using the presented algorithm and chose a comfortable initial solution (Note that, we have obtained two and there real solution of Examples 5.2 and 5.3 via two and one initial solution x_0, respectively). Here, the algorithm is performed by maple with 15 digits precision. In this section, we set $m = 5$, $\varepsilon = 10^{-12}$ and $\alpha = 0.1$, for all examples.

Example 5.1. Consider the nonlinear equation,

$$x^6 - 5x^5 + 3x^4 + x^3 + 2x^2 - 8x - 0.5 = 0, \tag{18}$$

which has two real solutions,

$$x^1 = -0.0615753511597450, \tag{19}$$

and

$$x^2 = 4.23471316736242, \tag{20}$$

In [3], the first solution, (19), was obtained by a modification of ADM. Here, we obtain all real solutions (19) and (20) by the presented algorithm. For this reason, we rewrite (18) as below,

$$x = \frac{x^6 - 5x^5 + 3x^4 + x^3 + 2x^2 - 8x + \beta x}{\beta} - \frac{0.5}{\beta}.$$

Now, by using algorithm (4.I), we obtain,

$$\beta = 8.0, \beta = 8.888\ldots, \beta = 8.888\ldots.$$

Choosing β =8.0, we obtain

$$\begin{aligned} x_0 &= -0.0625, \\ x_1 &= 0.000952370464801789, \\ x_2 &= -0.0000287613653337411, \\ x_3 &= 0.00000108346254799556, \\ x_4 &= -0.456933501001624\, e - 7, \\ x_5 &= 0.206479468869854\, e - 8, \end{aligned}$$

So, we obtain initial approximation $\bar{x} = \sum_{i=0}^{5} x_i = -0.0615753510665393$, with $|\, f(\bar{x})\,| = 9.3\, e - 11$ which it shows that this initial approximation is suitable. Now, by using algorithm (4.II), we have,

$$\beta = 8.23809310631965, \beta = 9.153436786\ldots, \beta = 9.153436788\ldots.$$

where through choosing β =8.23809310631965, we obtain,

$$\begin{aligned} x_0 &= -0.0615753510665395, \\ x_1 &= -0.932055\, e - 10, \\ x_2 &= -0.2\, e - 16, \\ x_3 &= 0.2\, e - 20, \\ x_4 &= -0.2\, e - 23, \\ x_5 &= -0.3\, e - 23, \end{aligned}$$

and $\tilde{x} = \bar{x} = \sum_{i=0}^{5} x_i = -0.0615753511597450$, with $|\, f(\bar{x})\,| = 0$.
To obtain, second real solution (20), corrected part of the presented algorithm with initial approximation $\bar{x} = 5.0$ is applied and Table (5.) shows the results. The obtained results show the advantage using proposed algorithm for this example.

Table 1: Second approximate solution for Example 5.1

Real β's	Approximated Solution, $\bar{x}$	$\lvert f(\bar{x}) \rvert$
-5235.55...	$x_0 = 5.0$ $x_1 = -0.38381791171477$ $x_2 = -0.038381791171477$ $x_3 = -0.104908590182653$ $\bar{x} = \sum_{i=0}^{2} x_i = 4.57780029711375$ (Note that $\lvert x_3 \rvert > \lvert x_2 \rvert$)	0.3431
-2564.12748600513	$x_0 = 4.57780029711375$ $x_1 = -0.22207317064549$ $x_2 = -0.022207317064394$ $x_3 = -0.044221101677761$ $\bar{x} = \sum_{i=0}^{2} x_i = 4.33351980940387$	0.0988
-1550.33665295112	$x_0 = 4.33351980940387$ $x_1 = -0.07946514158093$ $x_2 = -0.0079465141581112$ $x_3 = -0.007205623329186$ $x_4 = -0.0017627060015005$ $x_5 = -0.001335550382624$ $\bar{x} = \sum_{i=0}^{5} x_i = 4.23580427395152$	0.0011
-1125.62395151785, -1231.440..., -1214.112...	$x_0 = 4.23580427395152$, $\ldots$, $\bar{x} = \sum_{i=0}^{5} x_i = 4.23471316771527$	3.53 e-10
-1105.31666185694, -1228.129618..., -1228.129614...	$x_0 = 4.23471316771527$, $\ldots$, $\tilde{x} = \sum_{i=0}^{5} x_i = 4.23471316736245$ (Note that $\tilde{x} = \bar{x}$)	3.10 e-14

Example 5.2. Consider the nonlinear equation [7, 8],

$$e^x - 3x^2 = 0, \tag{21}$$

which has three real solutions,

$$x^1 = -0.458962267536950, \tag{22}$$

$$x^2 = 0.910007572488716, \tag{23}$$

$$x^3 = 3.73307902863280. \tag{24}$$

Here, all real solutions (21) are obtained by using purposed algorithm. For this reason, algorithm (4.II) with initial approximation $\bar{x} = 0$ is applied to this problem. Table 2 reports the obtained results. Now, consider Table 2, in the

Table 2: First approximate solution for Example 5.2

Real β's	Approximated Solution, $\bar{x}$	$\mid f(\bar{x}) \mid$
-2.72314789108930, -1.11..., 0.612036779978184	$\bar{x} = \sum_{i=0}^{3} x_i = -0.622829090834269$	0.1639
-4.74822137742396	$\bar{x} = \sum_{i=0}^{5} x_i = -0.460911148553635$	0.0019
-3.45881275663384, -3.7735..., -3.7108...	$\bar{x} = \sum_{i=0}^{5} x_i = -0.458962268550887$	1.01e-9
-3.38571272694698, -3.76190299..., -3.76190297...	$\tilde{x} = \sum_{i=0}^{5} x_i = -0.458962267536948$ (Note that $\tilde{x} = \bar{x}$)	-2.10 e-15

first iteration, if we choose $\beta = -1.11\ldots$ then the same results will obtain but if we choose $\beta = 0.612036779978184$ then the approximated solution converges to solution (23), the obtained results are shown in Table 3. Here, to obtain the

Table 3: Second approximate solution for Example 5.2

Real β's	Approximated Solution, $\bar{x}$	$\mid f(\bar{x}) \mid$
-2.72314789108930, -1.11..., 0.612036779978184	$\bar{x} = \sum_{i=0}^{1} x_i = 1.63388873465357$	0.7238...
-5.19952381530444	$\bar{x} = \sum_{i=0}^{3} x_i = 0.992050636325429$	0.0820...
3.61727215566294	$\bar{x} = \sum_{i=0}^{5} x_i = 0.910186137072770$	0.000178...
2.97950031256122, 3.3070..., 3.3038...	$\tilde{x} = \sum_{i=0}^{5} x_i = 0.910007572488732$ (Note that $\tilde{x} = \bar{x}$)	1.61 e-14

third real solution (24), it is used algorithm (4.II), with initial approximation $\bar{x} = 5.0$ and the obtained results are shown in Table 4.

Table 4: Third approximate solution for Example 5.2

Real β's	Approximated Solution, $\bar{x}$	$\mid f(\bar{x}) \mid$
-131.570176780641	$\bar{x} = \sum_{i=0}^{2} x_i = 4.38622507783453$	0.6531...
-60.0213678340246	$\bar{x} = \sum_{i=0}^{2} x_i = 3.97167875646619$	0.2385...
-32.4927568083889	$\bar{x} = \sum_{i=0}^{2} x_i = 3.77699096825473$	0.0439...
-23.3582780309814	$\bar{x} = \sum_{i=0}^{5} x_i = 3.73314769138646$	0.000068...
-19.4239860213644, -21.56..., -21.55...	$\tilde{x} = \sum_{i=0}^{5} x_i = 3.73307902863282$ (Note that $\tilde{x} = \bar{x}$)	2.10 e-14

Example 5.3. Consider the nonlinear equation,

$$e^x - 5\,\sin(x) = 0, \tag{25}$$

which has infinite real solutions (Figure 1 shows a piece of its graph),

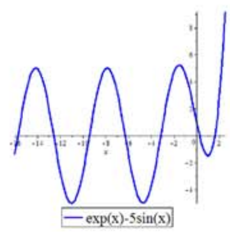

Figure 1. Plot of $e^x - 5sin(x)$

To solve the above problem by standard ADM and their modifications [3-8] is difficult. Here, we obtain three solutions of nonlinear equation (24) by choosing only one initial approximation $\bar{x} = -5.0$. This work is done by using different β's which are obtained by applying algorithm (4.II) to this problem. Tables 5., 5. and 5. show the results. Now, consider Table 5., in the first iteration, if we choose β =1.56841442257449 then the obtained approximations tend to the other solution, the obtained results are shown in Table 5., Again, consider Table 5., in the first iteration, if we choose β =-2.08456286550767 then the obtained approximation solutions tend to the other solution, the obtained results are shown in Table 5.,

The advantage of using the proposed algorithm of this paper is clearly demonstrated for these examples.

6. Conclusion

In this paper, an efficient algorithm for solving nonlinear equations is proposed. This method can be found more than one zero of a given nonlinear equation (if exist) by using only one initial solution. This algorithm is based on the Adomian decomposition method and the Banach fixed point theorem. The proposed method is tested by several examples and the results show the efficiency of the proposed algorithm.

Table 5: First approximate solution for Example 5.3

Real β's	Approximated Solution, $\bar{x}$	$\lvert f(\bar{x}) \rvert$
5.06455026839923, 1.56841442257449 -2.08456286550767,	$\bar{x} = \sum_{i=0}^{3} x_i = -6.69544129140790$	0.4126
3.31953835868750, 5.08..., 6.34...	$\bar{x} = \sum_{i=0}^{3} x_i = -6.34375878562871$	0.0609
4.90886367208948, 5.62..., 5.54...	$\bar{x} = \sum_{i=0}^{5} x_i = -6.28281146508226$	2.14e-7
4.99813150561590, 5.553479455..., 5.553479459...	$\tilde{x} = \sum_{i=0}^{5} x_i = -6.28281167905265$ (Note that $\tilde{x} = \bar{x}$)	0

Table 6: Second approximate solution for Example 5.3

Real β's	Approximated Solution, $\bar{x}$	$\lvert f(\bar{x}) \rvert$
5.06455026839923, 1.56841442257449, -2.08456286550767	$\bar{x} = \sum_{i=0}^{2} x_i = -8.35795928240905$	1.067
-5.81513280517790, 0.71..., -2.68...	$\bar{x} = \sum_{i=0}^{3} x_i = -9.59523735001112$	0.1704
-5.89027775471385, -4.51..., -5.47...	$\bar{x} = \sum_{i=0}^{5} x_i = -9.42488591243072$	0.000092
-5.00008037684925, -5.55564521..., -5.55564522...	$\tilde{x} = \sum_{i=0}^{5} x_i = -9.42479410041240$ (Note that $\tilde{x} = \bar{x}$)	0

Table 7: Third approximate solution for Example 5.3

Real β's	Approximated Solution, $\bar{x}$	$\mid f(\bar{x}) \mid$
5.06455026839923, 1.56841442257449, -2.08456286550767	$\bar{x} = \sum_{i=0}^{1} x_i = -2.70317148715466$	0.4470
-3.24760167799114, -5.10..., -6.45...	$\bar{x} = \sum_{i=0}^{3} x_i = -3.06988632699550$	0.0802
-4.92813051931722, -5.59..., -5.69...	$\bar{x} = \sum_{i=0}^{5} x_i = -3.15016231584163$	5.19 e-7
-5.04266135301120, -5.6029573..., -5.6029575...	$\tilde{x} = \sum_{i=0}^{5} x_i = -3.15016179727586$ (Note that $\tilde{x} = \bar{x}$)	1.10 e-14

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